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Positive Solutions Of Nonlocal Second-Order Differential Equations With Derivative Functions Under Robin Conditions*

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Abstract

In this paper, we investigate the second-order differential equations with derivative functions under Robin conditions

$$\begin{cases} -u''(t) = f(t, u(t), u'(t)),\ t \in [0, 1], \\ au(0) - bu'(0) = \alpha[u],\ cu(1) + du'(1) = \beta[u] + \lambda[u'], \end{cases}$$

where $f : [0, 1] \times \mathbb{R}^+ \times \mathbb{R} \longrightarrow \mathbb{R}^+$ is continuous, $\alpha[u]$ and $\beta[u]$ are linear functionals involving Stieltjes integral. The existence of positive solutions of the differential equation in the case of nonlinear terms $f(t, x_1, x_2)$ with super (sub) linear growth with respect to x_1, x_2 under this condition is proved by giving some inequality conditions and conditions on the spectral radius of the linear operator, using the theory of the index of immobile points on the cone in $C[0, 1]$. Some examples are given to illustrate the theorems respectively under multi-point and integral boundary conditions with sign-changing coefficients.

1 Introduction

In recent years, the fixed point theorems of cone mapping have been extensively applied to two-point boundary value problems and some results of existence and multiplicity of positive solutions have been obtained, see [1]–[6]. The existence of positive solutions of equations with Stieltjes integrals in the boundary value condition was discussed by Ming et al. [3]

$$\begin{cases} -u''(t) = f(t, u(t), u'(t)),\ t \in [0, 1], \\ au(0) - bu'(0) = \alpha[u],\ cu(1) + du'(1) = \beta[u], \end{cases} \tag{1}$$

using the theory of fixed point exponents on cones, where

$$\alpha[u] = \int_0^1 u(t)dA(t),\ \beta[u] = \int_0^1 u(t)dB(t),$$

$A,\ B$ are bounded variational functions, $a,\ b,\ c$ and d are nonnegative contants with $\rho = ac + ad + bc > 0$, and the derivative of the unknown function is not contained in the Stieltjes integrals.

There are also some research results on the boundary values problems in the Stieltjes integrals containing derivatives of unknown functions, see [5]–[9]. The authors in [5] studied the existence of positive solutions for the boundary value problem (BVP)

$$\begin{cases} -u''(t) = g(t)f(t, u(t)),\ t \in (0, 1), \\ u'(0) = \alpha[u],\ u(1) = \beta[u] + \lambda[u'], \end{cases} \tag{2}$$

where $\lambda[u'] = \int_0^1 u'(t)d\Lambda(t),\ A,\ B$ and Λ are bounded variational functions and the nonlinear term function does not contain the derivative of the unknown function.

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In this paper, we discuss the existence of positive solutions for the general derivative dependent BVP subject to Stieltjes integral boundary conditions and Robin conditions

$$\begin{cases} -u''(t) = f(t, u(t), u'(t)), \ t \in (0,1), \\ au(0) - bu'(0) = \alpha[u], \ cu(1) + du'(1) = \beta[u] + \lambda[u'], \end{cases} \tag{3}$$

where $\lambda[u'] = \rho \int_0^1 u'(t) d\Lambda(t)$, Λ are bounded variational functions, $a, \ b, \ c, \ d$ are nonnegative constants with $\rho = ac + ad + bc > 0$. Our work is different from [5, 9, 10, 11] about discussion on nonlinearity and Stieltjes integral boundary conditions.

2 Preliminaries

Let $C^1[0,1]$ denote the Banach space of all continuously differentiable functions on $[0,1]$ with the norm

$$\|u\|_{C^1} = \max\{\|u\|_C, \|u'\|_C\} = \max\left\{\max_{0\le t\le 1} |u(t)|, \max_{0\le t\le 1} |u'(t)|\right\}.$$

We first make the assumptions:

(C_1) $f : [0,1] \times \mathbb{R}^+ \times \mathbb{R} \longrightarrow \mathbb{R}^+$ is continuous where $\mathbb{R}^+ = [0, \infty)$.

(C_2) $\Lambda(1) = \Lambda(0) = 0, \ \Lambda(s) \ge 0, \ \forall s \in [0,1]$.

Lemma 1 *Under the conditions of (C_1) and (C_2), when $\alpha[u] = \beta[u] = 0$, i.e.*

$$\begin{cases} -u''(t) = f(t, u(t), u'(t)), \ t \in [0,1], \\ au(0) - bu'(0) = 0, \ cu(1) + du'(1) = \lambda[u'], \end{cases} \tag{4}$$

the solution of (4) in $C^1[0,1]$ is given by the fixed point of the operator H defined as follows,

$$\begin{aligned}(Hu)(t) &= \int_0^1 \Lambda(s)(at+b) f(s, u(s), u'(s)) ds + \int_0^1 k(t,s) f(s, u(s), u'(s)) ds \\ &:= \int_0^1 k_H(t,s) f(s, u(s), u'(s)) ds \end{aligned} \tag{5}$$

where

$$k(t,s) = \frac{1}{\rho}\begin{cases} (as+b)(c+d-ct), 0 \le s \le t \le 1, \\ (at+b)(c+d-cs), 0 \le t \le s \le 1, \end{cases} \tag{6}$$

$$k_H(t,s) = \begin{cases} \Lambda(s)(at+b) + \frac{1}{\rho}(as+b)(c+d-ct), 0 \le s \le t \le 1, \\ \Lambda(s)(at+b) + \frac{1}{\rho}(at+b)(c+d-cs), 0 \le t \le s \le 1. \end{cases} \tag{7}$$

Proof. First, integrate twice for $-u''(t) = f(t, u(t), u'(t))$ in $[0,t]$ and $[t,1]$ and use the side value conditions in (4) to obtain equation (5). The details are as follows

$$\begin{aligned}(Hu)(t) &= \int_0^1 \Lambda(s)(at+b) f(s, u(s), u'(s)) ds + \int_0^1 k(t,s) f(s, u(s), u'(s)) ds \\ &= \int_0^1 \Lambda(s)(at+b) f(s, u(s), u'(s)) ds \\ &\quad + \int_0^t \frac{1}{\rho}(as+b)(c+d-ct) f(s, u(s), u'(s)) ds \\ &\quad + \int_t^1 \frac{1}{\rho}(at+b)(c+d-cs) f(s, u(s), u'(s)) ds, \end{aligned}$$

$$\begin{aligned}(Hu)'(t) &= a\int_0^1 \Lambda(s)f(s,u(s),u'(s))ds - \int_0^t \frac{c}{\rho}(as+b)f(s,u(s),u'(s))ds \\ &\quad + \int_t^1 \frac{a}{\rho}(c+d-cs)f(s,u(s),u'(s))ds,\end{aligned}$$

and

$$(Hu)''(t) = -f(t,u(t),u'(t)).$$

Then

$$(Hu)(0) = b\int_0^1 \Lambda(s)f(s,u(s),u'(s))ds + \int_0^1 \frac{b}{\rho}(c+d-cs)f(s,u(s),u'(s))ds,$$

$$(Hu)'(0) = a\int_0^1 \Lambda(s)f(s,u(s),u'(s))ds + \int_0^1 \frac{a}{\rho}(c+d-cs)f(s,u(s),u'(s))ds,$$

