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∗-Complex Valued Metric Spaces And Zamfirescu Type Contractions*

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Abstract

In this paper, we introduce the concept of a ∗-complex valued metric space and give some fixed point results for self-mappings with Zamfirescu type contractions on such spaces. Also, we present three nontrivial examples to validate our hypotheses and to show the effectiveness and applicability of an extension of Banach contraction principle discussed herein. Taking the identity function I instead of the generators α and β in the construction of the set $\mathbb{C}(N)$, ∗-complex valued metric spaces turn into the complex valued metric spaces, so our results are stronger than some existing facts in the literature.

1 Introduction

The classical Banach contraction principle [1], a power tool in nonlinear analysis, initiated a new era of research in fixed point theory due to its intensive applicability in wide areas of mathematics like numerical analysis and differential or integral equations. Many authors investigated some fixed point theorems and new contractive conditions on metric spaces or generalizations of metric spaces. For example, Zamfirescu [2] obtained a new contractive condition which combines the contractive conditions in Banach [1], Kannan [3] and Chatterjea [4] for classical metric spaces. In 2011, Azam et al. [5] introduced complex valued metric spaces as a generalization of metric spaces. They proved some fixed point theorems for mappings which satisfy a rational inequality in complex valued metric spaces. They also applied these results to the existence and uniqueness for a solution of an integral equation. After the publication of this work, some very important fixed point results have appeared in complex valued metric spaces; for several related examples, see [6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16].

In 1972, Grossman and Katz [17] pointed out that different calculus, called non-Newtonian calculus consisting of some special calculus such as geometric, bigeometric, quadratic, biquadratic calculus, and so forth, which modify the calculus created by Isaac Newton and Gottfried Wilhelm Leibnitz in the 17th century. The non-Newtonian calculus is a self-contained system independent of any other system of calculus and provides a wide diversity of mathematical tools for use in technology and mathematics. Also, it has wonderful applications in various areas including engineering, physics, finance, approximation theory, dynamical systems, tumor therapy, weighted calculus etc. It provides differentiation and integration tools based on non-Newtonian operations instead of classical operations. Every property in classical calculus has an analogue in non-Newtonian calculus.

Recently, Çakmak and Başar [18] built up some topological properties and sequence spaces with respect to non-Newtonian calculus. Tekin and Başar [19] obtained some sequence spaces over non-Newtonian complex field by defining non-Newtonian complex field. Duyar et al. [20] introduced and examined for non-Newtonian analysis some fundamenal concepts and usual algebraic and topologic properties related to this concepts. Binbaşıoğlu et al. [21] introduced a fixed point theory by defining some topological structures of the relevant non-Newtonian metric space.

In the rest of this section, we recall some necessary notions and basic facts.

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A complete ordered field is called an arithmetic if its realm is a subset of $\mathbb{R}$. A generator is a one-to-one function whose domain $\mathbb{R}$ and whose range is a subset of $\mathbb{R}$. Let α be a generator with range A. We denote by $\mathbb{R}_\alpha$ the range of generator α. Also, the elements of $\mathbb{R}_\alpha$ (or $\mathbb{R}\,(N)_\alpha$) are called non-Newtonian real numbers. Taking $\alpha = I$, the generator α generates the classical arithmetic and $\mathbb{R}_\alpha = \mathbb{R}$.

Let α be a generator with range A. An arithmetic with range A, and its basic operations and ordering relation are defined as follows, is called α-arithmetic:

$$\begin{aligned}
\alpha\text{-addition} \quad &: \quad y \dot{+} z = \alpha\left\{\alpha^{-1}(y) + \alpha^{-1}(z)\right\}, \\
\alpha\text{-subtraction} \quad &: \quad y \dot{-} z = \alpha\left\{\alpha^{-1}(y) - \alpha^{-1}(z)\right\}, \\
\alpha\text{-multiplication} \quad &: \quad y \dot{\times} z = \alpha\left\{\alpha^{-1}(y) \times \alpha^{-1}(z)\right\}, \\
\alpha\text{-division}\left(z \neq \dot{0}\right) \quad &: \quad y \dot{/} z = \frac{y}{z}\alpha = \alpha\left\{\frac{\alpha^{-1}(y)}{\alpha^{-1}(z)}\right\}, \\
\alpha\text{-ordering} \quad &: \quad y \dot{<} z \Longleftrightarrow \alpha^{-1}(y) < \alpha^{-1}(z), \\
&\quad\;\; \left(y \dot{\leq} z \Longleftrightarrow \alpha^{-1}(y) \leq \alpha^{-1}(z)\right),
\end{aligned}$$

for all $y, z \in A$. With the above new operations, $\left(A, \dot{+}, \dot{-}, \dot{\times}, \dot{/}, \dot{\leq}\right)$ is an α-arithmetic.

A α-positive number is a number x with $\dot{0} \dot{<} x$ and a α-negative number is a number with $x \dot{<} \dot{0}$. α-zero and α-one numbers are denoted by $\dot{0} = \alpha(0)$ and $\dot{1} = \alpha(1)$. Also, $\alpha(-p) = \alpha\left\{-\alpha^{-1}\left(\dot{p}\right)\right\} = \dot{-}\dot{p}$ for all $p \in \mathbb{Z}^+$. Thus, the set of all α-integers is $\{..., \alpha(-2), \alpha(-1), \alpha(0), \alpha(1), \alpha(2), ...\}$. The α-absolute value of $x \in A$ is defined by

$$\dot{|}\, x \,\dot{|} = |x|_\alpha = \begin{cases} x & \text{if } \dot{0} \dot{<} x, \\ \dot{0} & \text{if } \dot{0} = x, \\ \dot{0} \dot{-} x & \text{if } x \dot{<} \dot{0}. \end{cases}$$

Let $\ddot{b} \in B \subseteq \mathbb{R}$. Then, the number $\ddot{b} \ddot{\times} \ddot{b}$ is called the β-square of $\ddot{b}$ and is denoted by $b^{\ddot{2}}$. Let $\ddot{b}$ be a nonnegative number in B. Then, $\beta\left[\sqrt{\beta^{-1}\left(\ddot{b}\right)}\right]$ is called the β-square root of $\ddot{b}$ and is denoted by $\ddot{\sqrt{b}}$ [17, 18].

An infinite sum

$$a_1 \dot{+} a_2 \dot{+} ... \dot{+} a_n \dot{+} ... =_\alpha \sum_{n=1}^{\infty} a_n$$

is called the non-Newtonian real number series or α-series. If ${}_\alpha\sum_{n=1}^{\infty} a_n$ is non-Newtonian real number series, then a sequence $\{S_m\}$ with the general term $S_m =_\alpha \sum_{n=1}^{m} a_n$ is called non-Newtonian partial sums sequence of the series ${}_\alpha\sum_{n=1}^{\infty} a_n$. If the sequence $\{S_m\}$ is a α-convergent, then it is said that the series ${}_\alpha\sum_{n=1}^{\infty} a_n$ is α-convergent. If ${}^\alpha\lim_{m\to\infty} S_m = S$, then it is written ${}_\alpha\sum_{n=1}^{\infty} a_n = S$. If the limit ${}^\alpha\lim_{m\to\infty} S_m$ is not available or equal to $-\infty$ or $+\infty$, then it is said that the series ${}_\alpha\sum_{n=1}^{\infty} a_n$ is α-divergent. For example, if $\dot{|}\, r \,\dot{|} \dot{<} \dot{1}$, then ${}_\alpha\sum_{n=1}^{\infty} r^{(k-1)_\alpha}$ is α-convergent and ${}_\alpha\sum_{n=1}^{\infty} r^{(k-1)_\alpha} = \frac{\dot{1}}{\dot{1} \dot{-} r}\alpha$ [22].

Let α and β be arbitrarily chosen generators which image the set $\mathbb{R}$ to A and B respectively and $*$-("star-") calculus also be the ordered pair of arithmetics (α-arithmetic, β-arithmetic). The following notations will be used.

	α-arithmetic	β-arithmetic
Realm	$A\left(=\mathbb{R}_\alpha=\mathbb{R}\left(N\right)_\alpha\right)$	$B\left(=\mathbb{R}_\beta=\mathbb{R}\left(N\right)_\beta\right)$
Summation	$\dot{+}$	$\ddot{+}$
Subtraction	$\dot{-}$	$\ddot{-}$
Multiplication	$\dot{\times}$	$\ddot{\times}$
Division	$\dot{/}$	$\ddot{/}$
Ordering	$\dot{\leq}$	$\ddot{\leq}$

There are the following three properties for the isomorphism from $a-$arithmetic to $\beta-$arithmetic that is the unique function $\imath$(iota).

1. $\imath$ is one-to-one.

2. $\imath$ is on A and onto B.

3. For all u,$v \in A$,

$$\begin{aligned}
\iota\left(u\dot{+}v\right) &= \iota\left(u\right)\ddot{+}\iota\left(v\right), \ \iota\left(u\dot{-}v\right)=\iota\left(u\right)\ddot{-}\iota\left(v\right), \\
\iota\left(u\dot{\times}v\right) &= \iota\left(u\right)\ddot{\times}\iota\left(v\right), \ \iota\left(u\dot{/}v\right)=\iota\left(u\right)\ddot{/}\iota\left(v\right), v\neq\dot{0} \\
u &\dot{<} v \Longleftrightarrow \iota\left(u\right)\ddot{<}\iota\left(v\right), \\
\iota\left(u\right) &= \beta\left\{\alpha^{-1}\left(u\right)\right\}.
\end{aligned}$$

Also, for every integer n, we set $\iota\left(\dot{n}\right)=\ddot{n}$ [17, 18].

The concepts of α-convergence sequence, α-Cauchy sequence, non-Newtonian metric space, non-Newtonian completeness, $*$-limit, $*$-continuity, $*$-boundedness, $*$-derivative and $*$-integral of the function $f: X\subset A\rightarrow B$ are found in [17, 18, 22].

The following definitions which appeared in [19] will be a crucial tool in our study.

