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Existence And Uniqueness Of Solution For A Mixed-Type Fractional Differential Equation And Ulam-Hyers Stability*

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Abstract

In this paper, we have discussed a special type of nonlinear boundary value problems (BVPs) which involves both the right-sided Caputo-Katugampola (CK) and the left-sided Katugampola fractional derivatives (FDs). Based on some new techniques and some properties of the Mittag-Leffler functions, we have introduced a formula of the solution for the aforementioned problem. To study the existence and uniqueness results of the solution for this problem, we have applied some known fixed point theorems (i.e., Banach's contraction principle, Schauder's fixed point theorem, nonlinear alternative of Leray-Schauder type and Schaefer's fixed point theorem). We have also studied the Ulam-Hyers stability of this problem. To illustrate the theoretical results in this work, we have given two examples.

1 Introduction

In mathematical analysis, fractional calculus (FC) is a subject that studies different approaches of defining non-integer order derivatives (i.e., fractional differential calculus (FDC)) and integrals (i.e., fractional integral calculus (FIC)). For more details in the subject, the reader may refer to (Samko et al. 1993 [1], Podlubny 1999 [2], Kilbas et al. 2006 [3], Diethelm 2010 [4]).

Considering FC, classical integer order differential equations (IODEs) commonly known as ordinary DEs have been generalized to get fractional order differential equations (FODEs) or FDEs simply. FDEs find their applications in may different academic and research fields of engineering and science including biology, mathematical physics, control theory, bio- and bio-inspired engineering, fluid mechanics, and signals and systems.

In [5, 6, 7] Katugampola introduced a new fractional integro-differential operator which generalized both the Reimann-Liouville (RL) and Hadamard operators.

Almeida et al. in [8] proposed a Caputo-type modification of Katugampola fractional derivative (FD) of order $\alpha \in (0,1)$ which represents in turn a generalization of the Caputo and Caputo-Hadamard FDs. This new FD was named as the Caputo-Katugampola (CK) FD. In the same paper, some fundamental results have been presented and proved. The authors have also developed an existence and uniqueness theorem for a BVP with dependence on the CKFD. Later, in [9] Ricardo Almeida has defined a CKFD of arbitrary real order $\alpha > 0$, and has studied its properties.

Different fixed-point theorems have been used by researchers to develop solutions and their existence for BVPs of FDEs (see [10, 11, 12])

The stability problem introduced by Ulam in [13] has attracted the attention and efforts of many famous researchers (see [14, 15]). Not long ago, Ulam-Hyers stability problem for FDEs has gained much research attention (see [16, 17, 18, 19]).

Recently, some BVPs involving nonlinear mixed FDEs have been studied (see [20, 21, 22, 23]).

In this paper, we investigate the existence and uniqueness of solution for the following BVP involving a nonlinear FDE with two different fractional derivatives

$$^{CK}D^{\beta,\rho}_{1^-}\left(^{K}D^{\alpha,\rho}_{0^+}+\lambda\right)u(t)=f(t,u(t)),\ t\in J=[0,1], \tag{1}$$

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with the boundary conditions

$$({}^{K}I_{0^+}^{1-\alpha,\rho}u)(0) = u_0, \tag{2}$$

$$\left({}^{K}D_{0^+}^{\alpha,\rho} + \lambda\right)u(1) = u_1, \tag{3}$$

where $\alpha, \beta,\ \alpha+\beta \in (0,1),\ \lambda, \rho > 0,\ u_0, u_1 \in \mathbb{R},\ {}^{CK}D_{1^-}^{\beta,\rho}$ is the right CKFD of order β. ${}^{K}D_{0^+}^{\alpha,\rho}$ is the left Katugampola FD of order α and ${}^{K}I_{0+}^{1-\alpha,\rho}$ is the Katugampola FI.

The rest of this paper is structured as follows. In section 2, we re-call some useful definitions, theorems and lemmas of FC. We also present some properties of classical and generalized Mittag-Leffler functions. In section 3, we give the formula of solution to the studied problem. This result plays a crucial role in the coming analysis throughout this paper. In section 4, we study the existence and uniqueness of solution to the problem (1)–(3) by using the Banach's contraction principle, Schauder's Fixed Point Theorem, Schaefer's Fixed Point Theorem and the nonlinear alternative of Leray-Schauder type. In section 5, we present the Ulam-Hyers stability result for the nonlinear mixed FDE (1)–(3). We also give two examples to demonstrate our theoretical results. Finally, this paper is ended with a conclusion.

2 Background Materials and Preliminaries

In this section, we present the necessary definitions and notations from FC theory which will be used through the whole of this work. Let $J = [0, 1]$ be a finite interval of $\mathbb{R}$. We denote by $C([0, 1], \mathbb{R})$ the Banach space of all continuous functions $y : [0, 1] \to \mathbb{R}$ with the norm

$$\|y\|_\infty = \sup_{t\in[0,1]} |y(t)|,$$

where $y \in C([0, 1], \mathbb{R})$. We denote also by $C^n([0, 1], \mathbb{R})$ with $n \in \mathbb{N}_0$ the set of mappings having n times continuously differentiable on J.

As in [3], for $1 \le p \le \infty$ and $c \in \mathbb{R}$, consider the space $X_c^p[0, 1]$ as follows

$$X_c^p[0,1] = \left\{ y : [0,1] \to \mathbb{R} : \|y\|_{X_c^p} = \left(\int_0^1 |s^c y(s)|^p \frac{ds}{s} \right)^{\frac{1}{p}} < \infty \right\},$$

for $1 \le p < \infty$, c $\in \mathbb{R}$. For the case $p = \infty$,

$$\|y\|_{X_c^\infty} = ess \sup_{0\le t\le 1} \left[t^c |y(t)|\right],\ \text{c} \in \mathbb{R}.$$

Now, we give some basic definitions and properties of the FC theory that we will need later in this work.