$$a(Hu)(0) - b(Hu)'(0) = 0,$$

$$(Hu)(1) = \int_0^1 \Lambda(s)(a+b)f(s,u(s),u'(s))ds + \int_0^1 \frac{d}{\rho}(as+b)f(s,u(s),u'(s))ds,$$

$$(Hu)'(1) = a\int_0^1 \Lambda(s)f(s,u(s),u'(s))ds - \int_0^1 \frac{c}{\rho}(as+b)f(s,u(s),u'(s))ds,$$

$$c(Hu)(1) + d(Hu')(1) = \rho\int_0^1 \Lambda(s)ds,$$

$$\begin{aligned}\lambda[(Hu)'] &= \rho\int_0^1 \left[a\int_0^1 \Lambda(s)f(s,u(s),u'(s))ds - \int_0^t \frac{c}{\rho}(as+b)f(s,u(s),u'(s))ds\right. \\ &\quad \left. + \int_t^1 \frac{a}{\rho}(c+d-cs)f(s,u(s),u'(s))ds\right] d\Lambda(t) \\ &= a\rho\int_0^1\int_0^1 \Lambda(s)f(s,u(s),u'(s))dsd\Lambda(t) - \int_0^1\int_0^t c(as+b)f(s,u(s),u'(s))dsd\Lambda(t) \\ &\quad + \int_0^1\int_t^1 a(c+d-cs)f(s,u(s),u'(s))dsd\Lambda(t) \\ &= \int_0^1 (-acs-bc)(-\Lambda(s))ds + \int_0^1 (ac+ad-acs)\Lambda(s)ds \\ &= \int_0^1 \rho\Lambda(s)ds,\end{aligned}$$

and

$$c(Hu)(1) + d(Hu)'(1) = \lambda[(Hu)'].$$

Therefore, $(Hu)''(t) = -f(t,u(t),u'(t)),\ a(Hu)(0) - b(Hu)'(0) = 0,\ c(Hu)(1) + d(Hu)'(1) = \lambda[(Hu)']$. ∎

Lemma 2 *If (C_2) is satisfied, then there exists a non-negative function $\Phi_H(s) = \Lambda(s)(a+b) + \frac{1}{\rho}(as+b)(c+d-cs)$ such that $\forall t,s \in [0,1]$ has*

$$c(t)\Phi_H(s) \le k_H(t,s) \le \Phi_H(s),$$

where

$$c(t) = \min\left\{\frac{b(c+d-ct)}{\rho M_0(a+b)+(a+b)(c+d)},\ \frac{d(at+b)}{\rho M_0(a+b)+(a+b)(c+d)}\right\}.$$

Proof. Viewing $k_H(t,s)$ as a function about t, and according to (7),

$$k_H(t,s) \leq \Phi_H(s) = \Lambda(s)(a+b) + \frac{1}{\rho}(as+b)(c+d-cs).$$

Denote

$$M_0 := \sup_{s\in[0,1]} \Lambda(s). \tag{8}$$

When $0 \leq s \leq t \leq 1$,

$$\begin{aligned} \frac{k_H(t,s)}{\Phi_H(s)} &= \frac{\Lambda(s)(at+b) + \frac{1}{\rho}(as+b)(c+d-ct)}{\Lambda(s)(a+b) + \frac{1}{\rho}(as+b)(c+d-cs)} \\ &\geq \frac{\frac{1}{\rho}b(c+d-ct)}{M_0(a+b) + \frac{1}{\rho}(a+b)(c+d)} \\ &\geq \frac{b(c+d-ct)}{\rho M_0(a+b) + (a+b)(c+d)} := c_0(t). \end{aligned}$$

When $0 \leq t \leq s \leq 1$,

$$\begin{aligned} \frac{k_H(t,s)}{\Phi_H(s)} &= \frac{\Lambda(s)(at+b) + \frac{1}{\rho}(at+b)(c+d-cs)}{\Lambda(s)(a+b) + \frac{1}{\rho}(as+b)(c+d-cs)} \\ &\geq \frac{\frac{1}{\rho}d(at+b)}{M_0(a+b) + \frac{1}{\rho}(a+b)(c+d)} \\ &\geq \frac{d(at+b)}{\rho M_0(a+b) + (a+b)(c+d)} := c_1(t). \end{aligned}$$

Therefore,

$$\frac{k_H(t,s)}{\Phi_H(s)} \geq \min\{c_0(t), c_1(t)\} =: c(t).$$

■

It is easy to prove that BVP (3) has a solution if and only if there exists a solution in $C^1[0,1]$, for the following integral equation

$$u(t) = \gamma_1(t)\alpha[u] + \gamma_2(t)\beta[u] + (Hu)(t) := (Tu)(t), \tag{9}$$

$$\gamma_1(t) = \frac{c(1-t+d)}{\rho} \quad \text{and} \quad \gamma_2(t) = \frac{at+b}{\rho}.$$

Because the mixed side value condition of equation (3) contains the derivative of the unknown function, similar to the method used by Webb in [5], we need to give the corresponding Green's function. We also impose the following hypotheses:

(C_3) A and B are of bounded variation and for $s \in [0,1]$,

$$\mathcal{K}_A(s) := \int_0^1 k_H(t,s)dA(t) \geq 0, \quad \mathcal{K}_B(s) := \int_0^1 k_H(t,s)dB(t) \geq 0;$$

(C_4) $0 \leq \alpha[\gamma_1] < 1,\ \beta[\gamma_1] \geq 0,\ 0 \leq \beta[\gamma_2] < 1,\ \alpha[\gamma_2] \geq 0$, and

$$D := (1-\alpha[\gamma_1])(1-\beta[\gamma_2]) - \alpha[\gamma_2]\beta[\gamma_1] > 0.$$

Define the operator as follows

$$\begin{aligned}(Su)(t) &= \frac{\gamma_1(t)}{D}\left[(1-\beta[\gamma_2])\int_0^1 \mathcal{K}_A(s)f(s,u(s),u'(s))ds\right.\\ &\left.+\alpha[\gamma_2]\int_0^1 \mathcal{K}_B(s)f(s,u(s),u'(s))ds\right]\\ &+\frac{\gamma_2(t)}{D}\left[\beta[\gamma_1]\int_0^1 \mathcal{K}_A(s)f(s,u(s),u'(s))ds\right.\\ &\left.+(1-\alpha[\gamma_1])\int_0^1 \mathcal{K}_B(s)f(s,u(s),u'(s))ds\right]\\ &+\int_0^1 k_H(t,s)f(s,u(s),u'(s))ds\\ &= \int_0^1 k_S(t,s)f(s,u(s),u'(s))ds, \end{aligned} \tag{10}$$

i.e,

$$(Su)(t) = \int_0^1 k_S(t,s)f(s,u(s),u'(s))ds, \tag{11}$$

where

$$k_S(t,s) = \frac{\gamma_1(t)}{D}\left[(1-\beta[\gamma_2])\mathcal{K}_A(s)+\alpha[\gamma_2]\mathcal{K}_B(s)\right] + \frac{\gamma_2(t)}{D}\left[\beta[\gamma_1]\mathcal{K}_A(s)+(1-\alpha[\gamma_1])\mathcal{K}_B(s)\right] + k_H(t,s). \tag{12}$$