Let $\dot{a}\in\left(A,\dot{+},\dot{-},\dot{\times},\dot{/},\dot{\leq}\right)$ and $\ddot{b}\in\left(B,\ddot{+},\ddot{-},\ddot{\times},\ddot{/},\ddot{\leq}\right)$ be arbitrarily chosen elements from corresponding arithmetics. Then, the ordered pair $\left(\dot{a},\ddot{b}\right)$ is called as a $*$-point. The set of all $*$-points is called the set of $*$-complex numbers and is denoted by $\mathbb{C}^*$ or $\mathbb{C}\left(N\right)$; that is,

$$\mathbb{C}\left(N\right)=\left\{\left(\dot{a},\ddot{b}\right):\dot{a}\in A\subseteq\mathbb{R},\ddot{b}\in B\subseteq\mathbb{R}\right\}.$$

The set $\mathbb{C}\left(N\right)$ forms a field and a Banach space with the algebraic operations $\oplus$ and $\otimes$ defined on $\mathbb{C}\left(N\right)$ and $*$-norm $\ddot{\|}\ .\ \ddot{\|}$ defined by

$$\begin{aligned}
\oplus &: \mathbb{C}\left(N\right)\times\mathbb{C}\left(N\right)\rightarrow\mathbb{C}\left(N\right), \\
\left(z^*,w^*\right) &\rightarrow z^*\oplus w^*=\left(\dot{a_1},\ddot{b_1}\right)\oplus\left(\dot{a_2},\ddot{b_2}\right)=\left(\dot{a_1}\dot{+}\dot{a_2},\ddot{b_1}\ddot{+}\ddot{b_2}\right), \\
\otimes &: \mathbb{C}\left(N\right)\times\mathbb{C}\left(N\right)\rightarrow\mathbb{C}\left(N\right), \\
\left(z^*,w^*\right) &\rightarrow z^*\otimes w^*=\left(\dot{a_1},\ddot{b_1}\right)\otimes\left(\dot{a_2},\ddot{b_2}\right)=\left(\alpha\left(a_1a_2-b_1b_2\right),\beta\left(a_1b_2+b_1a_2\right)\right), \\
\ddot{\|}\ .\ \ddot{\|} &: \mathbb{C}\left(N\right)\rightarrow\mathbb{R}_\beta, z^*\rightarrow\ddot{\|}\ z^*\ \ddot{\|}=\ddot{\sqrt{\left[\imath\left(\dot{a}_1\dot{-}\dot{0}\right)\right]^{\ddot{2}}\ddot{+}\left(\ddot{b}_1\ddot{-}\ddot{0}\right)^{\ddot{2}}}}=\beta\left(\sqrt{a_1^2+b_1^2}\right)
\end{aligned}$$

for all $z^*=\left(\dot{a}_1,\ddot{b}_1\right), w^*=\left(\dot{a}_2,\ddot{b}_2\right)\in\mathbb{C}\left(N\right)$ where $0_{\mathbb{C}(N)}=\left(\dot{0},\ddot{0}\right)$. The set $\mathbb{C}\left(N\right)$ is also called $*$-complex field. In addition, $\|z^*\otimes w^*\|=\|z^*\|\,\|w^*\|$ for all $z^*,w^*\in\mathbb{C}\left(N\right)$ [19].

For $z^* = \left(\dot{a}, \ddot{b}\right) \in \mathbb{C}(N)$ and $w^* = \left(\dot{c}, \ddot{d}\right) \in \mathbb{C}(N) - \left\{0_{\mathbb{C}(N)}\right\}$, the $*$-division $z \oslash w$ is defined by

$$z^* \oslash w^* = \left(\dot{a}, \ddot{b}\right) \oslash \left(\dot{c}, \ddot{d}\right) = \left(\alpha\left\{\frac{ac+bd}{c^2+d^2}\right\}, \beta\left\{\frac{bc-ad}{c^2+d^2}\right\}\right)$$

[23].

Remark 1 *If for $z^*, w^* \in \mathbb{C}(N)$ we have $z^* \oplus w^* = w^* \oplus z^* = 0_{\mathbb{C}(N)}$, then $*$-complex number w^* is called inverse element with respect to the addition of z^* and is denoted by $\ominus z^*$. We can denote $*$-complex number $\left(\dot{a}, \ddot{b}\right)$ by $\left(\dot{a}, \ddot{0}\right) \oplus i^* \otimes \left(\dot{b}, \ddot{0}\right)$ where $i^* = \left(\dot{0}, \ddot{1}\right)$, $(i^*)^2 = \ominus 1_{\mathbb{C}(N)} = \left(\dot{-1}, \ddot{0}\right)$ and also define $\dot{a}$ and $\ddot{b}$ by $\Re\, z^*$ and $\Im\, z^*$, respectively.*

In 1972, Zamfirescu [2] introduce the following existence and uniqueness theorem for a generalized contraction mapping.

Theorem 1 *Let (X, d) be a complete metric space, α, β, γ real numbers with $\alpha < 1$, $\beta < \frac{1}{2}$, $\gamma < \frac{1}{2}$ and $T : X \to X$ be a function such that for each couple of different points $x, y \in M$, at least one of the following conditions is satisfied:*

(i) $d(Tx, Ty) \leq \alpha d(x, y)$,

(ii) $d(Tx, Ty) \leq \beta\left(d(x, Tx) + d(y, Ty)\right)$,

(iii) $d(Tx, Ty) \leq \gamma\left(d(x, Ty) + d(y, Tx)\right)$.

Then, T has a unique fixed point.

Following the same line, we first define a concept of a $*$-complex valued metric space as a generalization of complex valued metric spaces. Also, we give some topological concepts and properties for this new space. Moreover, we extend some fixed point theorems in complex valued metric spaces and classical metric spaces to some fixed point theorems for Zamfirescu type contractions on $*$-complex valued metric spaces and verify the usability of our results by solving a specially chosen non-Newtonian Volterra integral equation and a specially chosen bigeometric Volterra equation.

Now, we are ready to present and discuss our main results.

2 Some Topological Concepts in $*$-Complex Valued Metric Spaces

This section is fundamentals of $*$-complex valued metric spaces.

At first, we introduce a partial order on $\mathbb{C}(N)$ and a concept of positive $*$-complex number.

Definition 1 *Let $z_1^*, z_2^* \in \mathbb{C}(N)$. Define a partial order $\overset{*}{\precsim}$ on $\mathbb{C}(N)$ as follows:*

$$z_1^* \overset{*}{\precsim} z_2^* \text{ if and only if } \Re\, z_1^* \dot{\leq} \Re\, z_2^*,\ \Im\, z_1^* \ddot{\leq} \Im\, z_2^*.$$

We will write $z_1^* \overset{*}{\prec} z_2^*$ if $\Re\, z_1^* \dot{<} \Re\, z_2^*$, $\Im\, z_1^* \ddot{<} \Im\, z_2^*$. An element $z^* \in \mathbb{C}(N)$ is said to be a positive $*$-complex number if $0_{\mathbb{C}(N)} \overset{*}{\prec} z^*$. The set $\left\{z^* \in \mathbb{C}(N) : 0_{\mathbb{C}(N)} \overset{*}{\prec} z^*\right\}$ will be denoted by $\mathbb{C}(N)^+$.

Now, we give some properties about partial order on $\mathbb{C}(N)$.

Theorem 2 *The following statements are true for $z_1^*, z_2^*, z_3^*, z_4^* \in \mathbb{C}(N)$:*

(i) *If $z_1^*, z_2^* \in \mathbb{C}(N)$ and $0_{\mathbb{C}(N)} \overset{*}{\precsim} z_1^* \overset{*}{\precsim} z_2^*$, then $\ddot{0} \ddot{\leq} \ddot{\|} z_1^* \ddot{\|} \ddot{\leq} \ddot{\|} z_2^* \ddot{\|}$.*

(ii) *If* $0_{\mathbb{C}(N)} \overset{*}{\prec} z_1^* \overset{*}{\prec} z_2^*$, *then* $\ddot{0} \ddot{<} \ddot{\|} z_1^* \ddot{\|} \ddot{<} \ddot{\|} z_2^* \ddot{\|}$.

(iii) *If* $z_1^* \overset{*}{\precsim} z_2^*$, *then* $z_1^* \oplus z_3^* \overset{*}{\precsim} z_2^* \oplus z_3^*$.

(iv) *If* $z_1^* \overset{*}{\prec} z_2^*$, *then* $z_1^* \oplus z_3^* \overset{*}{\prec} z_2^* \oplus z_3^*$.

(v) *If* $z_1^* \overset{*}{\precsim} z_2^*$ *and* $z_3^* \overset{*}{\precsim} z_4^*$, *then* $z_1^* \oplus z_3^* \overset{*}{\precsim} z_2^* \oplus z_4^*$.

Proof. (i) Let $z_1^* = \left(\dot{a_1}, \ddot{b_1}\right), z_2^* = \left(\dot{a_2}, \ddot{b_2}\right)$. If $0_{\mathbb{C}(N)} \overset{*}{\precsim} z_1^* \overset{*}{\precsim} z_2^*$, then $\dot{0} \dot{\leq} \dot{a_1} \dot{\leq} \dot{a_2}, \ddot{0} \ddot{\leq} \ddot{b_1} \ddot{\leq} \ddot{b_2}$. This implies that $\beta(0) \ddot{\leq} \beta\sqrt{a_1^2+b_1^2} \ddot{\leq} \beta\sqrt{a_2^2+b_2^2}$ and so $\ddot{0} \ddot{\leq} \ddot{\|} z_1^* \ddot{\|} \ddot{\leq} \ddot{\|} z_2^* \ddot{\|}$.

Similarly, (ii), (iii), (iv) and (v) can be easily proved using the definitions of the relations $\overset{*}{\precsim}$, $\overset{*}{\prec}$. ■

We continue this section with the concept of an $*$-complex absolutely value of a $*$-complex number and a related result.

Definition 2 *Let* $z^* \in \mathbb{C}(N)$. *Then,* $*$*-complex absolutely value of* z^* *is defined by*

$$\begin{cases} z^* & \text{if } 0_{\mathbb{C}(N)} \overset{*}{\precsim} z^*, \\ 0_{\mathbb{C}(N)} & \text{if } 0_{\mathbb{C}(N)} = z^*, \\ 0_{\mathbb{C}(N)} \ominus z^* & \text{if } z^* \overset{*}{\precsim} 0_{\mathbb{C}(N)}, \end{cases} \quad \text{and denoted by } \overset{*}{|}\, z \,\overset{*}{|}.$$

Theorem 3 *If* $z^* \in \mathbb{C}(N)$, *then* $\ddot{\|} \overset{*}{|}\, z \,\overset{*}{|} \ddot{\|} = \ddot{\|} z^* \ddot{\|}$.