Definition 1 (Katugampola fractional integrals [5, 6, 9]) *Let $a, b > 0$ be two reals, and $y : [a, b] \to \mathbb{R}$ be an integrable function. The left-sided and right-sided Katugampola fractional integrals of order $\alpha > 0$, and parameter $\rho > 0$ are defined respectively by*

$$\left({}^{K}I_{a^+}^{\alpha,\rho}y\right)(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t s^{\rho-1} (t^\rho - s^\rho)^{\alpha-1} y(s)\, ds,\ \ t > a,$$

and

$$\left({}^{K}I_{b^-}^{\alpha,\rho}y\right)(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_t^b s^{\rho-1} (s^\rho - t^\rho)^{\alpha-1} y(s)\, ds,\ \ t < b, \tag{4}$$

where $\Gamma(.)$ is the Gamma function.

Definition 2 (Katugampola fractional derivatives [6, 9]) *The left-sided and right-sided Katugampola fractional derivatives for a differential operator of order $\alpha > 0$ with dependence on a parameter $\rho > 0$ are defined respectively as*

$$\begin{aligned}\left({}^{K}D_{a^+}^{\alpha,\rho}y\right)(t) &= \left(t^{1-\rho}\frac{d}{dt}\right)^n \ {}^{K}I_{a^+}^{n-\alpha,\rho}y(t)\\ &= \frac{\rho^{1-n+\alpha}}{\Gamma(n-\alpha)}\left(t^{1-\rho}\frac{d}{dt}\right)^n\int_a^t s^{\rho-1}\left(t^\rho - s^\rho\right)^{n-\alpha-1}y(s)\,ds,\end{aligned}$$

and

$$\begin{aligned}\left({}^{K}D_{b^-}^{\alpha,\rho}y\right)(t) &= \left(-t^{1-\rho}\frac{d}{dt}\right)^n \ {}^{K}I_{b^-}^{n-\alpha,\rho}y(t)\\ &= \frac{\rho^{1-n+\alpha}}{\Gamma(n-\alpha)}\left(-t^{1-\rho}\frac{d}{dt}\right)^n\int_t^b s^{\rho-1}\left(s^\rho - t^\rho\right)^{n-\alpha-1}y(s)\,ds,\end{aligned}$$

where $n-1<\alpha<n$, $n=[\alpha]+1$ and $[\alpha]$ denotes the integer part of α.

Definition 3 (Caputo-Katugampola fractional derivatives [8]) *Let $0<a<b<\infty$, $y:[a,b]\to\mathbb{R}$ be an integrable function, and $\gamma\in(0,1)$ and $\rho>0$ are two fixed reals. The left-sided and right-sided Caputo-Katugampola fractional derivatives of order γ are defined respectively by*

$$\begin{aligned}{}^{CK}D_{a^+}^{\gamma,\rho}y(t) &= {}^{K}D_{a^+}^{\gamma,\rho}\left[y(t)-y(a)\right]\\ &= \frac{\rho^\gamma}{\Gamma(1-\gamma)}\left(t^{1-\rho}\frac{d}{dt}\right)\int_a^t \tau^{\rho-1}\left(t^\rho-\tau^\rho\right)^{-\gamma}\left[y(\tau)-y(a)\right]d\tau,\end{aligned}$$

and

$$\begin{aligned}{}^{CK}D_{b^-}^{\gamma,\rho}y(t) &= {}^{K}D_{b^-}^{\gamma,\rho}\left[y(t)-y(b)\right]\\ &= \frac{-\rho^\gamma}{\Gamma(1-\gamma)}\left(t^{1-\rho}\frac{d}{dt}\right)\int_t^b \tau^{\rho-1}\left(\tau^\rho-t^\rho\right)^{-\gamma}\left[y(\tau)-y(b)\right]d\tau.\end{aligned}$$

Lemma 1 (See [8]) *Let $J=[a,b]$, $\gamma,\ \rho>0$ and $y\in C(J,\mathbb{R})\cap C^1(J,\mathbb{R})$. Then, the Caputo-Katugampola fractional deferential equation*

$${}^{CK}D_{b^-}^{\gamma,\rho}y(t)=0,$$

has solutions

$$y(t)=c_0+c_1\left(\frac{b^\rho-t^\rho}{\rho}\right)+c_2\left(\frac{b^\rho-t^\rho}{\rho}\right)^2+...+c_{n-1}\left(\frac{b^\rho-t^\rho}{\rho}\right)^{n-1},$$

where $c_i\in\mathbb{R}$, $i=0,1,2,...,n-1$ and $n=[\alpha]+1$.

Theorem 1 (See [9]) *Given a function $y\in C^n[a,b]$, we have*

$${}^{K}I_{b^-}^{\gamma,\rho}\ {}^{CK}D_{b^-}^{\gamma,\rho}y(t)=y(t)-\sum_{j=0}^{n-1}\frac{\rho^{-j}(-1)^j}{j!}\left(b^\rho-t^\rho\right)^j y^{(j)}(b).$$

If $0<\gamma<1$, then

$${}^{K}I_{b^-}^{\gamma,\rho}\ {}^{CK}D_{b^-}^{\gamma,\rho}y(t)=y(t)-y(b).$$

Theorem 2 (See [24]) *The Cauchy problem*

$$\begin{cases}\left({}^{K}D_{0^+}^{\mu,\rho}-\lambda\right)u(t)=f(t),\ t>0,\ 0<\mu\le 1,\ \lambda\in\mathbb{R},\\ \left({}^{K}I_{0^+}^{1-\mu,\rho}u\right)(0)=k,\ k\in\mathbb{R},\end{cases}$$

has the solution

$$
\begin{aligned}
u(t) &= k\left(\frac{t^{\rho}}{\rho}\right)^{\mu-1} E_{\mu,\mu}\left(\lambda\left(\frac{t^{\rho}}{\rho}\right)^{\mu}\right) \\
&\quad + \int_0^t \left(\frac{t^{\rho}-\tau^{\rho}}{\rho}\right)^{\mu-1} E_{\mu,\mu}\left(\lambda\left(\frac{t^{\rho}-\tau^{\rho}}{\rho}\right)^{\mu}\right)\tau^{\rho-1} f(\tau)d\tau.
\end{aligned}
$$

Definition 4 *Let E be a Banach space. We say that a part P in $C(E)$ is equicontinuous if*

$$\forall\varepsilon>0,\ \exists\delta>0, \forall u,v\in E, \forall A\in P,\ \ \|u-v\|<\delta \Rightarrow \|A(u)-A(v)\|<\varepsilon.$$

Definition 5 *We say $A:E\to E$ is Completely continuous if for any bounded subset P of E, the set $A(P)$ is relatively compact.*