Lemma 3 *If (C_2)–(C_4) hold, then there exists a nonnegative continuous function $\Phi(s)$ satisfying*

$$v(t)\Phi(s) \le k_S(t,s) \le \Phi(s) \qquad \text{for } t,\ s\in[0,1],$$

where

$$v(t) = \min\left\{\frac{b(c+d-ct)}{\rho\left[\widetilde{\Phi_1}+\widetilde{\Phi_2}+M_0(a+b)\right]+(a+b)(c+d)}, \frac{d(at+b)}{\rho\left[\widetilde{\Phi_1}+\widetilde{\Phi_2}+M_0(a+b)\right]+(a+b)(c+d)}\right\},$$

$$\widetilde{\Phi_1} := \frac{c(1+d)}{\rho D}\left[(1-\beta[\gamma_2])\left(\sup_{s\in[0,1]}\mathcal{K}_A(s)\right)+\alpha[\gamma_2]\left(\sup_{s\in[0,1]}\mathcal{K}_B(s)\right)\right],$$

$$\widetilde{\Phi_2} := \frac{a+b}{\rho D}\left[\beta[\gamma_1]\left(\sup_{s\in[0,1]}\mathcal{K}_A(s)\right)+(1-\alpha[\gamma_1])\left(\sup_{s\in[0,1]}\mathcal{K}_B(s)\right)\right],$$

and

$$\Phi(s) = \frac{c(1+d)}{\rho D}\left[(1-\beta[\gamma_2])\mathcal{K}_A(s)+\alpha[\gamma_2]\mathcal{K}_B(s)\right] + \frac{a+b}{\rho D}\left[\beta[\gamma_1]\mathcal{K}_A(s)+(1-\alpha[\gamma_1])\mathcal{K}_B(s)\right] + \Phi_H(s).$$

Proof. Viewing $k_S(t,s)$ as a function about t, and according to (7), (8), (12), (C_3),

$$\Phi_1(s) = \frac{c(1+d)}{\rho D}\left[(1-\beta[\gamma_2])\mathcal{K}_A(s)+\alpha[\gamma_2]\mathcal{K}_B(s)\right],$$

$$\Phi_2(s) = \frac{a+b}{\rho D}\left[\beta[\gamma_1]\mathcal{K}_A(s)+(1-\alpha[\gamma_1])\mathcal{K}_B(s)\right].$$

Therefore,

$$k_S(t,s) \le \Phi(s) = \Phi_1(s)+\Phi_2(s)+\Phi_H(s).$$

When $0 \le s \le t \le 1$,

$$\begin{aligned}\frac{k_S(t,s)}{\Phi(s)} &= \frac{\Phi_1(s)+\Phi_2(s)+\Lambda(s)(at+b)+\frac{1}{\rho}(as+b)(c+d-ct)}{\Phi_1(s)+\Phi_2(s)+\Lambda(s)(a+b)+\frac{1}{\rho}(as+b)(c+d-cs)}\\ &\ge \frac{\frac{1}{\rho}b(c+d-ct)}{\widetilde{\Phi_1}+\widetilde{\Phi_2}+M_0(a+b)+\frac{1}{\rho}(a+b)(c+d)}\\ &\ge \frac{b(c+d-ct)}{\rho\left[\widetilde{\Phi_1}+\widetilde{\Phi_2}+M_0(a+b)\right]+(a+b)(c+d)} := v_0(t).\end{aligned}$$

When $0 \le t \le s \le 1$,

$$\begin{aligned}\frac{k_S(t,s)}{\Phi(s)} &= \frac{\Phi_1(s)+\Phi_2(s)+\Lambda(s)(at+b)+\frac{1}{\rho}(at+b)(c+d-cs)}{\Phi_1(s)+\Phi_2(s)+\Lambda(s)(a+b)+\frac{1}{\rho}(as+b)(c+d-cs)}\\ &\ge \frac{\frac{1}{\rho}d(at+b)}{\widetilde{\Phi_1}+\widetilde{\Phi_2}+M_0(a+b)+\frac{1}{\rho}(a+b)(c+d)}\\ &\ge \frac{d(at+b)}{\rho\left[\widetilde{\Phi_1}+\widetilde{\Phi_2}+M_0(a+b)\right]+(a+b)(c+d)} := v_1(t).\end{aligned}$$

Therefore,

$$\frac{k_S(t,s)}{\Phi(s)} = \min\{v_0(t), v_1(t)\} := v(t).$$

■

By (10), then

$$\begin{aligned}\left|\frac{\partial k_S(t,s)}{\partial t}\right| &\le \left|\frac{-c}{\rho D}[(1-\beta[\gamma_2])\mathcal{K}_A(s)+\alpha[\gamma_2]\mathcal{K}_B(s)]+\frac{a}{\rho D}[\beta[\gamma_1]\mathcal{K}_A(s)+(1-\alpha[\gamma_1])\mathcal{K}_B(s)]\right|\\ &\quad +\left|\frac{\partial k_H(t,s)}{\partial t}\right|\\ &\le \left|\frac{-c}{\rho D}[(1-\beta[\gamma_2])\mathcal{K}_A(s)+\alpha[\gamma_2]\mathcal{K}_B(s)]+\frac{a}{\rho D}[\beta[\gamma_1]\mathcal{K}_A(s)+(1-\alpha[\gamma_1])\mathcal{K}_B(s)]\right|\\ &\quad +\max\left\{a\Lambda(s)+\frac{1}{\rho}a(c+d-cs),\ a\Lambda(s)+\frac{1}{\rho}c(as+b)\right\} := \Psi(s),\end{aligned} \tag{13}$$

where

$$\frac{\partial k_H(t,s)}{\partial t} = \begin{cases} a\Lambda(s)+\frac{1}{\rho}(-acs-bc), & 0\le s\le t\le 1,\\ a\Lambda(s)+\frac{1}{\rho}(ac+ad-acs), & 0\le t\le s\le 1.\end{cases}$$

Define two cones in $C^1[0,1]$ and two linear operators in $C[0,1]$ as follows

$$P = \left\{u \in C^1[0,1] : u(t) \ge 0,\ \forall t \in [0,1]\right\}, \tag{14}$$

$$K = \{u \in P : u(t) \ge c\,\|u\|_C,\ \forall t\in[0,1];\ \alpha[u]\ge 0,\ \beta[u]\ge 0\}, \tag{15}$$

$$(Lu)(t) = \int_0^1 k_S(t,s)u(s)ds,\ u \in C[0,1], \tag{16}$$

and

$$(L^*u)(s) = \int_0^1 k_S(t,s)u(t)dt,\ u \in C[0,1]. \tag{17}$$

We write $u \preceq v$ equivalently $v \succeq u$ if and only if $v-u \in P$, to denote the cone ordering induced by P.