Proof. If $0_{\mathbb{C}(N)} \overset{*}{\precsim} z^*$ or $0_{\mathbb{C}(N)} = z^*$, the proof is clear. We assume that $z^* \overset{*}{\precsim} 0_{\mathbb{C}(N)}$ and $z^* = \left(\dot{a}, \ddot{b}\right)$. This implies that $\overset{*}{|}\, z \,\overset{*}{|} = 0_{\mathbb{C}(N)} \ominus z^*$. Then, we obtain that

$$\ddot{\|} \overset{*}{|}\, z \,\overset{*}{|} \ddot{\|} = \ddot{\|} \left(\dot{-}\dot{a}, \ddot{-}\ddot{b}\right) \ddot{\|} = \beta\left(\sqrt{(-a)^2 + (-b)^2}\right) = \beta\left(\sqrt{a^2+b^2}\right) = \ddot{\|} \left(\dot{a}, \ddot{b}\right) \ddot{\|} = \ddot{\|} z^* \ddot{\|}.$$

■

Next, we define a new concept as follows:

Definition 3 *Let* X *be a nonempty set and* $d^* : X \times X \to \mathbb{C}(N)$ *be a function such that for any* $x, y, z \in X$, *the following properties hold:*

(i) $0_{\mathbb{C}(N)} \overset{*}{\precsim} d^*(x,y)$ *and* $d^*(x,y) = 0_{\mathbb{C}(N)}$ *if and only if* $x = y$,

(ii) $d^*(x,y) = d^*(y,x)$,

(iii) $d^*(x,z) \overset{*}{\precsim} d^*(x,y) \oplus d^*(y,z)$.

Then d^* *is called a* $*$*-complex valued metric on* X *and the pair* (X, d^*) *is called a* $*$*-complex valued metric space.*

Example 1 *Let* $X = \mathbb{C}(N)$ *and*

$$d^*(x,y) = \begin{cases} 0_{\mathbb{C}(N)} = \left(\dot{0}, \ddot{0}\right) & \text{if } x = y, \\ 1_{\mathbb{C}(N)} = \left(\dot{1}, \ddot{0}\right) & \text{if } x \neq y. \end{cases}$$

It is easy to verify d^* *is a* $*$*-complex valued metric on* $\mathbb{C}(N)$ *and* $(\mathbb{C}(N), d^*)$ *is a* $*$*-complex valued metric space.*

Remark 2 *If we take the identity function I instead of the generators α and β in the construction of the set $\mathbb{C}(N)$, then the set of $*$-complex numbers $\mathbb{C}(N)$ turn into the set of complex numbers $\mathbb{C}$. Thus, $*$-complex valued metric spaces generalize the concept of a complex valued metric space defined by Azam et al. [5].*

In the rest of this section, we give some notions about $*$-complex valued metric spaces and discuss some relevant findings.

Definition 4 *Let (X, d^*) be a $*$-complex valued metric space, $x \in X$ and $r^* \in \mathbb{C}(N)^+$. We define a set*

$$B^*(x,r) = \left\{ y \in X : d^*(x,y) \overset{*}{\prec} r^* \right\}$$

which is called a $*$-complex open ball of $*$-complex radius r^* with center x. Similarly, a $*$-complex closed ball of $*$-complex radius r^* with center x is defined by

$$\overline{B^*}(x,r) = \left\{ y \in X : d^*(x,y) \overset{*}{\precsim} r^* \right\}.$$

Definition 5 *Let (X, d^*) be a $*$-complex valued metric space and $A \subset X$. A point $x \in X$ is called a $*$-complex interior point of A if there exists $r^* \in \mathbb{C}(N)^+$ such that $B^*(x,r) \subset A$. A point $x \in X$ is called a $*$-complex limit point of A if $(B^*(x,r) - \{x\}) \cap A \neq \varnothing$ for every $r^* \in \mathbb{C}(N)^+$. The set of $*$-complex interior points of A is denoted by A°_* and the set of $*$-complex limit points of A is denoted by A'_*.*

We say that the subset A is a $*$-complex open set if each element of A belong to A°_*. Also, we say that the subset A is a $*$-complex closed set if each $*$-complex limit point of A belong to A.

Proposition 1 *Let (X, d^*) be a $*$-complex valued metric space. Then, we have the following inequality for all $x, y, z \in X$.*

$$\ddot{\|}\, d^*(x,z) \ominus d^*(y,z)\, \ddot{\|} \ddot{\leq} \ddot{\|}\, d^*(x,y)\, \ddot{\|}\,.$$

Proof. Since d^* is $*$-complex valued metric on X, we can write $d^*(x,z) \overset{*}{\precsim} d^*(x,y) \oplus d^*(y,z)$ and $d^*(y,z) \overset{*}{\precsim} d^*(y,x) \oplus d^*(x,z)$ for all $x, y, z \in X$. Thus, we have

$$0^* \ominus d^*(x,y) \overset{*}{\precsim} d^*(x,z) \ominus d^*(y,z) \overset{*}{\precsim} d^*(x,y)$$

and so $\overset{*}{|}\, d^*(x,z) \ominus d^*(y,z)\, \overset{*}{|} \overset{*}{\precsim} d^*(x,y)$. This implies that

$$\ddot{\|}\, d^*(x,z) \ominus d^*(y,z)\, \ddot{\|} \ddot{\leq} \ddot{\|}\, d^*(x,y)\, \ddot{\|}$$

by Theorems 2(i) and 3. ■

Definition 6 *Let (X, d^*) be a $*$-complex valued metric space, (x_n) be any sequence in X and $x \in X$. If for every $\varepsilon^* \in \mathbb{C}(N)^+$ there exists $n_0 \in \mathbb{N}$ depending on ε^* such that for all $n \geq n_0$, $d^*(x_n, x) \overset{*}{\prec} \varepsilon^*$ then we say that (x_n) is convergent with respect to the metric d^*. We denote this by $\lim\limits_{n\to\infty}{}^* x_n = x$ or $x_n \xrightarrow{d^*} x$ as $n \to \infty$.*

If for every $\varepsilon^* \in \mathbb{C}(N)^+$ there exists $n_0 \in \mathbb{N}$ depending on ε^* such that for all $n, m \geq n_0$, $d^*(x_n, x_m) \overset{*}{\prec} \varepsilon^*$ then we say that (x_n) is a Cauchy sequence with respect to the metric d^*.

If every Cauchy sequence with respect to the metric d^* is convergent with respect to the metric d^* in (X, d^*), then we say that (X, d^*) is a complete $*$-complex valued metric space.

Lemma 1 *Let (X, d^*) be a $*$-complex valued metric space, (x_n) be any sequence in X and $x \in X$. Then, the sequence (x_n) converges to the x with respect to the metric d^* if and only if $\ddot{\|}\, d^*(x_n, x)\, \ddot{\|} \to \ddot{0}$ as $n \to \infty$ (β-convergent).*

Proof. Suppose that (x_n) converges to the x with respect to the metric d^*. We choose a positive $*$-*complex* number

$$\varepsilon^* = \left(\frac{\dot{c}}{\dot{\sqrt{2}}}\alpha, \frac{\ddot{c}}{\ddot{\sqrt{2}}}\beta \right)$$

with a positive real number c. Thus, there is a natural number n_0 such that $d^*(x_n, x) \overset{*}{\prec} \varepsilon^*$ whenever $n \geq n_0$. From Theorem 2(ii),

$$\ddot{\|} d^*(x_n, x) \ddot{\|} \ddot{<} \ddot{\|} \varepsilon^* \ddot{\|} = \beta(c) = \ddot{c}$$

for all $n \geq n_0$. This implies that $\ddot{\|} d^*(x_n, x) \ddot{\|} \to \ddot{0}$ as $n \to \infty$ (β-convergent).

Conversely, suppose that $\ddot{\|} d^*(x_n, x) \ddot{\|} \to \ddot{0}$ as $n \to \infty$ (β-convergent). We claim that for a given positive $*$-complex number $\varepsilon^* = \left(\dot{\varepsilon_1}, \ddot{\varepsilon_2}\right)$ with positive real numbers $\varepsilon_1, \varepsilon_2$, there is a real number $\delta > 0$ such that for any $z^* = \left(\dot{a}, \ddot{b}\right) \in \mathbb{C}(N)$

$$z^* \overset{*}{\prec} \varepsilon^* \text{ whenever } \ddot{\|} z^* \ddot{\|} \ddot{<} \ddot{\delta}.$$

In fact, set $\delta = \min\left\{\frac{\varepsilon_1}{2}, \frac{\varepsilon_2}{2}\right\}$ for every positive $*$-complex number ε^*. If $\ddot{\|} z^* \ddot{\|} \ddot{<} \ddot{\delta}$, then $\beta\left(\sqrt{a^2+b^2}\right) \ddot{<} \beta(\delta)$ and so $|a| < \delta$ and $|b| < \delta$. By the definition of δ we obtain that $a < \varepsilon_1$ and $b < \varepsilon_2$. Then, we can write $\dot{a} \dot{<} \dot{\varepsilon_1}$ and $\ddot{b} \ddot{<} \ddot{\varepsilon_2}$. This shows that $z^* \overset{*}{\prec} \varepsilon^*$. Also, for this δ there is a natural number n_0 such that $\ddot{\|} d^*(x_n, x) \ddot{\|} \ddot{<} \ddot{\delta}$ for all $n \geq n_0$. This implies that $d^*(x_n, x) \overset{*}{\prec} \varepsilon^*$ for all $n \geq n_0$. Then, we conclude that (x_n) converges to the x with respect to the metric d^*. ■

Lemma 2 *Let (X, d^*) be a $*$-complex valued metric space and (x_n) be any sequence in X. Then, the sequence (x_n) is a Cauchy sequence with respect to the metric d^* if and only if for all $m \in \mathbb{N}$, $\ddot{\|} d^*(x_n, x_{n+m}) \ddot{\|} \to \ddot{0}$ as $n \to \infty$ (β-convergent).*

Proof. Suppose that (x_n) is a Cauchy sequence with respect to the metric d^*. We choose a positive $*$-complex number

$$\varepsilon^* = \left(\frac{\dot{c}}{\dot{\sqrt{2}}}\alpha, \frac{\ddot{c}}{\ddot{\sqrt{2}}}\beta \right)$$

with a positive real number c. Thus, there is a natural number n_0 such that $d^*(x_n, x_k) \overset{*}{\prec} \varepsilon^*$ whenever $n, k \geq n_0$. Since there exists a natural number m such that $k = m + n$ for each k greater than n, we can write $d^*(x_n, x_{n+m}) \overset{*}{\prec} \varepsilon^*$ for all $n \geq n_0$. Therefore, for all $n \geq n_0$, we get

$$\ddot{\|} d^*(x_n, x_{n+m}) \ddot{\|} \ddot{<} \ddot{\|} \varepsilon^* \ddot{\|} = \beta(c) = \ddot{c}.$$

This implies that for all $m \in \mathbb{N}$, $\ddot{\|} d^*(x_n, x_{n+m}) \ddot{\|} \to \ddot{0}$ as $n \to \infty$ (β-convergent).