Theorem 3 (Arzelà-Ascoli's Theorem [26]) *Let $B\subset C(E,\mathbb{R}^n)$, $(E=[a,b]\subset\mathbb{R})$. B is relatively compact (i.e., $\overline{B}$ is compact) if and only if*

1. *B is uniformly bounded.*
2. *B is equicontinuous.*

Recall that a function f is uniformly bounded in B if there exists a constant $M>0$ such that

$$\|f\| = \sup_{x\in E}|f(x)| \le M, \forall f\in B.$$

Theorem 4 (Banach's Fixed Point Theorem [27]) *Let X be a Banach space and $Q: X\to X$ is a contraction mapping. Then Q has a fixed point i.e.*

$$\exists! x\in X:\ \ Qx=x.$$

Theorem 5 (Schauder's Fixed Point Theorem [28]) *Let X be a Banach space, and let P be a closed, convex and non-empty subset of X. Let $T:P\to P$ be a continuous mapping such that $T(P)$ is a relatively compact subset of X. Then T has at least one fixed point in P.*

Theorem 6 (Nonlinear alternative of Leray-Schauder type [28]) *Let X be a Banach space with $P\subset X$ be a closed and convex. U be an open subset of P with $0\in U$. Assume that $A:\overline{U}\to P$ is a continuous, compact (that is, $A(\overline{U})$ is a relatively compact subset of P) map. Then either*

(i) A has a fixed point in $\overline{U}$; or

(ii) there is a point $u\in\partial U$ and $\sigma\in(0,1)$ with $u=\sigma A(u)$.

Properties of Mittag-Leffler functions

Here, we present some properties of the Mittag-Leffler functions.

Definition 6 (See [2, 3]) *For $\sigma,\gamma>0$, $z\in\mathbb{R}$, the classical Mittag-Leffler function $E_{\sigma}(z)$ and the generalized Mittag-Leffler function $E_{\sigma,\gamma}(z)$ are defined by*

$$E_{\sigma}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\sigma k+1)} \tag{5}$$

and

$$E_{\sigma,\gamma}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\sigma k+\gamma)} \tag{6}$$

It is clear from these two equations that $E_{\sigma,1}(z)=E_{\sigma}(z)$.

Lemma 2 (see [20]) *Let $\alpha \in (0,1)$, $\theta > \alpha$ be arbitrary. The functions E_α, $E_{\alpha,\alpha}$ and $E_{\alpha,\theta}$ are nonnegative and have the following properties:*

(i) For any $t \in J$, $E_\alpha(-\lambda t^\alpha) \leq 1$, $E_{\alpha,\alpha}(-\lambda t^\alpha) \leq \frac{1}{\Gamma(\alpha)}$ and $E_{\alpha,\theta}(-\lambda t^\alpha) \leq \frac{1}{\Gamma(\theta)}$.

(ii) For any t_1, $t_2 \in J$,

$$|E_\alpha(-\lambda t_2^\alpha) - E_\alpha(-\lambda t_1^\alpha)| = O(|t_2 - t_1|^\alpha), \text{ as } t_2 \to t_1,$$
$$|E_{\alpha,\alpha}(-\lambda t_2^\alpha) - E_{\alpha,\alpha}(-\lambda t_1^\alpha)| = O(|t_2 - t_1|^\alpha), \text{ as } t_2 \to t_1,$$
$$|E_{\alpha,\alpha+1}(-\lambda t_2^\alpha) - E_{\alpha,\alpha+1}(-\lambda t_1^\alpha)| = O(|t_2 - t_1|^\alpha), \text{ as } t_2 \to t_1.$$

3 Solutions for BVP

In this section, we give the formula of the solution to the problem (1)–(3). We start by solving the following linear problem

$$^{CK}D_{1^-}^{\beta,\rho}\left(^{K}D_{0^+}^{\alpha,\rho} + \lambda\right)u(t) = h(t), \ t \in J = [0,1], \tag{7}$$

with

$$\left(^{K}I_{0^+}^{1-\alpha,\rho}u\right)(0) = u_0, \tag{8}$$

$$\left(^{K}D_{0^+}^{\alpha,\rho} + \lambda\right)u(1) = u_1. \tag{9}$$

Lemma 3 *Let $\alpha, \beta, \rho, \lambda \in \mathbb{R}$ be such that $0 < \alpha, \beta < 1$ and $\lambda, \rho > 0$. For a given $h \in C([0,1], \mathbb{R})$, the solution u to the linear BVP (7)–(9) is given by*

$$u(t) = u_0\left(\frac{t^\rho}{\rho}\right)^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda\left(\frac{t^\rho}{\rho}\right)^\alpha\right) + u_1\left(\frac{t^\rho}{\rho}\right)^\alpha E_{\alpha,\alpha+1}\left(-\lambda\left(\frac{t^\rho}{\rho}\right)^\alpha\right) + \int_0^1 K(t,\tau)\ h(\tau)d\tau, \tag{10}$$

where

$$K(t,\tau) = \frac{\tau^{\rho-1}}{\Gamma(\beta)} \times \begin{cases} \int_0^\tau \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1}\left(\frac{\tau^\rho - s^\rho}{\rho}\right)^{\beta-1} E_{\alpha,\alpha}\left(-\lambda\left(\frac{t^\rho - s^\rho}{\rho}\right)^\alpha\right)s^{\rho-1}ds, & 0 < \tau < t \leq 1, \\ \int_0^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1}\left(\frac{\tau^\rho - s^\rho}{\rho}\right)^{\beta-1} E_{\alpha,\alpha}\left(-\lambda\left(\frac{t^\rho - s^\rho}{\rho}\right)^\alpha\right)s^{\rho-1}ds, & 0 < t < \tau \leq 1. \end{cases} \tag{11}$$

Proof. First, by applying the right-sided Katugampola fractional integral $^{K}I_{1^-}^{\beta,\rho}$ defined by (4) to both sides of equation (7), using Theorem 1 and (9), we get

$$\left(^{K}D_{0^+}^{\alpha,\rho} + \lambda\right)u(t) = u_1 + \left(^{K}I_{1^-}^{\beta,\rho}h\right)(t). \tag{12}$$