Lemma 4 *If (C_1)–(C_4) hold, then $S:\ P \longrightarrow K$ and $L,\ L^*:\ C[0,1] \longrightarrow C[0,1]$ are completely continuous operators with $L(P) \subset K$.*

Proof. From (11), (12) and (C_1)–(C_4) we have for $u \in P$ that $(Su)(t) \geq 0$. It is easy to see from (C_1) that $S: P \longrightarrow C^1[0,1]$ is continuous. Let I be a bounded set in P, then there exists $M > 0$ such that $\|u\|_{C^1} \leq M$ for all $u \in I$. By (C_1) and Lemma 3, we have that $\forall u \in I$ and $t \in [0,1]$,

$$(Su)(t) \leq \left(\max_{(s,x,y)\in[0,1]\times[0,M]\times[-M,M]} f(s,x,y)\right)\int_0^1 \Phi(s)ds,$$

$$\begin{aligned}\left|(Su)'(t)\right| &\leq \left(\max_{(s,x,y)\in[0,1]\times[0,M]\times[-M,M]} f(s,x,y)\right)\int_0^1 \left|\frac{\partial k_S(t,s)}{\partial t}\right| ds \\ &\leq \left(\max_{(s,x,y)\in[0,1]\times[0,M]\times[-M,M]} f(s,x,y)\right)\int_0^1 \Psi(s)ds.\end{aligned}$$

Then $S(I)$ is uniformly bounded in $C^1[0,1]$. Moreover $\forall u \in I$ and $t_1,\ t_2 \in [0,1]$ with $t_1 < t_2$,

$$\begin{aligned}|(Su)(t_1) - (Su)(t_2)| &\leq \int_0^1 |k_S(t_1,s) - k_S(t_2,s)|\, f(s,u(s),u'(s))ds \\ &\leq \left(\max_{(s,x,y)\in[0,1]\times[0,M]\times[-M,M]} f(s,x,y)\right)\int_0^1 |k_S(t_1,s) - k_S(t_2,s)|\, ds,\end{aligned}$$

and

$$\begin{aligned}\left|(Su)'(t_1) - (Su)'(t_2)\right| &\leq \int_0^1 |k'_S(t_1,s) - k'_S(t_2,s)|\, f(s,u(s),u'(s))ds \\ &\leq \int_{t_1}^{t_2} |k'_S(t_1,s) - k'_S(t_2,s)|\, f(s,u(s),u'(s))ds \\ &\leq 2\left(\max_{(s,x,y)\in[0,1]\times[0,M]\times[-M,M]} f(s,x,y)\right)\int_{t_1}^{t_2} \Psi(s)ds.\end{aligned}$$

Thus $S(I)$ and $S'(I) := \left\{\delta' : \delta'(t) = (Su)'(t), u \in I\right\}$ are equicontinuous. Therefore $S : P \longrightarrow C^1[0,1]$ is completely continuous by the Arzela-Ascoli theorem.

For $u \in P$, it follows from Lemma 3 that

$$\|Su\|_C = \max_{0\leq t\leq 1}\left(\int_0^1 k_S(t,s)f(s,u(s),u'(s))ds\right) \leq \int_0^1 \Phi(s)f(s,u(s),u'(s))ds,$$

and hence for $t \in [0,1]$,

$$(Su)(t) = \int_0^1 k_S(t,s)f(s,u(s),u'(s))ds \geq v(t)\int_0^1 \Phi(s)f(s,u(s),u'(s))ds \geq v(t)\,\|Su\|_C\,.$$

From (C_1)–(C_4) it can easily be checked that $\alpha[Su] \geq 0$ and $\beta[Su] \geq 0$. Thus $S : P \longrightarrow K$.

Similarly, $L,\ L^* : C[0,1] \longrightarrow C[0,1]$ are completely continuous operators with $L(P) \subset K$. ■

Lemma 5 *If (C_1)–(C_4) hold, then S and T have the same fixed points in K. As a result, BVP (3) has a solution if and only if S has a fixed point.*

Proof. First, assume that u is a fixed point of T, i.e.

$$u(t) = (Tu)(t) = \gamma_1(t)\alpha[u] + \gamma_2(t)\beta[u] + (Hu)(t),$$

$$\alpha[Tu]=\alpha[\gamma_1]\alpha[u]+\alpha[\gamma_2]\beta[u]+\alpha[Hu],\ \beta[Tu]=\beta[\gamma_1]\alpha[u]+\beta[\gamma_2]\beta[u]+\beta[Hu]$$

and

$$\begin{pmatrix}\alpha[Hu]\\ \beta[Hu]\end{pmatrix}=\begin{pmatrix}1-\alpha[\gamma_1] & -\alpha[\gamma_2]\\ -\beta[\gamma_1] & 1-\beta[\gamma_2]\end{pmatrix}\begin{pmatrix}\alpha[u]\\ \beta[u]\end{pmatrix}=Q\begin{pmatrix}\alpha[u]\\ \beta[u]\end{pmatrix}.$$

From (C_4),

$$\begin{pmatrix}\alpha[u]\\ \beta[u]\end{pmatrix}=\frac{1}{D}\begin{pmatrix}1-\beta[\gamma_2] & \alpha[\gamma_2]\\ \beta[\gamma_1] & 1-\alpha[\gamma_1]\end{pmatrix}\begin{pmatrix}\alpha[Hu]\\ \beta[Hu]\end{pmatrix}=Q^{-1}\begin{pmatrix}\alpha[Hu]\\ \beta[Hu]\end{pmatrix}.$$

$$\begin{aligned}
u(t) &= \frac{\gamma_1(t)}{D}\left[(1-\beta[\gamma_2])\alpha[Hu]+\alpha[\gamma_2]\beta[Hu]\right]\\
&\quad+\frac{\gamma_2(t)}{D}\left[\beta[\gamma_1]\alpha[Hu]+(1-\alpha[\gamma_1])\beta[Hu]\right]+(Hu)(t)\\
&= \frac{\gamma_1(t)}{D}\left[(1-\beta[\gamma_2])\int_0^1\int_0^1 k_H(t,s)f(s,u(s),u'(s))dsdA(t)\right.\\
&\quad\left.+\alpha[\gamma_2]\int_0^1\int_0^1 k_H(t,s)f(s,u(s),u'(s))dsdB(t)\right]\\
&\quad+\frac{\gamma_2(t)}{D}\left[\beta[\gamma_1]\int_0^1\int_0^1 k_H(t,s)f(s,u(s),u'(s))dsdA(t)\right.\\
&\quad\left.+(1-\alpha[\gamma_1])\int_0^1\int_0^1 k_H(t,s)f(s,u(s),u'(s))dsdB(t)\right]\\
&\quad+\int_0^1 k_H(t,s)f(s,u(s),u'(s))ds\\
&= \frac{\gamma_1(t)}{D}\left[(1-\beta[\gamma_2])\int_0^1 \mathcal{K}_A(s)f(s,u(s),u'(s))ds\right.\\
&\quad\left.+\alpha[\gamma_2]\int_0^1 \mathcal{K}_B(s)f(s,u(s),u'(s))ds\right]\\
&\quad+\frac{\gamma_2(t)}{D}\left[\beta[\gamma_1]\int_0^1 \mathcal{K}_A(s)f(s,u(s),u'(s))ds\right.\\
&\quad\left.+(1-\alpha[\gamma_1])\int_0^1 \mathcal{K}_B(s)f(s,u(s),u'(s))ds\right]\\
&\quad+\int_0^1 k_H(t,s)f(s,u(s),u'(s))ds\\
&= \int_0^1 k_S(t,s)f(s,u(s),u'(s))ds=(Su)(t).
\end{aligned}$$

Therefore, u is a fixed point of S.