Conversely, suppose that for all $m \in \mathbb{N}$, $\ddot{\|} d^*(x_n, x_{n+m}) \ddot{\|} \to \ddot{0}$ as $n \to \infty$ (β-convergent). Then, for a given positive $*$-complex number $\varepsilon^* = \left(\dot{\varepsilon_1}, \ddot{\varepsilon_2}\right)$ with positive real numbers $\varepsilon_1, \varepsilon_2$, there is a real number $\delta > 0$ such that for any $z^* = \left(\dot{a}, \ddot{b}\right) \in \mathbb{C}(N)$

$$z^* \overset{*}{\prec} \varepsilon^* \text{ whenever } \ddot{\|} z^* \ddot{\|} \ddot{<} \ddot{\delta}.$$

Also, for this δ there is a natural number n_0 such that $\ddot{\|} d^*(x_n, x_{n+m}) \ddot{\|} \ddot{<} \ddot{\delta}$ for all $n \geq n_0$. This means that $d^*(x_n, x_{n+m}) \overset{*}{\prec} \varepsilon^*$ for all $n \geq n_0$. Then, we conclude that (x_n) is a Cauchy sequence with respect to the metric d^*. ■

3 Some Fixed Point Theorems for $*$-Complex Valued Metric Spaces

In this part, to obtain some fixed point results in $*$-complex valued metric spaces we modify the Zamfirescu type contractions which are combination the contractive conditions of Banach [1], Kannan [3] and Chatterjea [4] in metric spaces.

Recall that a fixed point of a self-mapping $T : X \to X$ is a point $x \in X$ such that $Tx = x$.

Now, we give the extension of Zamfirescu's fixed point theorem for self-mappings defined on classical metric spaces, which guarantees the existence and uniqueness of fixed point as follows:

Theorem 4 *Let (X, d^*) be a complete $*$-complex valued metric space, $z^*, u^*, v^* \in \mathbb{C}(N)^+$ such that $\ddot{\|}\, z^* \,\ddot{\|} \ddot{<} \ddot{1}$, $\ddot{\|}\, u^* \,\ddot{\|} \ddot{<} \frac{\ddot{1}}{\ddot{2}}\beta$, $\ddot{\|}\, v^* \,\ddot{\|} \ddot{<} \frac{\ddot{1}}{\ddot{2}}\beta$ and $T : X \to X$ be a self-mapping. If the mapping T satisfies at least one of the following conditions*

$$d^*(Tx, Ty) \overset{*}{\precsim} z^* \otimes d^*(x, y) \tag{1}$$

$$d^*(Tx, Ty) \overset{*}{\precsim} u^* \otimes [d^*(x, Tx) \oplus d^*(y, Ty)] \tag{2}$$

$$d^*(Tx, Ty) \overset{*}{\precsim} v^* \otimes [d^*(x, Ty) \oplus d^*(y, Tx)] \tag{3}$$

for all $x, y \in X$ with $x \neq y$. Then, T has a unique fixed point in X.

Proof. Consider the non-Newtonian real number $P =^\beta \max\left\{\ddot{\|}\, z^* \,\ddot{\|}, \frac{\ddot{\|} u^* \ddot{\|}}{\ddot{1} \ddot{-} \ddot{\|} u^* \ddot{\|}}\beta, \frac{\ddot{\|} v^* \ddot{\|}}{\ddot{1} \ddot{-} \ddot{\|} v^* \ddot{\|}}\beta\right\}$. It is clear that $P \ddot{<} \ddot{1}$. Choose $x_0 \in X$ arbitrarily. For $n \in \mathbb{N} \cup \{0\}$, take $x = T^n x_0$ and $y = T^{n+1}x_0$. If $x = y$, then x is fixed point of T and the proof is completed. So, let $x \neq y$.

If (1) holds, then

$$d^*\left(T^{n+1}x_0, T^{n+2}x_0\right) \overset{*}{\precsim} z^* \otimes d^*\left(T^n x_0, T^{n+1}x_0\right).$$

This implies that

$$\begin{aligned} & \ddot{\|}\, d^*\left(T^{n+1}x_0, T^{n+2}x_0\right) \ddot{\|} \overset{*}{\ddot{\precsim}} \ddot{\|}\, z^* \otimes d^*\left(T^n x_0, T^{n+1}x_0\right) \ddot{\|} \\ = \; & \ddot{\|}\, z^* \,\ddot{\|} \ddot{\times} \ddot{\|}\, d^*\left(T^n x_0, T^{n+1}x_0\right) \ddot{\|} \ddot{\leq} P \ddot{\times} \ddot{\|}\, d^*\left(T^n x_0, T^{n+1}x_0\right) \ddot{\|}. \end{aligned}$$

If (2) holds, then

$$\begin{aligned} & d^*\left(T^{n+1}x_0, T^{n+2}x_0\right) \overset{*}{\precsim} u^* \otimes \left[d^*\left(T^n x_0, T^{n+1}x_0\right) \oplus d^*\left(T^{n+1}x_0, T^{n+2}x_0\right)\right] \\ = \; & u^* \otimes \left[d^*\left(T^n x_0, T^{n+1}x_0\right)\right] \oplus u^* \otimes \left[d^*\left(T^{n+1}x_0, T^{n+2}x_0\right)\right] \end{aligned}$$

So, we have

$$\begin{aligned} & \ddot{\|}\, d^*\left(T^{n+1}x_0, T^{n+2}x_0\right) \ddot{\|} \ddot{\leq} \ddot{\|}\, u^* \otimes \left[d^*\left(T^n x_0, T^{n+1}x_0\right)\right] \ddot{\|} \ddot{+} \ddot{\|}\, u^* \otimes \left[d^*\left(T^{n+1}x_0, T^{n+2}x_0\right)\right] \ddot{\|} \\ = \; & \ddot{\|}\, u^* \,\ddot{\|} \ddot{\times} \ddot{\|}\, d^*\left(T^n x_0, T^{n+1}x_0\right) \ddot{\|} \ddot{+} \ddot{\|}\, u^* \,\ddot{\|} \ddot{\times} \ddot{\|}\, d^*\left(T^{n+1}x_0, T^{n+2}x_0\right) \ddot{\|} \end{aligned}$$

which means that

$$\ddot{\|}\, d^*\left(T^{n+1}x_0, T^{n+2}x_0\right) \ddot{\|} \ddot{\leq} \frac{\ddot{\|}\, u^* \,\ddot{\|}}{\ddot{1} \ddot{-} \ddot{\|}\, u^* \,\ddot{\|}}\beta \ddot{\times} \ddot{\|}\, d^*\left(T^n x_0, T^{n+1}x_0\right) \ddot{\|} \ddot{\leq} P \ddot{\times} \ddot{\|}\, d^*\left(T^n x_0, T^{n+1}x_0\right) \ddot{\|}.$$

If (3) holds, then

$$d^*\left(T^{n+1}x_0, T^{n+2}x_0\right) \overset{*}{\precsim} v^* \otimes \left[d^*\left(T^n x_0, T^{n+2}x_0\right) \oplus d^*\left(T^{n+1}x_0, T^{n+1}x_0\right)\right]$$

$$= \quad v^* \otimes d^*\left(T^n x_0, T^{n+2} x_0\right) \overset{*}{\precsim} v^* \otimes \left[d^*\left(T^n x_0, T^{n+1} x_0\right) \oplus d^*\left(T^{n+1} x_0, T^{n+2} x_0\right)\right]$$

which implies that

$$\begin{aligned} & \ddot{\|}\ d^*\left(T^{n+1} x_0, T^{n+2} x_0\right) \ddot{\|} \ddot{\leq} \ddot{\|}\ v^* \otimes \left[d^*\left(T^n x_0, T^{n+1} x_0\right)\right] \ddot{\|} \ddot{+} \ddot{\|}\ v^* \otimes \left[d^*\left(T^{n+1} x_0, T^{n+2} x_0\right)\right] \ddot{\|} \\ = \quad & \ddot{\|}\ v^* \ddot{\|} \ddot{\times} \ddot{\|}\ d^*\left(T^n x_0, T^{n+1} x_0\right) \ddot{\|} \ddot{+} \ddot{\|}\ v^* \ddot{\|} \ddot{\times} \ddot{\|}\ d^*\left(T^{n+1} x_0, T^{n+2} x_0\right) \ddot{\|}. \end{aligned}$$

Thus, we deduce that

$$\ddot{\|}\ d^*\left(T^{n+1} x_0, T^{n+2} x_0\right) \ddot{\|} \ddot{\leq} \frac{\ddot{\|}\ v^* \ddot{\|}}{\ddot{1} \ddot{-} \ddot{\|}\ v^* \ddot{\|}}\beta \ddot{\times} \ddot{\|}\ d^*\left(T^n x_0, T^{n+1} x_0\right) \ddot{\|} \ddot{\leq} P \ddot{\times} \ddot{\|}\ d^*\left(T^n x_0, T^{n+1} x_0\right) \ddot{\|}.$$

Consequently, in each case, we obtain $\ddot{\|}\ d^*\left(T^{n+1} x_0, T^{n+2} x_0\right) \ddot{\|} \ddot{\leq} P \ddot{\times} \ddot{\|}\ d^*\left(T^n x_0, T^{n+1} x_0\right) \ddot{\|}$ for all $n \in \mathbb{N} \cup \{0\}$. Continuing in the same manner, we get

$$\ddot{\|}\ d^*\left(T^{n+1} x_0, T^{n+2} x_0\right) \ddot{\|} \ddot{\leq} P^{(n+1)_\beta} \ddot{\times} \ddot{\|}\ d^*\left(x_0, T x_0\right) \ddot{\|}.$$