Following the same idea of Theorem 2, with (8), equation (12) can be written as

$$\begin{aligned} u(t) &= u_0\left(\frac{t^\rho}{\rho}\right)^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda\left(\frac{t^\rho}{\rho}\right)^\alpha\right) \\ &+ \int_0^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda\left(\frac{t^\rho - s^\rho}{\rho}\right)^\alpha\right)[u_1 + {}^{K}I_{1^-}^{\beta,\rho}h(s)]s^{\rho-1}ds, \end{aligned}$$

so

$$\begin{aligned} u(t) &= u_0\left(\frac{t^\rho}{\rho}\right)^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda\left(\frac{t^\rho}{\rho}\right)^\alpha\right) \\ &+ u_1\int_0^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda\left(\frac{t^\rho - s^\rho}{\rho}\right)^\alpha\right)s^{\rho-1}ds \end{aligned}$$

$$+\frac{1}{\Gamma(\beta)}\int_0^t\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda\left(\frac{t^\rho-s^\rho}{\rho}\right)^\alpha\right)s^{\rho-1}\left[\int_s^1\left(\frac{\tau^\rho-s^\rho}{\rho}\right)^{\beta-1}\tau^{\rho-1}h(s)d\tau\right]ds. \tag{13}$$

Applying Fubini's Theorem, (13) can be re-written as

$$\begin{aligned} u(t) &= u_0\left(\frac{t^\rho}{\rho}\right)^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda\left(\frac{t^\rho}{\rho}\right)^\alpha\right)+u_1\left(\frac{t^\rho}{\rho}\right)^\alpha E_{\alpha,\alpha+1}\left(-\lambda\left(\frac{t^\rho}{\rho}\right)^\alpha\right)\\ &+\frac{1}{\Gamma(\beta)}\int_0^t\tau^{\rho-1}h(\tau)\int_0^\tau\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda\left(\frac{t^\rho-s^\rho}{\rho}\right)^\alpha\right)\left(\frac{\tau^\rho-s^\rho}{\rho}\right)^{\beta-1}s^{\rho-1}dsd\tau\\ &+\frac{1}{\Gamma(\beta)}\int_t^1\tau^{\rho-1}h(\tau)\int_0^t\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda\left(\frac{t^\rho-s^\rho}{\rho}\right)^\alpha\right)\left(\frac{\tau^\rho-s^\rho}{\rho}\right)^{\beta-1}s^{\rho-1}dsd\tau, \end{aligned}$$

which can be simplified to

$$\begin{aligned} u(t) &= u_0\left(\frac{t^\rho}{\rho}\right)^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda\left(\frac{t^\rho}{\rho}\right)^\alpha\right)+u_1\left(\frac{t^\rho}{\rho}\right)^\alpha E_{\alpha,\alpha+1}\left(-\lambda\left(\frac{t^\rho}{\rho}\right)^\alpha\right)\\ &+\int_0^1 K(t,\tau)\,h(\tau)d\tau. \end{aligned}$$

The proof is finished. ■

Let us define

$$K(t,\tau)=\begin{cases} K_1(t,\tau), & 0<\tau<t\le 1,\\ K_2(t,\tau), & 0<t<\tau\le 1.\end{cases}$$

where

$$K_1(t,\tau)=\frac{\tau^{\rho-1}}{\Gamma(\beta)}\int_0^\tau\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha-1}\left(\frac{\tau^\rho-s^\rho}{\rho}\right)^{\beta-1}E_{\alpha,\alpha}\left(-\lambda\left(\frac{t^\rho-s^\rho}{\rho}\right)^\alpha\right)s^{\rho-1}ds,$$

$$K_2(t,\tau)=\frac{\tau^{\rho-1}}{\Gamma(\beta)}\int_0^t\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha-1}\left(\frac{\tau^\rho-s^\rho}{\rho}\right)^{\beta-1}E_{\alpha,\alpha}\left(-\lambda\left(\frac{t^\rho-s^\rho}{\rho}\right)^\alpha\right)s^{\rho-1}ds.$$

In the next lemma, we present some properties of the function $K(t,\tau)$, that form the basis of our main work.

Lemma 4 *For $\alpha,\beta\in(0,1)$, ρ, $\lambda>0$, the function $K(t,\tau)$ mentioned in Lemma 3 satisfies the following estimates.*

1) *The function $K(t,\tau)$ is nonnegative.*

2)
$$|K_1(t,\tau)|\le\frac{\tau^{\rho-1}\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{\alpha-1}}{\Gamma(\beta+1)\Gamma(\alpha)},\ 0<\tau<t\le 1\ . \tag{14}$$

3)
$$|K_2(t,\tau)|\le\frac{\tau^{\rho-1}\left(\frac{\tau^\rho-t^\rho}{\rho}\right)^{\beta-1}}{\Gamma(\alpha+1)\Gamma(\beta)},\ 0<t<\tau\le 1. \tag{15}$$

4)
$$\int_0^1 K(t,\tau)\,d\tau\le\frac{\rho^\alpha+\rho^\beta}{\rho^{\alpha+\beta}\Gamma(\alpha+1)\Gamma(\beta+1)}=\Lambda_0. \tag{16}$$

Proof. 1) It is obvious that $K(t,\tau) \geq 0$.

2) For $0 < \tau < t \leq 1$, by Lemma 2, we obtain

$$\begin{aligned} & \left|K_1(t,\tau)\right| \\ = & \left|\frac{\tau^{\rho-1}}{\Gamma(\beta)}\int_0^\tau \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1}\left(\frac{\tau^\rho - s^\rho}{\rho}\right)^{\beta-1} E_{\alpha,\alpha}\left(-\lambda\left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha}\right)s^{\rho-1}ds\right| \\ \leq & \frac{\tau^{\rho-1}}{\Gamma(\beta)\Gamma(\alpha)}\left|\int_0^\tau \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1}\left(\frac{\tau^\rho - s^\rho}{\rho}\right)^{\beta-1} s^{\rho-1}ds\right|. \end{aligned}$$

