Applied Mathematics E-Notes, 22(2022), 656-659 © ISSN 1607-2510
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Odd/Even r-Free Numbers

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Received 19 October 2021

Abstract

In this paper we use an elementary method to give an asymptotical ratio of odd to even r-free numbers and show that it is asymptotically $2^r : 2^r - 2$.

1 Introduction and Results

Let $r > 1$ be a fixed integer. A positive integer n is *r-free* if each of its prime factors appears to the power at most $r - 1$. A positive integer n is *r-full* if each of its prime factors appears to the power at least r. As usual, 2-full and 3-full numbers are called square-full and cube-full, respectively.

Let $N_r(x)$ be the number of r-free integers $\leq x$. It is well know that for r fixed

$$N_r(x) = \frac{1}{\zeta(r)}x + O(x^{1/r}). \tag{1}$$

For a study of these asymptotic formulae, we refer to [2, Equation 14.24].

In this paper, we study the odd/even dichotomy for the set of r-free numbers. The motivation follows from work of Scott [5] and Jameson [3], where it was shown that the ratio of odd to even square-free numbers is asymptotically $2 : 1$. In 2020, the second author [6] used an elementary method to prove the odd/even dichotomy for the set of square-full numbers. In 2021, Tippawan Puttasontiphot and Teerapat Srichan [7] extended the method in [6] to the case of cube-full numbers. Very recently, Jameson [4] used this to give a new proof in [3]. Thus, it would be interesting to generalize these results to the odd/even dichotomy for the set of r-free numbers by using the method in [6].

Here we prove the following results.

Theorem 1 *As $x \to \infty$, we have*

$$\frac{O(x)}{E(x)} \sim \frac{2^r}{2^r - 2}, \tag{2}$$

where $O(x)$ and $E(x)$ denote the number of odd and even r-free positive integers not greater than x, respectively.

Corollary 2 *As $x \to \infty$, we have*

$$\frac{C_{odd}(x)}{C_{even}(x)} \sim 2, \tag{3}$$

where $C_{odd}(x)$ and $C_{even}(x)$ denote the number of odd and even square-free positive integers not greater than x, respectively.

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Notations

Let A be a given set, for $x > 1$, we denote $A(x)$ be the number of elements in A. $f(x) \sim g(x)$ means $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$ and we say that $f(x)$ is asymptotic to $g(x)$ as $x \to \infty$.

2 Proofs

Proof of Theorem 1. Let E and O be the set of all even and odd r-free integers, respectively. We assume that,

$$O(x) \sim ax \qquad \text{and} \qquad E(x) \sim bx, \qquad \text{for some } a, b \in \mathbb{R}^+. \tag{4}$$

Denote

$$A_1 = \{n \in E : 4 \nmid n\}, \quad A_1^{'} = \{n \in E : 4 \mid n\},$$

for $2 \le k \le r-2$,

$$A_k = \{n \in A_{k-1}^{'} : 2^{k+1} \nmid n\}, \quad A_k^{'} = \{n \in A_{k-1}^{'} : 2^{k+1} \mid n\}.$$

We note that $A_k^{'} = A_{k+1} \cup A_{k+1}^{'}$. Thus,

$$E = \Big(\bigcup_{k=1}^{r-2} A_k\Big) \cup A_{r-2}^{'}. \tag{5}$$

For $1 \le k \le r-2$, set A_k are disjoint set. Thus, from (5), we have

$$E(x) = \Big(\sum_{k=1}^{r-2} A_k(x)\Big) + A_{r-2}^{'}(x). \tag{6}$$

Now, we note that the element $n \in A_k$ is the form $n = 2^{k+1}m + 2^k$, for some $m \in \mathbb{N} \cup \{0\}$. Thus, $\frac{n}{2^k}$ is odd and r-free. This implies that, for $1 \le k \le r-2$,

$$A_k(x) = O\Big(\frac{x}{2^k}\Big). \tag{7}$$

Similary, the element $h \in A_{r-2}^{'}$ is the form $h = 2^r m_1 + 2^{r-1}$, for some $m_1 \in \mathbb{N} \cup \{0\}$. Thus, $\frac{h}{2^{r-1}}$ is odd and r-free. This implies that,

$$A_{r-2}^{'}(x) = O\Big(\frac{x}{2^{r-1}}\Big). \tag{8}$$

Inserting (7) and (8) in (6), we have

$$E(x) = \sum_{k=1}^{r-2} O\Big(\frac{x}{2^k}\Big) + O\Big(\frac{x}{2^{r-1}}\Big) = \sum_{k=1}^{r-1} O\Big(\frac{x}{2^k}\Big). \tag{9}$$

In view of (4) and (9), we have

$$bx = \sum_{k=1}^{r-1} a\frac{x}{2^k} = ax\Big(1 - 2^{1-r}\Big).$$

This proves (2).

Now it remains to prove the existence of a and b.