Let $n, m \in \mathbb{N}$ and $m > n$. Then, we have

$$d^*\left(T^n x_0, T^m x_0\right) \overset{*}{\precsim} d^*\left(T^n x_0, T^{n+1} x_0\right) \oplus d^*\left(T^{n+1} x_0, T^{n+2} x_0\right) \oplus \cdots \oplus d^*\left(T^{m-2} x_0, T^{m-1} x_0\right) \oplus d^*\left(T^{m-1} x_0, T^m x_0\right)$$

which means that

$$\begin{aligned} \ddot{\|}\ d^*\left(T^n x_0, T^m x_0\right) \ddot{\|} \quad & \ddot{\leq} \ddot{\|}\ d^*\left(T^n x_0, T^{n+1} x_0\right) \ddot{\|} \ddot{+} \ddot{\|}\ d^*\left(T^{n+1} x_0, T^{n+2} x_0\right) \ddot{\|} \\ & \ddot{+} \ddot{\|}\ d^*\left(T^{m-2} x_0, T^{m-1} x_0\right) \ddot{\|} \ddot{+} \ddot{\|}\ d^*\left(T^{m-1} x_0, T^m x_0\right) \ddot{\|} \\ & \ddot{\leq} \left[P^{(n)_\beta} \ddot{\times} \ddot{\|}\ d^*\left(x_0, T x_0\right) \ddot{\|}\right] \ddot{+} \left[P^{(n+1)_\beta} \ddot{\times} \ddot{\|}\ d^*\left(x_0, T x_0\right) \ddot{\|}\right] \ddot{+} \ldots \\ & \ddot{+} \left[P^{(m-2)_\beta} \ddot{\times} \ddot{\|}\ d^*\left(x_0, T x_0\right) \ddot{\|}\right] \ddot{+} \left[P^{(m-1)_\beta} \ddot{\times} \ddot{\|}\ d^*\left(x_0, T x_0\right) \ddot{\|}\right] \\ & = \left[P^{(n)_\beta} \ddot{+} P^{(n+1)_\beta} \ddot{+} \ldots \ddot{+} P^{(m-1)_\beta}\right] \ddot{\times} \ddot{\|}\ d^*\left(x_0, T x_0\right) \ddot{\|} \\ & \ddot{\leq} \left[{}_\beta \sum_{k=n}^{\infty} P^{(k)_\beta}\right] \ddot{\times} \ddot{\|}\ d^*\left(x_0, T x_0\right) \ddot{\|}. \end{aligned}$$

Since $P \ddot{<} \ddot{1}$, we say that the β-series ${}_\beta \sum_{k=1}^{\infty} P^{(k)_\beta}$ is β-convergent and $\lim_{n \to \infty} {}_\beta \sum_{k=n}^{\infty} P^{(k)_\beta} = \ddot{0}$. Therefore, $\lim_{m,n \to \infty} d^*\left(T^n x_0, T^m x_0\right) = 0_{\mathbb{C}(N)}$. This implies that $\left(T^n x_0\right)$ is a Cauchy sequence and by the completeness of X, it converges to some point $z \in X$.

Now, we prove that $Tz = z$. Suppose this is not true: $Tz \neq z$. Taking $d^*(Tz, z) = \varepsilon$ we set

$$B = \left\{ x \in X : \ddot{\|}\ d^*(x, z) \ddot{\|} \ddot{\leq} \frac{\ddot{1}}{\ddot{4}}\beta \ddot{\times} \ddot{\|}\ \varepsilon \ddot{\|} \right\}.$$

For $x \in B$, we can write

$$\varepsilon = d^*(z, Tz) \overset{*}{\precsim} d^*(z, x) \oplus d^*(x, Tz)$$

and then,

$$\ddot{\|}\ \varepsilon \ddot{\|} = \ddot{\|}\ d^*(z, Tz) \ddot{\|} \ddot{\leq} \ddot{\|}\ d^*(z, x) \ddot{\|} \ddot{+} \ddot{\|}\ d^*(x, Tz) \ddot{\|} \ddot{\leq} \frac{\ddot{1}}{\ddot{4}}\beta \ddot{\times} \ddot{\|}\ \varepsilon \ddot{\|} \ddot{+} \ddot{\|}\ d^*(x, Tz) \ddot{\|}$$

and so $\frac{\ddot{3}}{\ddot{4}}\beta \ddot{\times} \ddot{\|}\ \varepsilon \ddot{\|} \ddot{\leq} \ddot{\|}\ d^*(x, Tz) \ddot{\|}$. Since $\left(T^n x_0\right)$ converges to z, there exists a natural number n_0 such that $\ddot{\|}\ d^*\left(T^n x_0, z\right) \ddot{\|} \ddot{<} \frac{\ddot{1}}{\ddot{4}}\beta \ddot{\times} \ddot{\|}\ \varepsilon \ddot{\|}$ for all $n \geq n_0$. That means $T^n x_0 \in B$ for all $n \geq n_0$. Take $x = T^{n_0} x_0$ and $y = z$. Then, by (1) we have

$$\ddot{\|}\ d^*\left(T^{n_0+1} x_0, Tz\right) \ddot{\|} \ddot{\leq} \ddot{\|}\ z^* \otimes d^*\left(T^{n_0} x_0, z\right) \ddot{\|}$$

$$
\begin{aligned}
&= \ddot{\|}\, z^{*}\, \ddot{\|} \,\ddot{\times}\, \ddot{\|}\, d^{*}\left(T^{n_0}x_0, z\right) \ddot{\|} \\
&\ddot{\leq} \ddot{\|}\, d^{*}\left(T^{n_0}x_0, z\right) \ddot{\|} \ddot{<} \frac{\ddot{1}}{\ddot{4}}\beta \ddot{\times}\, \ddot{\|}\, \varepsilon\, \ddot{\|} \\
&\ddot{<} \frac{\ddot{3}}{\ddot{4}}\beta \ddot{\times}\, \ddot{\|}\, \varepsilon\, \ddot{\|} \\
&\ddot{\leq} d^{*}\left(T^{n_0+1}x_0, Tz\right)
\end{aligned}
$$

and this is a contradiction. By (2) we get

$$
\begin{aligned}
\ddot{\|}\, d^{*}\left(T^{n_0+1}x_0, Tz\right) \ddot{\|} &\ddot{\leq} \ddot{\|}\, u^{*} \otimes \left[d^{*}\left(T^{n_0}x_0, T^{n_0+1}x_0\right) \oplus d^{*}\left(z, Tz\right)\right] \ddot{\|} \\
&= \ddot{\|}\, u^{*}\, \ddot{\|} \,\ddot{\times}\, \ddot{\|}\, d^{*}\left(T^{n_0}x_0, T^{n_0+1}x_0\right) \oplus d^{*}\left(z, Tz\right) \ddot{\|} \\
&\ddot{<} \frac{\ddot{1}}{\ddot{2}}\beta \ddot{\times}\, \ddot{\|}\, d^{*}\left(T^{n_0}x_0, z\right) \oplus d^{*}\left(z, T^{n_0+1}x_0\right) \oplus d^{*}\left(z, Tz\right) \ddot{\|} \\
&\ddot{\leq} \frac{\ddot{1}}{\ddot{2}}\beta \,\ddot{\times} \left[\ddot{\|}\, d^{*}\left(T^{n_0}x_0, z\right) \ddot{\|} \,\ddot{+}\, \ddot{\|}\, d^{*}\left(z, T^{n_0+1}x_0\right) \ddot{\|} \,\ddot{+}\, \ddot{\|}\, d^{*}\left(z, Tz\right) \ddot{\|}\right] \\
&\ddot{<} \frac{\ddot{1}}{\ddot{2}}\beta \,\ddot{\times} \left[\frac{\ddot{1}}{\ddot{4}}\beta \ddot{\times}\, \ddot{\|}\, \varepsilon\, \ddot{\|} \,\ddot{+} \frac{\ddot{1}}{\ddot{4}}\beta \ddot{\times}\, \ddot{\|}\, \varepsilon\, \ddot{\|} \,\ddot{+}\, \ddot{\|}\, \varepsilon\, \ddot{\|}\right] \\
&= \frac{\ddot{3}}{\ddot{4}}\beta \ddot{\times}\, \ddot{\|}\, \varepsilon\, \ddot{\|} \\
&\ddot{\leq} \ddot{\|}\, d^{*}\left(T^{n_0+1}x_0, Tz\right) \ddot{\|}
\end{aligned}
$$

and this yields a contradiction. By (3) we obtain that

$$
\begin{aligned}
\ddot{\|}\, d^{*}\left(T^{n_0+1}x_0, Tz\right) \ddot{\|} &\ddot{\leq} \ddot{\|}\, v^{*} \otimes \left[d^{*}\left(T^{n_0}x_0, Tz\right) \oplus d^{*}\left(T^{n_0+1}x_0, z\right)\right] \ddot{\|} \\
&= \ddot{\|}\, v^{*}\, \ddot{\|} \,\ddot{\times}\, \ddot{\|}\, d^{*}\left(T^{n_0}x_0, Tz\right) \oplus d^{*}\left(T^{n_0+1}x_0, z\right) \ddot{\|} \\
&\ddot{<} \frac{\ddot{1}}{\ddot{2}}\beta \ddot{\times}\, \ddot{\|}\, d^{*}\left(T^{n_0}x_0, z\right) \oplus d^{*}\left(z, Tz\right) \oplus d^{*}\left(T^{n_0+1}x_0, z\right) \ddot{\|} \\
&\ddot{\leq} \frac{\ddot{1}}{\ddot{2}}\beta \,\ddot{\times} \left[\ddot{\|}\, d^{*}\left(T^{n_0}x_0, z\right) \ddot{\|} \,\ddot{+}\, \ddot{\|}\, d^{*}\left(z, Tz\right) \ddot{\|} \,\ddot{+}\, \ddot{\|}\, d^{*}\left(T^{n_0+1}x_0, z\right) \ddot{\|}\right] \\
&\ddot{<} \frac{\ddot{1}}{\ddot{2}}\beta \,\ddot{\times} \left[\frac{\ddot{1}}{\ddot{4}}\beta \ddot{\times}\, \ddot{\|}\, \varepsilon\, \ddot{\|} \,\ddot{+}\, \ddot{\|}\, \varepsilon\, \ddot{\|} \,\ddot{+} \frac{\ddot{1}}{\ddot{4}}\beta \ddot{\times}\, \ddot{\|}\, \varepsilon\, \ddot{\|}\right] \\
&= \frac{\ddot{3}}{\ddot{4}}\beta \ddot{\times}\, \ddot{\|}\, \varepsilon\, \ddot{\|} \\
&\ddot{\leq} \ddot{\|}\, d^{*}\left(T^{n_0+1}x_0, Tz\right) \ddot{\|}
\end{aligned}
$$

But this contradicts (3). Therefore, it must be $Tz = z$.