Since

$$\begin{aligned} & \int_0^\tau \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1}\left(\frac{\tau^\rho - s^\rho}{\rho}\right)^{\beta-1} s^{\rho-1}ds \\ \leq & \left(\frac{t^\rho - \tau^\rho}{\rho}\right)^{\alpha-1}\int_0^\tau \left(\frac{\tau^\rho - s^\rho}{\rho}\right)^{\beta-1} s^{\rho-1}ds \\ \leq & \left(\frac{t^\rho - \tau^\rho}{\rho}\right)^{\alpha-1}\left[\frac{-1}{\beta\rho^\beta}\int_0^\tau \frac{d}{ds}\left(\tau^\rho - s^\rho\right)^\beta ds\right] \\ \leq & \left(\frac{t^\rho - \tau^\rho}{\rho}\right)^{\alpha-1}\left[\frac{-\left(\tau^\rho - s^\rho\right)^\beta}{\beta\rho^\beta}\right]_0^\tau \leq \frac{\left(\frac{\tau^\rho}{\rho}\right)^\beta}{\beta}\left(\frac{t^\rho - \tau^\rho}{\rho}\right)^{\alpha-1}, \end{aligned}$$

we see that

$$\left|K_1(t,\tau)\right| \leq \frac{\tau^{\rho-1}}{\Gamma(\beta)\Gamma(\alpha)}\frac{\left(\frac{\tau^\rho}{\rho}\right)^\beta}{\beta}\left(\frac{t^\rho - \tau^\rho}{\rho}\right)^{\alpha-1} \leq \frac{\tau^{\rho-1}\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{\alpha-1}}{\Gamma(\beta+1)\Gamma(\alpha)}.$$

3) For $0 < t < \tau \leq 1$, similarly with 2), by Lemma 2, we get

$$\begin{aligned} & \left|K_2(t,\tau)\right| \\ = & \left|\frac{\tau^{\rho-1}}{\Gamma(\beta)}\int_0^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1}\left(\frac{\tau^\rho - s^\rho}{\rho}\right)^{\beta-1} E_{\alpha,\alpha}\left(-\lambda\left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha}\right)s^{\rho-1}ds\right| \\ \leq & \frac{\tau^{\rho-1}}{\Gamma(\beta)\Gamma(\alpha)}\left|\int_0^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1}\left(\frac{\tau^\rho - s^\rho}{\rho}\right)^{\beta-1} s^{\rho-1}ds\right|. \end{aligned}$$

We have

$$\begin{aligned} & \int_0^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1}\left(\frac{\tau^\rho - s^\rho}{\rho}\right)^{\beta-1} s^{\rho-1}ds \\ \leq & \left(\frac{\tau^\rho - t^\rho}{\rho}\right)^{\beta-1}\int_0^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1}ds \\ \leq & \left(\frac{\tau^\rho - t^\rho}{\rho}\right)^{\beta-1}\left[\frac{-\left(t^\rho - s^\rho\right)^\alpha}{\alpha\rho^\alpha}\right]_0^t \\ \leq & \left(\frac{\tau^\rho - t^\rho}{\rho}\right)^{\beta-1}\left[\frac{t^{\rho\alpha}}{\alpha\rho^\alpha}\right] \leq \frac{\left(\frac{t^\rho}{\rho}\right)^\alpha}{\alpha}\left(\frac{\tau^\rho - t^\rho}{\rho}\right)^{\beta-1}. \end{aligned}$$

Then

$$\left|K_2(t,\tau)\right| \leq \frac{\tau^{\rho-1}}{\Gamma(\beta)\Gamma(\alpha)}\frac{\left(\frac{t^\rho}{\rho}\right)^\alpha}{\alpha}\left(\frac{\tau^\rho - t^\rho}{\rho}\right)^{\beta-1} \leq \frac{\left(\frac{\tau^\rho-t^\rho}{\rho}\right)^{\beta-1}\tau^{\rho-1}}{\Gamma(\alpha+1)\Gamma(\beta)}.$$

4) By (14) and (15), we observe that

$$\begin{aligned}\int_0^1 K(t,\tau)\,d\tau &= \int_0^t K_1(t,\tau)\,d\tau + \int_t^1 K_2(t,\tau)\,d\tau \\ &\leq \frac{1}{\Gamma(\beta+1)\Gamma(\alpha)} \int_0^t \left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{\alpha-1} \tau^{\rho-1} d\tau \\ &\quad + \frac{1}{\Gamma(\alpha+1)\Gamma(\beta)} \int_t^1 \left(\frac{\tau^\rho-t^\rho}{\rho}\right)^{\beta-1} \tau^{\rho-1} d\tau \\ &\leq \frac{t^{\rho\alpha}}{\rho^\alpha\Gamma(\beta+1)\Gamma(\alpha+1)} + \frac{(1-t^\rho)^\beta}{\rho^\beta\Gamma(\alpha+1)\Gamma(\beta+1)} \\ &\leq \frac{1}{\rho^\alpha\Gamma(\beta+1)\Gamma(\alpha+1)} + \frac{1}{\rho^\beta\Gamma(\alpha+1)\Gamma(\beta+1)} \\ &\leq \frac{\rho^\alpha+\rho^\beta}{\rho^{\alpha+\beta}\Gamma(\beta+1)\Gamma(\alpha+1)} = \Lambda_0.\end{aligned}$$

■

Lemma 5 *The following inequalities hold*

$$\left|\int_0^\tau \left[\left(\frac{t_1^\rho-s^\rho}{\rho}\right)^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda\left(\frac{t_1^\rho-s^\rho}{\rho}\right)^\alpha\right) - \left(\frac{t_2^\rho-s^\rho}{\rho}\right)^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda\left(\frac{t_2^\rho-s^\rho}{\rho}\right)^\alpha\right)\right] \times \left(\frac{\tau^\rho-s^\rho}{\rho}\right)^{\beta-1} s^{\rho-1} ds\right| \leq O((t_2^\rho-t_1^\rho)^\alpha), \quad \textit{for} \quad 0<\tau<t_1<t_2\leq 1, \tag{17}$$

$$\begin{aligned}&\left|\int_0^{t_1} \left[\left(\frac{t_1^\rho-s^\rho}{\rho}\right)^{\alpha-1} - \left(\frac{t_2^\rho-s^\rho}{\rho}\right)^{\alpha-1}\right] \left(\frac{\tau^\rho-s^\rho}{\rho}\right)^{\beta-1} s^{\rho-1} ds\right| \\ &\leq \left(\frac{\tau^\rho-t_2^\rho}{\rho}\right)^{\beta-1} .O((t_2^\rho-t_1^\rho)^\alpha), \quad \textit{for} \quad 0<t_1<t_2<\tau\leq 1,\end{aligned} \tag{18}$$