Second, assuming that u is a fixed point of S,

$$\begin{pmatrix}\alpha_1\\ \beta_1\end{pmatrix}=Q^{-1}\begin{pmatrix}\alpha[Hu]\\ \beta[Hu]\end{pmatrix}\quad\text{and}\quad Q^{-1}=\begin{pmatrix}1-\beta[\gamma_2] & \alpha[\gamma_2]\\ \beta[\gamma_1] & 1-\alpha[\gamma_1]\end{pmatrix}.$$

We have $u(t)=(Su)(t)=\gamma_1\alpha_1+\gamma_2\beta_1+(Hu)(t)$. Therefore, $\alpha[u]=\alpha[\gamma_1]\alpha_1+\alpha[\gamma_2]\beta_1+\alpha[Hu]$ and $\beta[u]=\beta[\gamma_1]\alpha_1+\beta[\gamma_2]\beta_1+\beta[Hu]$.

Let $\alpha[u]=\alpha_1+\eta_1$ and $\beta[u]=\beta_1+\eta_2$. Substitute them into the above equation, so we have that

$$\begin{pmatrix}1-\alpha[\gamma_1] & -\alpha[\gamma_2]\\ -\beta[\gamma_1] & 1-\beta[\gamma_2]\end{pmatrix}\begin{pmatrix}\alpha_1\\ \beta_1\end{pmatrix}+\begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix}=\begin{pmatrix}\alpha[Hu]\\ \beta[Hu]\end{pmatrix}.$$

Hence,

$$\begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}\quad\text{and}\quad\begin{pmatrix}\alpha[u]\\ \beta[u]\end{pmatrix}=\begin{pmatrix}\alpha_1\\ \beta_1\end{pmatrix}.$$

Then $(Su)(t)=(Tu)(t)=u(t)$. That is, S and T have the same fixed point. ■

3 Main Results

In order to prove the main theorems, we need the following properties of fixed point index, see [1, 2, 7].

Lemma 6 *Let Ω be a bounded open subset of X with $0\in\Omega$ and K be a cone in X. If $A: K\cap\bar{\Omega}\longrightarrow K$ is a completely continuous operator and $\mu Au\neq u$ for $u\in K\cap\partial\Omega$ and $\mu\in[0,1]$, then the fixed point index $i(A,\ K\cap\Omega,\ K)=1$.*

Lemma 7 *Let Ω be a bounded open subset of X and K be a cone in X. If $A: K\cap\bar{\Omega}\longrightarrow K$ is a completely continuous operator and there exists $v_0\in K\setminus\{0\}$ such that $u-Au\neq vv_0$ for $u\in K\cap\partial\Omega$ and $v\geq 0$, then the fixed point index $i(A,\ K\cap\Omega,\ K)=0$.*

Recall that a cone P in Banach space X is said to be total if $X=\overline{P-P}$.

Lemma 8 (Krein-Rutman) *Let P be a total cone in Banach space X and $L: X\longrightarrow X$ be a completely continuous linear operator with $L(P)\subset P$. If the spectral radius $r(L)>0$, then there exists $\varphi\in P\setminus\{0\}$ such that $L\varphi=r(L)\varphi$, where 0 denotes the zero element in X.*

The following lemma comes from [7, Theorem 2.5] and is useful for later calculations of $r(L)$.

Lemma 9 *Let P be a cone in Banach space X and $L: X\longrightarrow X$ be a completely continuous linear operator with $L(P)\subset P$. If there exist $v_0\in P\setminus\{0\}$ and $\lambda_0>0$ such that $Lv_0\geq\lambda_0 v_0$ in the sense of partial ordering induced by P, then there exist $u_0\in P\setminus\{0\}$ and $\lambda_1\geq\lambda_0$ such that $Lu_0=\lambda_1 u_0$.*

In the sequel, let $X=C^1[0,1]$ and denote $\Omega_r=\{u\in X:\|u\|_{C^1}<r\}$ for $r>0$.

Theorem 1 *Under the hypotheses(C_1)–(C_4) suppose that*

(F_1) *there exist nonnegative constants $a_1,\ b_1,\ c_1$ satisfying*

$$a_1\int_0^1\Phi(s)ds+b_1\int_0^1\Psi(s)ds<1 \tag{18}$$

such that

$$f(t,x_1,x_2)\leq a_1x_1+b_1\left|x_2\right|+c_1, \tag{19}$$

for all $(t,x_1,x_2)\in[0,1]\times\mathbb{R}^+\times\mathbb{R}$;

(F_2) *there exist constants $a_2>0$ and $r>0$ such that*

$$f(t,\ x_1,\ x_2)\geq a_1x_1, \tag{20}$$

for all $(t,x_1,x_2)\in[0,1]\times[0,r]\times[-r,r]$, moreover the spectral radius $r(L)\geq\frac{1}{a_1}$, where L is defined by (16).

Then BVP (3) has at least one positive solution.

Proof. Let $W=\{u\in K:\ u=\mu Su,\ \mu\in[0,1]\}$ where S and K are respectively defined in (11) and (15). We first assert that W is a bounded set. In fact, if $u\in W$, then $u=\mu Su$ for some $\mu\in[0,1]$. From Lemma 3 and (19), we have that

$$\begin{aligned}\|u\|_C &= \mu\max_{0\le t\le 1}(\int_0^1 k_S(t,s)f(s,u(s),u'(s))ds)\\ &\le \int_0^1 \Phi(s)[a_1u(s)+b_1\left|u'(s)\right|+c_1]ds\\ &< (a_1\|u\|_C+b_1\|u'\|_C+c_1)\int_0^1\Phi(s)ds\end{aligned}$$

and

$$\begin{aligned}\|u'\|_C &= \mu\max_{0\le t\le 1}\left|\int_0^1\frac{\partial k_S(t,s)}{\partial t}f(s,u(s),u'(s))ds\right|\\ &\le \int_0^1 \Psi(s)[a_1u(s)+b_1\left|u'(s)\right|+c_1]ds\\ &< (a_1\|u\|_C+b_1\|u'\|_C+c_1)\int_0^1\Psi(s)ds.\end{aligned}$$

Thus

$$\|u\|_C\le(1-a_1\int_0^1\Phi(s)ds)^{-1}(b_1\|u'\|_C+c_1)\int_0^1\Phi(s)ds \tag{21}$$

and

$$\begin{aligned}\|u'\|_C \le\ & \frac{a_1b_1}{1-a_1\int_0^1\Phi(s)ds}\|u'\|_C\left(\int_0^1\Phi(s)ds\right)\left(\int_0^1\Psi(s)ds\right)\\ &+\frac{a_1c_1}{1-a_1\int_0^1\Phi(s)ds}\left(\int_0^1\Phi(s)ds\right)\left(\int_0^1\Psi(s)ds\right)\\ &+b_1\|u'\|_C\int_0^1\Psi(s)ds+c_1\int_0^1\Psi(s)ds.\end{aligned} \tag{22}$$

From (18), (21) and (22), then

$$\|u\|_C\le\frac{c_1\int_0^1\Phi(s)ds}{1-a_1\int_0^1\Phi(s)ds-b_1\int_0^1\Psi(s)ds},\quad \|u'\|_C\le\frac{c_1\int_0^1\Psi(s)ds}{1-a_1\int_0^1\Phi(s)ds-b_1\int_0^1\Psi(s)ds},$$

and hence W is bounded.