To show the uniqueness of fixed point, we assume that there is an element $w \neq z$ in X such that $w = Tw$. Then, we have

$$d^{*}\left(Tz, Tw\right) = d^{*}\left(z, w\right),$$

$$d^{*}\left(z, Tz\right) \oplus d^{*}\left(w, Tw\right) \overset{*}{\prec} d^{*}\left(Tz, Tw\right),$$

$$d^{*}\left(Tz, Tw\right) = \left(\frac{\dot{1}}{\dot{2}}\alpha, \ddot{0}\right) \otimes \left[d^{*}\left(z, Tw\right) \oplus d^{*}\left(w, Tz\right)\right].$$

This shows that none of the conditions (1), (2) and (3) of the theorem is satisfied. Hence, fixed point is unique. ■

As the first result of Theorem 4, extending the Banach contraction principle [1] in classical metric spaces, we state the following main theorem which implies the existence and uniqueness of fixed point on complete $*$-*complex* valued metric spaces.

Corollary 1 *Let (X, d^*) be a complete $*$-complex valued metric space, $z^* \in \mathbb{C}(N)^+$ such that $\ddot{\|}\, z^* \,\ddot{\|} \ddot{<} \ddot{1}$ and $T : X \to X$ be a self-mapping. If the mapping T satisfies the condition*

$$d^*(Tx, Ty) \overset{*}{\precsim} z^* \otimes d^*(x, y) \tag{4}$$

for all $x, y \in X$ with $x \neq y$, then, T has a unique fixed point in X.

The following result is an extension of Kannan's fixed point theorem [3] to $*$-*complex* valued metric spaces.

Corollary 2 *Let (X, d^*) be a complete $*$-complex valued metric space, $u^* \in \mathbb{C}(N)^+$ such that $\ddot{\|}\, u^* \,\ddot{\|} \ddot{<} \ddot{\frac{1}{2}}\beta$ and $T : X \to X$ be a self-mapping. If the mapping T satisfies the condition*

$$d^*(Tx, Ty) \overset{*}{\precsim} u^* \otimes [d^*(x, Tx) \oplus d^*(y, Ty)] \tag{5}$$

for all $x, y \in X$ with $x \neq y$, then, T has a unique fixed point in X.

The following result is a new version of Chatterjea's fixed point theorem [4] in $*$-*complex* valued metric spaces.

Corollary 3 *Let (X, d^*) be a complete $*$-complex valued metric space, $v^* \in \mathbb{C}(N)^+$ such that $\ddot{\|}\, v^* \,\ddot{\|} \ddot{<} \ddot{\frac{1}{2}}\beta$ and $T : X \to X$ be a self-mapping. If the mapping T satisfies the condition*

$$d^*(Tx, Ty) \overset{*}{\precsim} v^* \otimes [d^*(x, Ty) \oplus d^*(y, Tx)] \tag{6}$$

for all $x, y \in X$ with $x \neq y$, then, T has a unique fixed point in X.

Corollary 4 *By choosing $\alpha = \beta = I$ in the construction of the set $\mathbb{C}(N)$, Corollary 1 turns into Theorem 4 of Azam et al. [5] for $\mu = 0$ and $S = T$.*

Theorem 5 *Let (X, d^*) be a complete $*$-complex valued metric space, $x_0 \in X$, $z^*, u^*, v^* \in \mathbb{C}(N)^+$ such that $\ddot{\|}\, z^* \,\ddot{\|} \ddot{<} \ddot{1}$, $\ddot{\|}\, u^* \,\ddot{\|} \ddot{<} \ddot{\frac{1}{2}}\beta$, $\ddot{\|}\, v^* \,\ddot{\|} \ddot{<} \ddot{\frac{1}{2}}\beta$ and $T : X \to X$ be a self-mapping. If the mapping T satisfies at least one of (1), (2) and (3) of Theorem 4 and $(T^n x_0)$ has a limit point y_0 in X, that is, $\lim\limits_{n\to\infty}{}^* T^n x_0 = y_0$, then y_0 is the unique fixed point of T.*

Proof. One can easily show that $(T^n x_0)$ is a Cauchy sequence similar to the proof of Theorem 4. Since $(T^n x_0)$ has a limit point y_0 in X, by Theorem 4 the point y_0 is the unique fixed point of T. ■

The followings are consequences of Theorem 5.

Corollary 5 *Let (X, d^*) be a complete $*$-complex valued metric space, $x_0 \in X$, $z^* \in \mathbb{C}(N)^+$ such that $\ddot{\|}\, z^* \,\ddot{\|} \ddot{<} \ddot{1}$ and $T : X \to X$ be a self-mapping. If the mapping T satisfies the condition (1) and $(T^n x_0)$ has a limit point y_0 in X, then T has a unique fixed point in X.*

Corollary 6 *Let (X, d^*) be a complete $*$-complex valued metric space, $x_0 \in X$, $u^* \in \mathbb{C}(N)^+$ such that $\ddot{\|}\, u^* \,\ddot{\|} \ddot{<} \ddot{\frac{1}{2}}\beta$ and $T : X \to X$ be a self-mapping. If the mapping T satisfies the condition (2) and $(T^n x_0)$ has a limit point y_0 in X, then T has a unique fixed point in X.*

Corollary 7 *Let (X, d^*) be a complete $*$-complex valued metric space, $x_0 \in X$, $v^* \in \mathbb{C}(N)^+$ such that $\ddot{\|} v^* \ddot{\|} \ddot{<} \frac{\ddot{1}}{\ddot{2}}\beta$ and $T : X \to X$ be a self-mapping. If the mapping T satisfies the condition (3) and $(T^n x_0)$ has a limit point y_0 in X, then T has a unique fixed point in X.*

Now, we give an example which satisfies the requirements of Corollary 1 and emphasizes its importance.

Example 2 *Let*

$$X_1 = \left\{ z^* \in \mathbb{C}(N) : \dot{0} \dot{\leq} \Re\, z^* \dot{\leq} \dot{1}, \Im\, z^* = \ddot{0} \right\},$$

$$X_2 = \left\{ z^* \in \mathbb{C}(N) : \ddot{0} \ddot{\leq} \Im\, z^* \ddot{\leq} \ddot{1}, \Re\, z^* = \dot{0} \right\}$$

and $X = X_1 \cup X_2$. Define $d^* : X \times X \to \mathbb{C}(N)$ by

$$d^*(z_1^*, z_2^*) = \begin{cases} \left(\frac{\dot{3}}{\dot{5}}\alpha \dot{\times} \dot{|}\, \dot{x}_1 \dot{-} \dot{x}_2 \,\dot{|}, \frac{\ddot{1}}{\ddot{2}}\beta \ddot{\times} \ddot{|}\, \ddot{x}_1 \ddot{-} \ddot{x}_2 \,\ddot{|} \right), & z_1^*, z_2^* \in X_1, \\ \left(\frac{\dot{1}}{\dot{2}}\alpha \dot{\times} \dot{|}\, \dot{y}_1 \dot{-} \dot{y}_2 \,\dot{|}, \frac{\ddot{2}}{\ddot{5}}\beta \ddot{\times} \ddot{|}\, \ddot{y}_1 \ddot{-} \ddot{y}_2 \,\ddot{|} \right), & z_1^*, z_2^* \in X_2, \\ \left(\frac{\dot{3}}{\dot{5}}\alpha \dot{\times} \dot{x}_1 \dot{+} \frac{\dot{1}}{\dot{2}}\alpha \dot{\times} \dot{y}_2, \frac{\ddot{1}}{\ddot{2}}\beta \ddot{\times} \ddot{x}_1 \ddot{+} \frac{\ddot{2}}{\ddot{5}}\beta \ddot{\times} \ddot{y}_2 \right), & z_1^* \in X_1, z_2^* \in X_2, \\ \left(\frac{\dot{1}}{\dot{2}}\alpha \dot{\times} \dot{y}_1 \dot{+} \frac{\dot{3}}{\dot{5}}\alpha \dot{x}_2, \frac{\ddot{2}}{\ddot{5}}\beta \ddot{\times} \ddot{y}_1 \ddot{+} \frac{\ddot{1}}{\ddot{2}}\beta \ddot{\times} \ddot{x}_2 \right), & z_1^* \in X_2, z_2^* \in X_1, \end{cases}$$

where $z_1^* = (\dot{x}_1, \ddot{y}_1)$ and $z_2^* = (\dot{x}_2, \ddot{y}_2)$. Then, (X, d^*) is a complete $*$-complex valued metric space. Define $T : X \to X$ as

$$Tz^* = T((\dot{x}, \ddot{y})) = \begin{cases} (\dot{0}, \ddot{x}), & z^* \in X_1, \\ \left(\frac{\dot{1}}{\dot{3}}\alpha \dot{\times} \dot{y}, \ddot{0} \right), & z^* \in X_2. \end{cases}$$