$$\left|\int_0^{t_1} \left[\left(\frac{t_1^\rho-s^\rho}{\rho}\right)^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda\left(\frac{t_1^\rho-s^\rho}{\rho}\right)^\alpha\right) - \left(\frac{t_2^\rho-s^\rho}{\rho}\right)^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda\left(\frac{t_2^\rho-s^\rho}{\rho}\right)^\alpha\right)\right] \times \left(\frac{\tau^\rho-s^\rho}{\rho}\right)^{\beta-1} s^{\rho-1} ds\right| \leq \left[\left(\frac{\tau^\rho}{\rho}\right)^\beta - \left(\frac{\tau^\rho-t_1^\rho}{\rho}\right)^\beta\right] .O((t_2^\rho-t_1^\rho)^\alpha), \quad \textit{for}\ 0<t_1<\tau<t_2\leq 1. \tag{19}$$

Proof. For $0<\tau<t_1<t_2\leq 1$, it follows from Lemma 2 and the Mean Value Theorem that

$$\begin{aligned}&\left|\left(\frac{t_1^\rho-s^\rho}{\rho}\right)^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda\left(\frac{t_1^\rho-s^\rho}{\rho}\right)^\alpha\right) - \left(\frac{t_2^\rho-s^\rho}{\rho}\right)^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda\left(\frac{t_2^\rho-s^\rho}{\rho}\right)^\alpha\right)\right| \\ \leq\ & \left|\left(\frac{t_1^\rho-s^\rho}{\rho}\right)^{\alpha-1} - \left(\frac{t_2^\rho-s^\rho}{\rho}\right)^{\alpha-1}\right| E_{\alpha,\alpha}\left(-\lambda\left(\frac{t_1^\rho-s^\rho}{\rho}\right)^\alpha\right) \\ & + \left|E_{\alpha,\alpha}\left(-\lambda\left(\frac{t_1^\rho-s^\rho}{\rho}\right)^\alpha\right) - E_{\alpha,\alpha}\left(-\lambda\left(\frac{t_2^\rho-s^\rho}{\rho}\right)^\alpha\right)\right| \left(\frac{t_2^\rho-s^\rho}{\rho}\right)^{\alpha-1} \\ \leq\ & O\left((t_2^\rho-t_1^\rho)^\alpha\right),\end{aligned}$$

which yields

$$\begin{aligned}
&\left|\int_0^\tau \left[\left(\frac{t_1^\rho - s^\rho}{\rho}\right)^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda\left(\frac{t_1^\rho - s^\rho}{\rho}\right)^\alpha\right) - \left(\frac{t_2^\rho - s^\rho}{\rho}\right)^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda\left(\frac{t_2^\rho - s^\rho}{\rho}\right)^\alpha\right)\right]\right.\\
&\left.\times\left(\frac{\tau^\rho - s^\rho}{\rho}\right)^{\beta-1} s^{\rho-1} ds\right|\\
\leq\ & \int_0^\tau \left(\frac{\tau^\rho - s^\rho}{\rho}\right)^{\beta-1} s^{\rho-1} ds . O((t_2^\rho - t_1^\rho)^\alpha)\\
\leq\ & \frac{1}{\beta}\left(\frac{\tau^\rho}{\rho}\right)^\beta . O((t_2^\rho - t_1^\rho)^\alpha) \leq O((t_2^\rho - t_1^\rho)^\alpha).
\end{aligned}$$

For $0 < t_1 < t_2 < \tau \leq 1$, we obtain

$$\begin{aligned}
&\left|\int_0^{t_1} \left[\left(\frac{t_1^\rho - s^\rho}{\rho}\right)^{\alpha-1} - \left(\frac{t_2^\rho - s^\rho}{\rho}\right)^{\alpha-1}\right]\left(\frac{\tau^\rho - s^\rho}{\rho}\right)^{\beta-1} s^{\rho-1} ds\right|\\
\leq\ & \left|\int_0^{t_1} \left[\left(\frac{t_1^\rho - s^\rho}{\rho}\right)^{\alpha-1} - \left(\frac{t_2^\rho - s^\rho}{\rho}\right)^{\alpha-1}\right] s^{\rho-1} ds\right|\left(\frac{\tau^\rho - t_1^\rho}{\rho}\right)^{\beta-1}.
\end{aligned}$$

We have

$$\begin{aligned}
&\left[\left(\frac{t_1^\rho - s^\rho}{\rho}\right)^{\alpha-1} - \left(\frac{t_2^\rho - s^\rho}{\rho}\right)^{\alpha-1}\right] s^{\rho-1}\\
=\ & \rho^{1-\alpha} s^{\rho-1}\left[(t_1^\rho - s^\rho)^{\alpha-1} - (t_2^\rho - s^\rho)^{\alpha-1}\right] = \frac{-1}{\alpha\rho^\alpha}\frac{d}{ds}\left[(t_1^\rho - s^\rho)^\alpha - (t_2^\rho - s^\rho)^\alpha\right].
\end{aligned}$$

Then

$$\begin{aligned}
&\left|\int_0^{t_1} \left[\left(\frac{t_1^\rho - s^\rho}{\rho}\right)^{\alpha-1} - \left(\frac{t_2^\rho - s^\rho}{\rho}\right)^{\alpha-1}\right] s^{\rho-1} ds\right|\left(\frac{\tau^\rho - t_1^\rho}{\rho}\right)^{\beta-1}\\
\leq\ & \frac{1}{\alpha\rho^\alpha}\left|\int_0^{t_1} \frac{d}{ds}\left[(t_1^\rho - s^\rho)^\alpha - (t_2^\rho - s^\rho)^\alpha\right] ds\right|\left(\frac{\tau^\rho - t_1^\rho}{\rho}\right)^{\beta-1}\\
\leq\ & \frac{1}{\alpha\rho^\alpha}\left[t_2^{\rho\alpha} - t_1^{\rho\alpha} + (t_2^\rho - t_1^\rho)^\alpha\right]\left(\frac{\tau^\rho - t_1^\rho}{\rho}\right)^{\beta-1}\\
\leq\ & \left(\frac{\tau^\rho - t_2^\rho}{\rho}\right)^{\beta-1} . O((t_2^\rho - t_1^\rho)^\alpha).
\end{aligned}$$