Now select $R>\max\{r,\ \sup W\}$. Then $\mu Su\ne u$ for $u\in K\cap\partial\Omega_R$ and $\mu\in[0,1]$, and $i(S,\ K\cap\Omega_R,\ K)=1$ follows from Lemma 6. It is easy to see that $L(P^+)\subset P\subset P^+$, where

$$P^+[0,1]=\{u\in C[0,1]:u(t)\ge 0,\ \forall t\in[0,1]\}$$

is a total cone in $C[0,1]$. Since $r(L)\ge\frac{1}{a_2}>0$, it follows from Lemma 8 that there exists $\varphi_1\in P^+\setminus\{0\}$ such that $L\varphi_1=r(L)\varphi_1$. Furthermore, $\varphi_1=(r(L))^{-1}L\varphi_1\in K$ by Lemma 4.

We may suppose that S has no fixed points in $K\cap\partial\Omega_r$ and will show that $u-Su\ne v\varphi_1$ for $u\in K\cap\partial\Omega_r$ and $v\ge 0$. Otherwise, there exist $u_1\in K\cap\partial\Omega_r$ and $\tau\ge 0$ such that $u_1-Su_1=\tau\varphi_1$, and it is clear that $\tau\ge 0$. Since $u_1\in K\cap\partial\Omega_r$, we have $0\le u_1(t)\le r,\ -r\le u_1'(t)\le r,\ \forall t\in[0,1]$. It follows from (20) that $(Su_1)(t)\ge a_1(Lu_1)(t)$ which implies that

$$u_1=\tau\varphi_1+Su_1\succeq\tau\varphi_1+a_1Lu_1\succeq\tau\varphi_1. \tag{23}$$

Set $\tau^* = \sup\{\tau > 0 : u_1 \geq \tau\varphi_1\}$. Then $\tau \leq \tau^* < +\infty$ and $u_1 \geq \tau^*\varphi_1$. Thus it follows from (23) that

$$u_1 \succeq \tau\varphi_1 + a_1 L u_1 \succeq \tau\varphi_1 + a_1\tau^* L\varphi_1 = \tau\varphi_1 + a_1\tau^* r(L)\varphi_1.$$

But $r(L) \geq \frac{1}{a_1}$, so $u_1 \succeq (\tau + \tau^*)\varphi_1$, which is a contradiction to the definition of τ^*. Therefore $u - Su \neq \tau\varphi_1$ for $u \in K \cap \partial\Omega_r$ and $\tau \geq 0$.

From Lemma 7, it follows that $i(S, K \cap \Omega_r, K) = 0$. Making use of the properties of fixed point index, we have that

$$i(S, K \cap \left(\Omega_R \setminus \overline{\Omega_r}\right), K) = i(S, K \cap \Omega_R, K) - i(S, K \cap \Omega_r, K) = 1$$

and hence S has at least one fixed point in K. Therefore, BVP(3) has at least one positive solution by Lemma 5. ■

Lemma 10 ([3, Lemma 5.1 of Chapter XII]) *Let $R > 0$, and let $\varphi : [0, \infty] \longrightarrow (0, \infty)$ be continuous and satisfy*

$$\int_0^\infty \frac{\rho d\rho}{\varphi(\rho)} = \infty. \tag{24}$$

Then there exists a number $M > 0$, depending only on φ, R such that if $v \in C^2[0, 1]$ which satisfies $\|v\|_C \leq R$ and $|v''(t)| \leq \varphi(|v'(t)|)$, $t \in [0, 1]$, then $\|v'\|_C \leq M$.

Theorem 2 *Under the hypotheses (C_1)–(C_4) suppose that*

(F_3) *there exist nonnegative constants a_1, b_1 and $r > 0$ satisfying*

$$(a_1 + b_1) \max\left\{\int_0^1 \Phi(s)ds, \int_0^1 \Psi(s)ds\right\} < 1, \tag{25}$$

such that

$$f(t, x_1, x_2) \leq a_1 x_1 + b_1 |x_2|, \tag{26}$$

for all $(t, x_1, x_2) \in [0, 1] \times [0, r] \times [-r, r]$;

(F_4) *there exist positive constants a_2, c_2 such that*

$$f(t, x_1, x_2) \geq a_2 x_1 - c_2, \tag{27}$$

for all $(t, x_1, x_2) \in [0, 1] \times \mathbb{R}^+ \times \mathbb{R}$, moreover the spectral radiu $r(L) \geq \frac{1}{a_2}, r(L^*) \geq \frac{1}{a_2}$ where L, L^* are defined by (16) and (17)respectively;

(F_5) *for any $M > 0$ there is a positive continuous function $\varphi(\rho)$ on $\mathbb{R}^+$ satisfying (24) such that*

$$f(t, x, y) \leq \varphi(|y|) - c_2,\ \forall (t, x, y) \in [0, 1] \times [0, M] \times \mathbb{R}, \tag{28}$$

Then BVP (3) has at least one positive solution.

Proof. (i) First we prove that $\mu Su \neq u$ for $u \in K \cap \partial\Omega_r$ and $\mu \in [0, 1]$. In fact, if there exist $u_1 \in K \cap \partial\Omega_r$ and $\mu_0 \in [0, 1]$ such that $u_1 = \mu_0 Su_1$, then we deduce from Lemma 3, (13), (25), (26)and $0 \leq u_1(t) \leq r$, $-r \leq u_1' \leq r$, $\forall t \in [0, 1]$ that

$$\begin{aligned} \|u_1\|_C &= \mu_0 \max_{0 \leq t \leq 1}\left(\int_0^1 k_S(t, s) f(s, u(s), u'(s))ds\right) \\ &\leq \int_0^1 \Phi(s)[a_1 u_1(s) + b_1 |u_1'(s)|]ds \\ &\leq (a_1 + b_1)\left(\int_0^1 \Phi(s)ds\right)\|u_1\|_{C^1} \leq \|u_1\|_{C^1} = r \end{aligned}$$

and

$$\begin{aligned}\|u_1'\|_C &= \mu_0 \max_{0\le t\le 1}\left|\int_0^1 \frac{\partial k_S(t,s)}{\partial t} f(s,u(s),u'(s))ds\right| \\ &\le \int_0^1 \Psi(s)[a_1u_1(s)+b_1\,|u_1'(s)|]ds \\ &\le (a_1+b_1)\left(\int_0^1 \Psi(s)ds\right)\|u_1\|_{C^1} \le \|u_1\|_{C^1} = r.\end{aligned}$$

Hence $\|u_1\|_{C^1} < r$ which contradicts $u_1 \in K\cap\partial\Omega_r$. Therefore, $i\,(S,\ K\cap\Omega_r,\ K) = 1$ follows from Lemma 6.