Thus,

$$\begin{aligned} d^*(Tz_1^*, Tz_2^*) &= \begin{cases} \left(\frac{\dot{1}}{\dot{2}}\alpha \dot{\times} \dot{|}\, \dot{x}_1 \dot{-} \dot{x}_2 \,\dot{|}, \frac{\ddot{2}}{\ddot{5}}\beta \ddot{\times} \ddot{|}\, \ddot{x}_1 \ddot{-} \ddot{x}_2 \,\ddot{|} \right), & z_1^*, z_2^* \in X_1, \\ \left(\frac{\dot{3}}{\dot{5}}\alpha \dot{\times} \dot{|}\, \frac{\dot{1}}{\dot{3}}\alpha \dot{\times} \dot{y}_1 \dot{-} \frac{\dot{1}}{\dot{3}}\alpha \dot{\times} \dot{y}_2 \,\dot{|}, \frac{\ddot{2}}{\ddot{5}}\beta \ddot{\times} \ddot{|}\, \frac{\ddot{1}}{\ddot{3}}\alpha \ddot{\times} \ddot{y}_1 \ddot{-} \frac{\ddot{1}}{\ddot{3}}\alpha \ddot{\times} \ddot{y}_2 \,\ddot{|} \right), & z_1^*, z_2^* \in X_2, \\ \left(\frac{\dot{1}}{\dot{2}}\alpha \dot{\times} \dot{x}_1 \dot{+} \frac{\dot{3}}{\dot{5}}\alpha \dot{\times} \frac{\dot{1}}{\dot{3}}\alpha \dot{\times} \dot{y}_2, \frac{\ddot{2}}{\ddot{5}}\beta \ddot{\times} \ddot{x}_1 \ddot{+} \frac{\ddot{1}}{\ddot{2}}\beta \ddot{\times} \frac{\ddot{1}}{\ddot{3}}\beta \ddot{\times} \ddot{y}_2 \right) & z_1^* \in X_1,\ z_2^* \in X_2, \\ \left(\frac{\dot{3}}{\dot{5}}\alpha \dot{\times} \frac{\dot{1}}{\dot{3}}\alpha \dot{\times} \dot{y}_1 \dot{+} \frac{\dot{1}}{\dot{2}}\alpha \dot{x}_2, \frac{\ddot{1}}{\ddot{2}}\beta \ddot{\times} \frac{\ddot{1}}{\ddot{3}}\beta \ddot{\times} \ddot{y}_1 \ddot{+} \frac{\ddot{2}}{\ddot{5}}\beta \ddot{\times} \ddot{x}_2 \right) & z_1^* \in X_2,\ z_2^* \in X_1, \end{cases} \\ &= \begin{cases} \left(\frac{\dot{1}}{\dot{2}}\alpha \dot{\times} \dot{|}\, \dot{x}_1 \dot{-} \dot{x}_2 \,\dot{|}, \frac{\ddot{2}}{\ddot{5}}\beta \ddot{\times} \ddot{|}\, \ddot{x}_1 \ddot{-} \ddot{x}_2 \,\ddot{|} \right), & z_1^*, z_2^* \in X_1, \\ \left(\frac{\dot{1}}{\dot{5}}\alpha \dot{\times} \dot{|}\, \dot{y}_1 \dot{-} \dot{y}_2 \,\dot{|}, \frac{\ddot{1}}{\ddot{6}}\beta \ddot{\times} \ddot{|}\, \ddot{y}_1 \ddot{-} \ddot{y}_2 \,\ddot{|} \right), & z_1^*, z_2^* \in X_2, \\ \left(\frac{\dot{1}}{\dot{2}}\alpha \dot{\times} \dot{x}_1 \dot{+} \frac{\dot{1}}{\dot{5}}\alpha \dot{\times} \dot{y}_2, \frac{\ddot{2}}{\ddot{5}}\beta \ddot{\times} \ddot{x}_1 \ddot{+} \frac{\ddot{1}}{\ddot{6}}\beta \ddot{\times} \ddot{y}_2 \right), & z_1^* \in X_1,\ z_2^* \in X_2, \\ \left(\frac{\dot{1}}{\dot{5}}\alpha \dot{\times} \dot{y}_1 \dot{+} \frac{\dot{1}}{\dot{2}}\alpha \dot{x}_2, \frac{\ddot{1}}{\ddot{6}}\beta \ddot{\times} \ddot{y}_1 \ddot{+} \frac{\ddot{2}}{\ddot{5}}\beta \ddot{\times} \ddot{x}_2 \right), & z_1^* \in X_2,\ z_2^* \in X_1, \end{cases} \\ &\precsim^{*} \begin{cases} \left(\frac{\dot{1}}{\dot{2}}\alpha \dot{\times} \dot{|}\, \dot{x}_1 \dot{-} \dot{x}_2 \,\dot{|}, \frac{\ddot{5}}{\ddot{12}}\beta \ddot{\times} \ddot{|}\, \ddot{x}_1 \ddot{-} \ddot{x}_2 \,\ddot{|} \right), & z_1^*, z_2^* \in X_1, \\ \left(\frac{\dot{5}}{\dot{12}}\alpha \dot{\times} \dot{|}\, \dot{y}_1 \dot{-} \dot{y}_2 \,\dot{|}, \frac{\ddot{1}}{\ddot{3}}\beta \ddot{\times} \ddot{|}\, \ddot{y}_1 \ddot{-} \ddot{y}_2 \,\ddot{|} \right), & z_1^*, z_2^* \in X_2, \\ \left(\frac{\dot{1}}{\dot{2}}\alpha \dot{\times} \dot{x}_1 \dot{+} \frac{\dot{5}}{\dot{12}}\alpha \dot{\times} \dot{y}_2, \frac{\ddot{5}}{\ddot{12}}\beta \ddot{\times} \ddot{x}_1 \ddot{+} \frac{\ddot{1}}{\ddot{3}}\beta \ddot{\times} \ddot{y}_2 \right), & z_1^* \in X_1,\ z_2^* \in X_2, \\ \left(\frac{\dot{5}}{\dot{12}}\alpha \dot{\times} \dot{y}_1 \dot{+} \frac{\dot{1}}{\dot{2}}\alpha \dot{x}_2, \frac{\ddot{1}}{\ddot{3}}\beta \ddot{\times} \ddot{y}_1 \ddot{+} \frac{\ddot{5}}{\ddot{12}}\beta \ddot{\times} \ddot{x}_2 \right), & z_1^* \in X_2,\ z_2^* \in X_1, \end{cases} \\ &= \left(\frac{\dot{5}}{\dot{6}}\alpha, \ddot{0} \right) \otimes \begin{cases} \left(\frac{\dot{3}}{\dot{5}}\alpha \dot{\times} \dot{|}\, \dot{x}_1 \dot{-} \dot{x}_2 \,\dot{|}, \frac{\ddot{1}}{\ddot{2}}\beta \ddot{\times} \ddot{|}\, \ddot{x}_1 \ddot{-} \ddot{x}_2 \,\ddot{|} \right), & z_1^*, z_2^* \in X_1, \\ \left(\frac{\dot{1}}{\dot{2}}\alpha \dot{\times} \dot{|}\, \dot{y}_1 \dot{-} \dot{y}_2 \,\dot{|}, \frac{\ddot{2}}{\ddot{5}}\beta \ddot{\times} \ddot{|}\, \ddot{y}_1 \ddot{-} \ddot{y}_2 \,\ddot{|} \right), & z_1^*, z_2^* \in X_2, \\ \left(\frac{\dot{3}}{\dot{5}}\alpha \dot{\times} \dot{x}_1 \dot{+} \frac{\dot{1}}{\dot{2}}\alpha \dot{\times} \dot{y}_2, \frac{\ddot{1}}{\ddot{2}}\beta \ddot{\times} \ddot{x}_1 \ddot{+} \frac{\ddot{2}}{\ddot{5}}\beta \ddot{\times} \ddot{y}_2 \right), & z_1^* \in X_1,\ z_2^* \in X_2, \\ \left(\frac{\dot{1}}{\dot{2}}\alpha \dot{\times} \dot{y}_1 \dot{+} \frac{\dot{3}}{\dot{5}}\alpha \dot{x}_2, \frac{\ddot{2}}{\ddot{5}}\beta \ddot{\times} \ddot{y}_1 \ddot{+} \frac{\ddot{1}}{\ddot{2}}\beta \ddot{\times} \ddot{x}_2 \right), & z_1^* \in X_2,\ z_2^* \in X_1, \end{cases} \end{aligned}$$

$$= \left(\frac{\dot{5}}{\dot{6}}\alpha, \ddot{0}\right) \otimes d^* \left(z_1^*, z_2^*\right)$$

that is, $d^* \left(Tz_1^*, Tz_2^*\right) \overset{*}{\precsim} \left(\frac{\dot{5}}{\dot{6}}\alpha, \ddot{0}\right) \otimes d^* \left(z_1^*, z_2^*\right)$ *holds for all* $z_1^*, z_2^* \in X$*. Since* $\left(\frac{\dot{5}}{\dot{6}}\alpha, \ddot{0}\right) \in \mathbb{C}(N)^+$ *and* $\ddot{\|} \left(\frac{\dot{5}}{\dot{6}}\alpha, \ddot{0}\right) \ddot{\|} = \frac{\ddot{5}}{\ddot{6}}\beta \ddot{<} \ddot{1}$*, the mapping* T *has a fixed point* $z^* = 0_{\mathbb{C}(N)} \in X$*, which is unique.*

In the closing of this section, consider the non-Newtonian Volterra integral equation given by Güngör [24] as follows:

$$v(x) = f(x) \ddot{+} \lambda \ddot{\times} * \int_{\dot{0}}^{x} K(x,s) \ddot{\times} v(s)\, d^* s \tag{7}$$

where f and K are specified $\mathbb{R}_\beta$-valued functions, v is unknown $\mathbb{R}_\beta$-valued function, λ is a $\mathbb{R}_\beta$-parameter, $\dot{0} \dot{\leq} x \dot{\leq} a$ for $a \in \mathbb{R}_\alpha$ and $\dot{0} \dot{\leq} s \dot{\leq} x$.