In a similar way as for (17) and (18), for $0 < t_1 < \tau < t_2 \leq 1$, we obtain

$$\begin{aligned}
&\left|\int_0^{t_1} \left[\left(\frac{t_1^\rho - s^\rho}{\rho}\right)^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda\left(\frac{t_1^\rho - s^\rho}{\rho}\right)^\alpha\right) -\right.\right.\\
&\left.\left.\left(\frac{t_2^\rho - s^\rho}{\rho}\right)^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda\left(\frac{t_2^\rho - s^\rho}{\rho}\right)^\alpha\right)\right]\left(\frac{\tau^\rho - s^\rho}{\rho}\right)^{\beta-1} s^{\rho-1} ds\right|\\
\leq\ & \int_0^{t_1} \left(\frac{\tau^\rho - s^\rho}{\rho}\right)^{\beta-1} s^{\rho-1} ds \cdot O((t_2^\rho - t_1^\rho)^\alpha)\\
\leq\ & \left[\left(\frac{\tau^\rho}{\rho}\right)^\beta - \left(\frac{\tau^\rho - t_1^\rho}{\rho}\right)^\beta\right] . O((t_2^\rho - t_1^\rho)^\alpha).
\end{aligned}$$

■

Lemma 6 *We have the following properties*

$$|K_1(t_2,\tau)-K_1(t_1,\tau)|=\tau^{\rho-1}\cdot O((t_2^\rho-t_1^\rho)^\alpha),\ \textit{for}\ \ 0<\tau<t_1<t_2\le 1, \tag{20}$$

$$|K_2(t_2,\tau)-K_2(t_1,\tau)|=\tau^{\rho-1}\left(\frac{\tau^\rho-t_2^\rho}{\rho}\right)^{\beta-1}.O((t_2^\rho-t_1^\rho)^\alpha), \tag{21}$$

for $0<t_1<t_2<\tau\le 1$, *and*

$$|K_1(t_2,\tau)-K_2(t_1,\tau)|=\tau^{\rho-1}\left[\left(\frac{\tau^\rho}{\rho}\right)^\beta-\left(\frac{\tau^\rho-t_1^\rho}{\rho}\right)^\beta\right]\cdot O((t_2^\rho-t_1^\rho)^\alpha), \tag{22}$$

for $0<t_1<\tau<t_2\le 1$.

Proof. For $0<\tau<t_1<t_2\le 1$, by (17), we get

$$\begin{aligned}
&|K_1(t_2,\tau)-K_1(t_1,\tau)|\\
\le\ & \frac{\tau^{\rho-1}}{\Gamma(\beta)}\left|\int_0^\tau\left[\left(\frac{t_1^\rho-s^\rho}{\rho}\right)^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda\left(\frac{t_1^\rho-s^\rho}{\rho}\right)^\alpha\right)-\right.\right.\\
&\left.\left.\left(\frac{t_2^\rho-s^\rho}{\rho}\right)^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda\left(\frac{t_2^\rho-s^\rho}{\rho}\right)^\alpha\right)\right]\left(\frac{\tau^\rho-s^\rho}{\rho}\right)^{\beta-1}s^{\rho-1}ds\right|\\
\le\ & \tau^{\rho-1}\cdot O((t_2^\rho-t_1^\rho)^\alpha).
\end{aligned}$$