(ii) It is easy to see that $L^*(P^+)\subset P^+$. Since $r(L^*)\ge \frac{1}{a_2} > 0$, it follows from Lemma 8 that there exists $\varphi^*\in P^+\setminus\{0\}$ such that $L^*\varphi^* = r(L^*)\varphi^*$. Let

$$M = \frac{c_2\int_0^1 \varphi^*(t)dt\int_0^1 k_S(t,s)ds}{(a_2r(L^*)-1)\int_0^1 v(t)\varphi^*(t)dt}, \tag{29}$$

where $v(t)$ comes from Lemma 3.

(iii) For $u\in P$, define

$$(S_1u)(t) = \int_0^1 k_S(t,s)\left(f(s,u(s),u'(s))+c_2\right)ds. \tag{30}$$

Similar to the proof in Lemma 4, we know that is completely continuous. If there exist $u_2\in K$ and $\lambda_0\in[0,1]$ such that

$$(1-\lambda_0)\,Su_2 + \lambda_0 S_1u_2 = u_2, \tag{31}$$

thus by (27) and (31), we obtain that

$$\begin{aligned}&\int_0^1 \varphi^*(t)u_2(t)dt \\ =\;& (1-\lambda_0)\int_0^1 \varphi^*(t)dt\int_0^1 k_S(t,s)f(s,u_2(s),u_2'(s))ds \\ &+\lambda_0\int_0^1 \varphi^*(t)dt\int_0^1 \left(k_S(t,s)\left(f(s,u_2(s),u_2'(s))+c_2\right)ds\right. \\ =\;& \int_0^1 \varphi^*(t)dt\int_0^1 k_S(t,s)\left(f(s,u_2(s),u_2'(s))+\lambda_0c_2\right)ds \\ \ge\;& \int_0^1 \varphi^*(t)dt\int_0^1 k_S(t,s)\left(a_2u_2(s)-c_2+\lambda_0c_2\right)ds \\ \ge\;& a_2\int_0^1 \varphi^*(t)dt\int_0^1 k_S(t,s)u_2(s)ds - c_2\int_0^1 \varphi^*(t)dt\int_0^1 k_S(t,s)ds \\ =\;& a_2\int_0^1 u_2(s)ds\int_0^1 k_S(t,s)\varphi^*(t)dt - c_2\int_0^1 \varphi^*(t)dt\int_0^1 k_S(t,s)ds \\ =\;& a_2r(L^*)\int_0^1 \varphi^*(s)u_2(s)ds - c_2\int_0^1 \varphi^*(t)dt\int_0^1 k_S(t,s)ds,\end{aligned}$$

which implies that

$$\|u_2\|_C\int_0^1 v(t)\varphi^*(t)dt \le \int_0^1 \varphi^*(t)u_2(t)dt \le \frac{c_2\int_0^1 \varphi^*(t)dt\int_0^1 k_S(t,s)ds}{a_2r(L^*)-1}$$

and thus

$$\|u_2\|_C \le \frac{c_2 \int_0^1 \varphi^*(t)dt \int_0^1 k_S(t,s)ds}{(a_2 r(L^*) - 1) \int_0^1 v(t)\varphi^*(t)dt} = M. \tag{32}$$

We can derive from (28), (31) and (32)that

$$\begin{aligned} |u_2''(t)| &= (1-\lambda_0) f(t, u_2(t), u_2'(t)) + \lambda_0 (f(t, u_2(t), u_2'(t)) + c_2) \\ &= f(t, u_2(t), u_2'(t)) + \lambda_0 c_2 \le f(t, u_2(t), u_2'(t)) + c_2 \\ &\le \varphi(|u_2'(t)|). \end{aligned} \tag{33}$$

By Lemma 10, there exists a constant $M_1 \ge 0$ such that $\|u_2'\|_C \le M_1$. Let $R > \max\{r,\ M,\ M_1\}$. Then

$$(1-\lambda) Su + \lambda S_1 u \ne u,\ \forall u \in K \cap \partial\Omega_R,\ \lambda \in [0,1]. \tag{34}$$

From (34) it follows that

$$i(S,\ K \cap \Omega_R,\ K) = i(S_1,\ K \cap \Omega_R,\ K) \tag{35}$$

by the homotopy invariance property of fixed point index.

(iv) Since $L(P^+) \subset P \subset P^+$ and $r(L) \ge \frac{1}{a_2} > 0$, it follows from Lemma 8 that there exists $\varphi_0 \in P^+ \setminus \{0\}$ such that $L\varphi_0 = r(L)\varphi_0$. Furthmore, $\varphi_0 = (r(L))^{-1} L\varphi_0 \in K$ by Lemma 3. Now we prove that $u - S_1 u \ne v\varphi_0$ for $u \in K \cap \partial\Omega_R$ and $v \ge 0$ and hence

$$i(S_1, K \cap \Omega_R, K) = 0 \tag{36}$$

holds by Lemma 7. Assume that there exist $u_0 \in K \cap \partial\Omega_R$ and $v_0 \ge 0$ such that $u_0 - S_1 u_0 = v_0\varphi_0$. Obviously $v_0 > 0$ by (34) and

$$u_0 = S_1 u_0 + v_0\varphi_0 \succeq v_0\varphi_0. \tag{37}$$

Set

$$v^* = \sup\{v > 0 : u_0 \succeq v\varphi_0\},$$

Then $v_0 \le v^* < +\infty$ and $u_0 \succeq v^*\varphi_0$. From (27) and (37) we have

$$\begin{aligned} u_0 &= S_1 u_0 + v_0\varphi_0 \succeq a_2 L u_0 + v_0\varphi_0 \\ &\succeq a_2 v^* L\varphi_0 + v_0\varphi_0 = a_2 v^* r(L)\varphi_0 + v_0\varphi_0. \end{aligned}$$

But $r(L) \ge \frac{1}{a_2}$, so $u_0 \succeq (v^* + v_0)\varphi_0$, which is a contradiction to the definition of v^*.

(v) From (35) and (36) it follows that $i(S,\ K \cap \Omega_R,\ K) = 0$ and

$$i\left(S,\ K \cap \left(\Omega_R \setminus \overline{\Omega_r}\right),\ K\right) = i(S,\ K \cap \Omega_R,\ K) - i(S,\ K \cap \Omega_r,\ K) = -1.$$

Hence S has at least one fixed point in K and BVP (3) has at least one positive solution by Lemma 5. ■

4 Examples

We consider second-order problem under mixed boundary conditions involving multi-point with coefficients of both signs and integral with sign-changing kernel

$$\begin{cases} -u''(t) = f(t, u(t), u'(t)),\ t \in [0,1] \\ u'(0) = \frac{1}{3}u(\frac{1}{3}) - \frac{1}{12}u(\frac{2}{3}), \\ u'(1) = \int_0^1 u(t)(\cos \pi t + \frac{2}{\pi})dt + \int_0^1 \rho u'(t)(\sin \pi t + \pi t \cos \pi t)\,dt. \end{cases} \tag{38}$$

that is, $\alpha[u] = \frac{1}{3}u(\frac{1}{3}) - \frac{1}{12}u(\frac{2}{3})$, $\beta[u] = \int_0^1 u(t)(\cos \pi t + \frac{2}{\pi})dt$, $\lambda[u'] = \int_0^1 u'(t)\ (\sin \pi t + \pi t \cos \pi t)\,dt$ and $a = c = 0,\ b = d = 1,\ \rho = 1$. Hence

$$k_H(t,s) = \begin{cases} \Lambda(s)t + s(1-t), 0 \le s \le t \le 1, \\ \Lambda(s)t + t(1-s), 0 \le t \le s \le 1, \end{cases}$$