Now, we prove the existence and uniqueness of solution of integral equation (8) given below, which is the special case of integral equation (7) with $f(x) = \ddot{0}$ for all $x \in \dot{[}\dot{0}, a\dot{]}$, $\lambda = \ddot{1}$ and $K(x,s) = \imath(x) \ddot{\times} \ddot{e}^{\left(-\alpha^{-1}(x)\alpha^{-1}(s)\right)_\beta}$ by using Corollary 1:

$$v(x) = * \int_{0}^{x} \imath(x) \ddot{\times} \ddot{e}^{\left(-\alpha^{-1}(x)\alpha^{-1}(s)\right)_\beta} \ddot{\times} v(s)\, d^* s. \tag{8}$$

Example 3 *Let*

$$X = C\left(\dot{[}\dot{0}, a\dot{]}, \mathbb{R}_\beta\right) = \left\{ v : \dot{[}\dot{0}, a\dot{]} \to \mathbb{R}_\beta \,|v \text{ is } *\text{-continuous} \right\} \tag{9}$$

and for every $v, u \in X$*,*

$$d^* : X \times X \to \mathbb{C}(N), \ \ (v,u) \to d^*(v,u) = \left(\dot{0}, {}^\beta\max_{x \in \dot{[}\dot{0}, a\dot{]}} \ddot{|}\, v(x) \ddot{-} u(x) \ddot{|}\right). \tag{10}$$

Then, it is easy to show that (X, d^*) *is a complete* $*$*-complex valued metric space by Lemma 2 and definition of the* $*$*-complex valued metric* d^**. Define a mapping* T *on* X *as*

$$T : C\left(\dot{[}\dot{0}, a\dot{]}, \mathbb{R}_\beta\right) \to C\left(\dot{[}\dot{0}, a\dot{]}, \mathbb{R}_\beta\right), \ v \to Tv, \tag{11}$$

$$Tv : \dot{[}\dot{0}, a\dot{]} \to \mathbb{R}_\beta, \ x \to Tv(x) = * \int_{\dot{0}}^{x} \imath(x) \ddot{\times} \ddot{e}^{\left(-\alpha^{-1}(x)\alpha^{-1}(s)\right)_\beta} \ddot{\times} v(s)\, d^* s.$$

Thus,

$$\begin{aligned} d^*(Tv, Tu) &= \left(\dot{0}, {}^\beta\max_{x \in \dot{[}\dot{0}, a\dot{]}} \ddot{|}\, Tv(x) \ddot{-} Tu(x) \ddot{|}\right) \\ &= \left(\dot{0}, {}^\beta\max_{x \in \dot{[}\dot{0}, a\dot{]}} \ddot{|} * \int_{\dot{0}}^{x} \imath(x) \ddot{\times} \ddot{e}^{\left(-\alpha^{-1}(x)\alpha^{-1}(s)\right)_\beta} \ddot{\times} \left(v(s) \ddot{-} u(s)\right) d^* s \,\ddot{|}\right) \\ &= \left(\dot{0}, {}^\beta\max_{x \in \dot{[}\dot{0}, a\dot{]}} \ddot{|}\, \beta\left(\int_{0}^{\alpha^{-1}(x)} \alpha^{-1}(x).e^{-\alpha^{-1}(x)\alpha^{-1}(s)}.\left(\beta^{-1}(v(s)) - \beta^{-1}(u(s))\right) d\alpha^{-1} s\right) \ddot{|}\right) \end{aligned}$$

$$
\begin{aligned}
&\overset{*}{\precsim} \left(\dot{0}, {}^{\beta}\max_{x\in[\dot{0},a]} \ddot{|}\, \beta\left(\int_{0}^{\alpha^{-1}(x)} \alpha^{-1}(x).e^{-\alpha^{-1}(x)\alpha^{-1}(s)} d\alpha^{-1}(s)\right) \ddot{|}\, \ddot{\times}^{\beta} \max_{x\in[\dot{0},a]} \ddot{|} \left(v(x) \ddot{-} u(x)\right) \ddot{|}\right) \\
= &\left(\dot{0}, {}^{\beta}\max_{x\in[\dot{0},a]} \ddot{|}\, \beta\left(\left.\frac{-\alpha^{-1}(x).e^{-\alpha^{-1}(x)\alpha^{-1}(s)}}{\alpha^{-1}(x)}\right|_{0}^{\alpha^{-1}(x)}\right) \ddot{|}\, \ddot{\times}^{\beta} \max_{x\in[\dot{0},a]} \ddot{|} \left(v(x) \ddot{-} u(x)\right) \ddot{|}\right) \\
= &\left(\dot{0}, {}^{\beta}\max_{x\in[\dot{0},a]} \ddot{|}\, \beta\left(1 - e^{-\left(\alpha^{-1}(x)\right)^{2}}\right) \ddot{|}\, \ddot{\times}^{\beta} \max_{x\in[\dot{0},a]} \ddot{|} \left(v(x) \ddot{-} u(x)\right) \ddot{|}\right) \\
= &\left(\dot{0}, \left(\ddot{1} \ddot{-} \ddot{e}^{\left(-\left(\alpha^{-1}(a)\right)^{2}\right)_{\beta}}\right) \ddot{\times}\, {}^{\beta}\max_{x\in[\dot{0},a]} \ddot{|}\, v(x) \ddot{-} u(x)\, \ddot{|}\right) \\
= &\left(\dot{1} \dot{-} \dot{e}^{\left(-\left(\alpha^{-1}(a)\right)^{2}\right)_{\alpha}}, \ddot{0}\right) \otimes d^{*}(v,u)
\end{aligned}
$$

holds for all $v, u \in X$. Therefore, since $\ddot{\|} \left(\dot{1} \dot{-} \dot{e}^{(-p^{2})_{\alpha}}, \ddot{0}\right) \ddot{\|} = \ddot{1} \ddot{-} \ddot{e}^{(-p^{2})_{\beta}} \ddot{<} \ddot{1}$, the mapping T has a unique fixed point. This implies that there exists a unique solution of the non-Newtonian Volterra integral equation (8).

Each choice of specific isomorphisms for α and β determines a $*$-calculus [25]. Bigeometric calculus obtained by choosing the exponential function exp instead of the generators α and β, that is, $\alpha(x) = \beta(x) = e^{x}$ for all $x \in \mathbb{R}$ is one of the most popular $*$-calculi and has some attractive applications.

In bigeometric calculus, the set of $*$-complex numbers turns into $\left\{\left(e^{a}, e^{b}\right) : a, b \in \mathbb{R}\right\}$ and $*$-complex valued metric spaces turn into the bigeometric valued metric spaces.

In the following discussion, we will use the symbols $\mathbb{C}_{\exp}$, $\mathbb{R}_{\exp}$, d_{BG}, $\overset{*}{\precsim}_{BG}$, $\otimes_{BG}$, ${}_{BG}\int_{a}^{x}$, $\|\cdot\|_{BG}$, $|\cdot|_{\exp}$, $+_{\exp}$ and $\times_{\exp}$ for the bigeometric versions of $\mathbb{C}(N)$, $\mathbb{R}_{\alpha}(\mathbb{R}_{\beta})$, d^{*}, $\overset{*}{\precsim}$, $\otimes$, $*\int_{a}^{x}$, $\ddot{\|} \cdot \ddot{\|}$, $\ddot{|} \cdot \ddot{|}$, $\ddot{+}$ and $\ddot{\times}$ respectively.

Now, we first restate Corollary 1 with respect to bigeometric calculus as follows:

Corollary 8 *Let (X, d_{BG}) be a complete bigeometric valued metric space, $z^{*} = \left(e^{a}, e^{b}\right) \in \mathbb{C}_{\exp}^{+}$ such that $\| z^{*} \|_{BG} = \sqrt{a^{2}+b^{2}} < 1$ and $T : X \to X$ be a self-mapping. If the mapping T satisfies the condition*

$$d_{BG}(Tx, Ty) \overset{*}{\precsim}_{BG} z^{*} \otimes_{BG} d_{BG}(x, y) \tag{12}$$

for all $x, y \in X$ with $x \neq y$, then, T has a unique fixed point in X.

The bigeometric version of non-Newtonian Volterra integral equation (7) introduced by Güngör [26] as follows:

$$v(x) = f(x) +_{\exp} \left(\lambda \times_{\exp} {}_{BG}\int_{a}^{x} K(x,s) \times_{\exp} v(s)\, ds^{BG}\right) \tag{13}$$

where f and K are known functions, v is unknown function and $\lambda \in \mathbb{R}_{\exp}$.

The following is an example of how to solve the integral equation (14) given below, which is the special case of integral equation (13) with $f(x) = 1$ for all $x \in [1, a] \subset \mathbb{R}_{\exp}$, $\lambda = e$ and $K(x,s) = e^{(\ln x) e^{-(\ln x \ln s)}}$ by using Corollary 8:

$$v(x) = e^{\int_{1}^{x} \frac{\ln x e^{-(\ln x \ln s)} \ln v(s)}{s} ds} \tag{14}$$

Example 4 *Consider the following statements (15), (16) and (17) which are bigeometric versions of the statements (9), (10) and (11) in Example 3, respectively:*

$$X = C\left([1,a], \mathbb{R}_{\exp}\right) = \left\{v : [1,a] \to \mathbb{R}_{\exp} \,|v \text{ is } *\text{-continuous}\right\}, \tag{15}$$

$$d_{BG} : X \times X \to \mathbb{C}_{\exp}, \;\; (v,u) \to d_{BG}(v,u) = \left(1, \max_{x\in[1,a]} \left| e^{(\ln v(x) - \ln u(x))} \right|_{\exp}\right), \tag{16}$$

$$\begin{aligned} T &: C\left([1,a], \mathbb{R}_{\exp}\right) \to C\left([1,a], \mathbb{R}_{\exp}\right), \; v \to Tv \\ Tv &: [1,a] \to \mathbb{R}_{\exp}, \; x \to Tv(x) = e^{\int\limits_1^x \frac{\ln x e^{-(\ln x \ln s)} \ln v(s)}{s} ds}. \end{aligned} \tag{17}$$

Thus, we have $d_{BG}(Tv,Tu) \overset{*}{\precsim}_{BG} \left(e^{1-e^{-(\ln a)^2}}, e^0\right) \otimes_{BG} d_{BG}(v,u)$. *Since*

$$\left\| \left(e^{1-e^{-(\ln a)^2}}, e^0\right) \right\|_{BG} = 1 - e^{-(\ln a)^2} < 1,$$

the condition (12) in Corollary 8 is satisfied. Consequently, the mapping T has a unique fixed point which is a unique solution of the bigeometric Volterra integral equation (14).

4 Conclusion

In this article, we proved the existence and uniqueness of a fixed point for Zamfirescu type contractions on complete $*$-complex valued metric spaces. We also discussed three examples that support our main result Corollary 1, two of which show the existence and uniqueness of the solutions of a specially chosen non-Newtonian Volterra integral equation and a specially chosen bigeometric Volterra integral equation. Our findings carry some known results from the literature to $*$-complex valued metric spaces.

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