In view of (9), we write

$$N_r(x) = O(x) + E(x) = \sum_{k=0}^{r-1} O\Big(\frac{x}{2^k}\Big). \tag{10}$$

We replace x in (10) by $x/2$ and subtract this with (10). We have

$$N_r(x) - N_r\Big(\frac{x}{2}\Big) = O(x) - O\Big(\frac{x}{2^r}\Big). \tag{11}$$

Replacing x in (11) by $x/2^r$, we have

$$N_r\Big(\frac{x}{2^r}\Big) - N_r\Big(\frac{x}{2^{r+1}}\Big) = O\Big(\frac{x}{2^r}\Big) - O\Big(\frac{x}{2^{2r}}\Big). \tag{12}$$

In view of (11) and (12), we have

$$N_r(x) - N_r\Big(\frac{x}{2}\Big) + N_r\Big(\frac{x}{2^r}\Big) - N_r\Big(\frac{x}{2^{r+1}}\Big) = O(x) - O\Big(\frac{x}{2^{2r}}\Big).$$

Repeating this, we have

$$O(x) - O\Big(\frac{x}{2^{r(k+1)}}\Big) = \sum_{i=0}^{k} N_r\Big(\frac{x}{2^{ri}}\Big) - \sum_{i=0}^{k} N_r\Big(\frac{x}{2^{ri+1}}\Big). \tag{13}$$

The asymptotic formula (1) implies $N_r(x) \sim cx$, where $c = 1/\zeta(r)$. Then, for $\epsilon > 0$, we take x_0 such that

$$(c-\epsilon)x \le N_r(x) \le (c+\epsilon)x, \text{ for } x \ge x_0. \tag{14}$$

To apply inequality (14) with (13), we take k such that $\frac{x}{2^{rk+r+1}} < x_0 \le \frac{x}{2^{rk+1}}$. Then, we have

$$(c-\epsilon)\frac{x}{2^{ri+1}} \le N_r(\frac{x}{2^{ri+1}}) \le (c+\epsilon)\frac{x}{2^{ri+1}}, \tag{15}$$

and

$$(c-\epsilon)\frac{x}{2^{ri}} \le N_r(\frac{x}{2^{ri}}) \le (c+\epsilon)\frac{x}{2^{ri}}, \tag{16}$$

for $0 \le i \le k$. In view of (13), (15) and (16), we have

$$\begin{aligned} O(x) - O\Big(\frac{x}{2^{r(k+1)}}\Big) &\le \sum_{i=0}^{k}(c+\epsilon)\frac{x}{2^{ri}} - \sum_{i=0}^{k}(c-\epsilon)\frac{x}{2^{ri+1}} \\ &= x\Big(\frac{c}{2} + \frac{3\epsilon}{2}\Big)\sum_{i=0}^{k}\frac{1}{2^{ri}} \\ &\le x\Big(\frac{c}{2} + \frac{3\epsilon}{2}\Big)\sum_{i=0}^{\infty}\frac{1}{2^{ri}} \\ &= x\Big(\frac{c}{2} + \frac{3\epsilon}{2}\Big)\frac{2^r}{2^r-1}. \end{aligned}$$

From the choosing k such that $\frac{x}{2^{rk+r+1}} < x_0 \le \frac{x}{2^{r(k+1)}}$, we have $O\Big(\frac{x}{2^{r(k+1)}}\Big) \le O(2x_0) < 2x_0$. Then, we have

$$O(x) \le x\Big(c + 3\epsilon\Big)\frac{2^{r-1}}{2^r-1} + 2x_0 \le x\Big(c+3\epsilon\Big)\frac{2^{r-1}}{2^r-1} + \frac{2^{r+1}}{2^r-1}x_0.$$

Thus, for $x > \frac{x_0}{\epsilon}$,

$$O(x) \le x\Big(c+3\epsilon\Big)\frac{2^{r-1}}{2^r-1} + \frac{2^{r-1}}{2^r-1}4x\epsilon = x\Big(c+7\epsilon\Big)\frac{2^{r-1}}{2^r-1}.$$

By the similar proof we deal with the lower bound. In view of (13), (15) and (16), we have

$$\begin{aligned} O(x) - O\Big(\frac{x}{2^{r(k+1)}}\Big) &\geq \sum_{i=0}^{k}(c-\epsilon)\frac{x}{2^{ri}} - \sum_{i=0}^{k}(c+\epsilon)\frac{x}{2^{ri+1}} \\ &= x\Big(\frac{c}{2} - \frac{3\epsilon}{2}\Big)\sum_{i=0}^{k}\frac{1}{2^{ri}} \\ &= x\Big(\frac{c}{2} - \frac{3\epsilon}{2}\Big)\frac{2^r}{2^r-1} - x\Big(\frac{c}{2} - \frac{3\epsilon}{2}\Big)\frac{2^{-rk}}{2^r-1} \\ &\geq x\Big(c-3\epsilon\Big)\frac{2^{r-1}}{2^r-1} - \frac{cx}{2^{rk+1}(2^r-1)}. \end{aligned}$$

We note that $2^r x_0 \geq \frac{x}{2^{rk+1}}$. Then, we have

$$O(x) \geq O(x) - O\Big(\frac{x}{2^{r(k+1)}}\Big) \geq x\Big(c-3\epsilon\Big)\frac{2^{r-1}}{2^r-1} - cx_0\frac{2^r}{2^r-1}.$$

Thus, for $x > \frac{x_0}{\epsilon}$,

$$O(x) \geq O(x) - O\Big(\frac{x}{2^{r(k+1)}}\Big) \geq x\Big(c-3\epsilon-2c\epsilon\Big)\frac{2^{r-1}}{2^r-1}.$$

This proves the existence of a and consequently b also exists, in fact $b = c - a$. ■

Acknowledgements. The authors are very grateful to the anonymous referee for her or his valuable remarks. This work was financially supported by Office of the Permanent Secretary, Ministry of Higher Education, Science, Research and Innovation, Grant No. RGNS 63-40.

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