$\Lambda(t) = t\sin \pi t$, $\Lambda(0) = \Lambda(1) = 0$, $\Lambda(s) \geq 0$, $\forall s \in [0,1]$. Then (C_2) is satisfied. Since

$$0 \leq \mathcal{K}_A(s) = \frac{1}{3}k_H(\frac{1}{3}, s) - \frac{1}{12}k_H(\frac{2}{3}, s) = \begin{cases} \left(\frac{1}{18}\sin \pi s + \frac{7}{36}\right) s & 0 \leq s \leq \frac{1}{3}, \\ \left(\frac{1}{18}\sin \pi s - \frac{5}{36}\right) s + \frac{1}{9} & \frac{1}{3} \leq s \leq \frac{2}{3}, \\ \frac{1}{18}\left[(\sin \pi s - 1)\, s + 1\right] & \frac{2}{3} \leq s \leq 1, \end{cases}$$

$$\mathcal{K}_B(s) = \int_0^1 k_H(t,s)\left(\cos \pi t + \frac{2}{\pi}\right) dt = \frac{\cos(\pi s) + 2s - 1}{\pi^2} + \frac{s - s^2}{\pi} \geq 0,$$

we see that (C_3) is satisfied. Since

$$0 \leq \alpha[\gamma_1] = \frac{7}{36} < 1, \alpha[\gamma_2] = \frac{1}{18} \geq 0,$$

$$\beta[\gamma_1] = \frac{1}{\pi} + \frac{2}{\pi^2} \geq 0, 0 \leq \beta[\gamma_2] = \frac{1}{\pi} - \frac{2}{\pi^2},$$

and

$$D = (1 - \alpha[\gamma_1])(1 - \beta[\gamma_2]) - \alpha[\gamma_2]\beta[\gamma_1] = \frac{29\pi^2 - 31\pi + 54}{36\pi^2} > 0,$$

(C_4) is also satisfied. Furthermore,

$$\Phi(s) = \frac{1}{D}\left[\frac{\pi^2 + 4}{\pi^2}\mathcal{K}_A(s) + \frac{31}{36}\mathcal{K}_B(s)\right] + s^2 \sin \pi s + s(1 - s),$$

$$\Psi(s) = \frac{1}{D}\left[\frac{2\pi - \pi^2}{\pi^2}\mathcal{K}_A(s) + \frac{31}{36}\mathcal{K}_B(s)\right] + \max\left\{s(\sin \pi s) + (1 - s),\ s(\sin \pi s) + s\right\}.$$

Example 1 *If $f(t, x_1, x_2) = tx^{\frac{1}{3}} + x^{\frac{2}{3}}$, take $a_1 = \frac{1}{4}$, $b_1 = \frac{1}{2}$ and thus*

$$\begin{aligned} & a_1 \int_0^1 \Phi(s)ds + b_1 \int_0^1 \Psi(s)ds \\ = \ & \frac{1}{4} \times \frac{144\pi^7 + 594\pi^6 + 3750\pi^5 + (108\sqrt{3} - 8352)\pi^4 + 12816\pi^3 + (432\sqrt{3} - 11664)\pi^2}{3132\pi^6 - 3348\pi^5 + 5832\pi^4} \\ & + \frac{1}{2} \times \frac{-432\pi^6 + 7434\pi^5 + 4032\pi^4 + (2374 - 324\sqrt{3})\pi^3 + 16596\pi^2 + 648\sqrt{3}\pi}{9396\pi^5 - 10044\pi^4 + 16596\pi^3} \\ < \ & 1. \end{aligned}$$

So F_1 holds for c_1 large enough. In addition, take $a_2 = 12$, $r = \frac{1}{24\sqrt{3}}$. From Lemma 3 and Lemma 4 we have that $v(t) \in C^+[0,1]$, and for $t \in [0,1]$,

$$Lv(t) \geq v(t)\int_0^1 \Phi(s)v(s)ds,$$

then by Lemma 9, the spectral radius

$$\begin{aligned} r(L) \ &\geq \int_0^1 \Phi(s)v(s)ds \\ &= \left[\frac{1872\pi^7 + 7722\pi^6 + 48750\pi^5 + (14043\sqrt{3} - 108576)\pi^4 + 166608\pi^3 + (5616\sqrt{3} - 151632)\pi^2}{56196\pi^6 + 60264\pi^5 + 104976\pi^4}\right] 0.93 \\ &\approx 0.264903 \\ &> \frac{1}{a_2}. \end{aligned} \tag{39}$$

Therefore, (F_2) holds since (20) can be inferred easily. By Theorem 1 we know that BVP (38) has at least one positive solution.

Example 2 *If* $f(t,x_1,x_2)=\frac{tx_1^4+x_2^4}{2(1+x_1^2+x_2^2)}$, *take* $a_1=\frac{1}{4}$, $b_1=\frac{1}{2}$ *and thus*

$$\begin{aligned}&(a_1+b_1)\int_0^1\Phi(s)ds\\ =\;&\frac{3}{4}\times\frac{144\pi^7+594\pi^6+3750\pi^5+(108\sqrt{3}-8352)\pi^4+12816\pi^3+(432\sqrt{3}-11664)\pi^2}{3132\pi^6-3348\pi^5+5832\pi^4}\\ <\;&1,\end{aligned}$$

$$\begin{aligned}&(a_1+b_1)\int_0^1\Psi(s)ds\\ =\;&\frac{3}{4}\times\frac{-432\pi^6+7434\pi^5+4032\pi^4+(2374-324\sqrt{3})\pi^3+16596\pi^2+648\sqrt{3}\pi}{9396\pi^5-10044\pi^4+16596\pi^3}\\ <\;&1.\end{aligned}$$

Therefore, (F_3) *holds since* (26) *can be inferred easily for* $r=1$. *Now take take* $a_2=12$. *From Lemma 3 and Lemma 4 we have* $\Phi\in P^+[0,1]$ *and for* $s\in[0,1]$,

$$(L^*\Phi)(s)\geq\Phi(s)\int_0^1 v(t)\Phi(t)dt,$$

then by Lemma 9, the spectral radius

$$\begin{aligned}r(L^*)\;&\geq\;\int_0^1 v(t)\Phi(t)dt\\ &=\;\left[\frac{1872\pi^7+7722\pi^6+48750\pi^5+(14043\sqrt{3}-108576)\pi^4+166608\pi^3+(5616\sqrt{3}-151632)\pi^2}{56196\pi^6+60264\pi^5+104976\pi^4}\right]0.93\\ &\approx\;0.264903\\ &>\;\frac{1}{a_2}.\end{aligned}$$

It is easy to see that (27) *holds for* c_2 *large enough. Therefore,* (F_4) *is satisfied if* (39) *is conbined with. As for* (F_5), *one can let* $\varphi(\rho)=M^2+\rho^2+c_2$. *By Theorem 2 we know that BVP* (38) *has at least one positive solution.*

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