For $0<t_1<t_2<\tau\le 1$, by (18) and Lemma 2, we find

$$\begin{aligned}
&|K_2(t_2,\tau)-K_2(t_1,\tau)|\\
=\ & \left|\frac{\tau^{\rho-1}}{\Gamma(\beta)}\left[\int_0^{t_2}\left(\frac{t_2^\rho-s^\rho}{\rho}\right)^{\alpha-1}\left(\frac{\tau^\rho-s^\rho}{\rho}\right)^{\beta-1}E_{\alpha,\alpha}\left(-\lambda\left(\frac{t_2^\rho-s^\rho}{\rho}\right)^\alpha\right)s^{\rho-1}ds\right.\right.\\
&\left.\left.-\int_0^{t_1}\left(\frac{t_1^\rho-s^\rho}{\rho}\right)^{\alpha-1}\left(\frac{\tau^\rho-s^\rho}{\rho}\right)^{\beta-1}E_{\alpha,\alpha}\left(-\lambda\left(\frac{t_1^\rho-s^\rho}{\rho}\right)^\alpha\right)s^{\rho-1}ds\right]\right|\\
\le\ & \frac{\tau^{\rho-1}}{\Gamma(\beta)}\left|\int_0^{t_1}\left(\frac{t_2^\rho-s^\rho}{\rho}\right)^{\alpha-1}\left(\frac{\tau^\rho-s^\rho}{\rho}\right)^{\beta-1}E_{\alpha,\alpha}\left(-\lambda\left(\frac{t_2^\rho-s^\rho}{\rho}\right)^\alpha\right)s^{\rho-1}ds\right|\\
&+\frac{\tau^{\rho-1}}{\Gamma(\beta)}\left|\int_{t_1}^{t_2}\left(\frac{t_2^\rho-s^\rho}{\rho}\right)^{\alpha-1}\left(\frac{\tau^\rho-s^\rho}{\rho}\right)^{\beta-1}E_{\alpha,\alpha}\left(-\lambda\left(\frac{t_2^\rho-s^\rho}{\rho}\right)^\alpha\right)s^{\rho-1}ds\right|\\
&-\frac{\tau^{\rho-1}}{\Gamma(\beta)}\left|\int_0^{t_1}\left(\frac{t_1^\rho-s^\rho}{\rho}\right)^{\alpha-1}\left(\frac{\tau^\rho-s^\rho}{\rho}\right)^{\beta-1}E_{\alpha,\alpha}\left(-\lambda\left(\frac{t_1^\rho-s^\rho}{\rho}\right)^\alpha\right)s^{\rho-1}ds\right|\\
\le\ & \frac{\tau^{\rho-1}}{\Gamma(\beta)}\left[\left|\int_0^{t_1}\left(\left(\frac{t_1^\rho-s^\rho}{\rho}\right)^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda\left(\frac{t_1^\rho-s^\rho}{\rho}\right)^\alpha\right)-\left(\frac{t_2^\rho-s^\rho}{\rho}\right)^{\alpha-1}\right.\right.\right.\\
&\left.\left.\left.\times E_{\alpha,\alpha}\left(-\lambda\left(\frac{t_2^\rho-s^\rho}{\rho}\right)^\alpha\right)\right)\left(\frac{\tau^\rho-s^\rho}{\rho}\right)^{\beta-1}s^{\rho-1}ds\right|\right]\\
&+\frac{\tau^{\rho-1}}{\Gamma(\beta)}\int_{t_1}^{t_2}\left(\frac{t_2^\rho-s^\rho}{\rho}\right)^{\alpha-1}\left(\frac{\tau^\rho-s^\rho}{\rho}\right)^{\beta-1}E_{\alpha,\alpha}\left(-\lambda\left(\frac{t_2^\rho-s^\rho}{\rho}\right)^\alpha\right)s^{\rho-1}ds\\
\le\ & \frac{\tau^{\rho-1}}{\Gamma(\beta)}\left[\frac{1}{\Gamma(\alpha)}\left|\int_0^{t_1}\left(\left(\frac{t_1^\rho-s^\rho}{\rho}\right)^{\alpha-1}-\left(\frac{t_2^\rho-s^\rho}{\rho}\right)^{\alpha-1}\right)\left(\frac{\tau^\rho-s^\rho}{\rho}\right)^{\beta-1}s^{\rho-1}ds\right|\right]
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\tau^{\rho-1}}{\Gamma(\beta)\Gamma(\alpha)}\int_{t_1}^{t_2}\left(\frac{t_2^\rho-s^\rho}{\rho}\right)^{\alpha-1}\left(\frac{\tau^\rho-s^\rho}{\rho}\right)^{\beta-1}s^{\rho-1}ds\\
\leq\ & \frac{\tau^{\rho-1}}{\Gamma(\alpha)\Gamma(\beta)}\left(\frac{\tau^\rho-t_2^\rho}{\rho}\right)^{\beta-1}\cdot O((t_2^\rho-t_1^\rho)^\alpha)\\
&+\frac{\tau^{\rho-1}\left(\frac{\tau^\rho-t_2^\rho}{\rho}\right)^{\beta-1}}{\Gamma(\alpha)\Gamma(\beta)}\int_{t_1}^{t_2}\left(\frac{t_2^\rho-s^\rho}{\rho}\right)^{\alpha-1}s^{\rho-1}ds\\
\leq\ & \tau^{\rho-1}\left(\frac{\tau^\rho-t_2^\rho}{\rho}\right)^{\beta-1}\cdot O((t_2^\rho-t_1^\rho)^\alpha).
\end{aligned}
$$

In the same way, for $0<t_1<\tau<t_2\leq 1$, by (19) and Lemma 2, we have

$$
\begin{aligned}
&\left|K_1(t_2,\tau)-K_2(t_1,\tau)\right|\\
=\ & \left|\frac{\tau^{\rho-1}}{\Gamma(\beta)}\left[\int_0^{\tau}\left(\frac{t_2^\rho-s^\rho}{\rho}\right)^{\alpha-1}\left(\frac{\tau^\rho-s^\rho}{\rho}\right)^{\beta-1}E_{\alpha,\alpha}\left(-\lambda\left(\frac{t_2^\rho-s^\rho}{\rho}\right)^{\alpha}\right)s^{\rho-1}ds\right.\right.\\
&\left.\left.-\int_0^{t_1}\left(\frac{t_1^\rho-s^\rho}{\rho}\right)^{\alpha-1}\left(\frac{\tau^\rho-s^\rho}{\rho}\right)^{\beta-1}E_{\alpha,\alpha}\left(-\lambda\left(\frac{t_1^\rho-s^\rho}{\rho}\right)^{\alpha}\right)s^{\rho-1}ds\right]\right|\\
\leq\ & \frac{\tau^{\rho-1}}{\Gamma(\beta)}\left|\left[\int_0^{t_1}\left(\frac{t_2^\rho-s^\rho}{\rho}\right)^{\alpha-1}\left(\frac{\tau^\rho-s^\rho}{\rho}\right)^{\beta-1}E_{\alpha,\alpha}\left(-\lambda\left(\frac{t_2^\rho-s^\rho}{\rho}\right)^{\alpha}\right)s^{\rho-1}ds\right.\right.\\
&\left.\left.-\int_0^{t_1}\left(\frac{t_1^\rho-s^\rho}{\rho}\right)^{\alpha-1}\left(\frac{\tau^\rho-s^\rho}{\rho}\right)^{\beta-1}E_{\alpha,\alpha}\left(-\lambda\left(\frac{t_1^\rho-s^\rho}{\rho}\right)^{\alpha}\right)s^{\rho-1}ds\right]\right|\\
&+\frac{\tau^{\rho-1}}{\Gamma(\beta)}\int_{t_1}^{\tau}\left(\frac{t_2^\rho-s^\rho}{\rho}\right)^{\alpha-1}\left(\frac{\tau^\rho-s^\rho}{\rho}\right)^{\beta-1}E_{\alpha,\alpha}\left(-\lambda\left(\frac{t_2^\rho-s^\rho}{\rho}\right)^{\alpha}\right)s^{\rho-1}ds\\
\leq\ & \frac{\tau^{\rho-1}}{\Gamma(\beta)}\left[\left|\int_0^{t_1}\left[\left(\frac{t_2^\rho-s^\rho}{\rho}\right)^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda\left(\frac{t_2^\rho-s^\rho}{\rho}\right)^{\alpha}\right)-\left(\frac{t_1^\rho-s^\rho}{\rho}\right)^{\alpha-1}\right.\right.\right.\\
&\left.\left.\left.\times\ E_{\alpha,\alpha}\left(-\lambda\left(\frac{t_1^\rho-s^\rho}{\rho}\right)^{\alpha}\right)\right]\left(\frac{\tau^\rho-s^\rho}{\rho}\right)^{\beta-1}s^{\rho-1}ds\right|\right]\\
&+\frac{\tau^{\rho-1}}{\Gamma(\beta)\Gamma(\alpha)}\int_{t_1}^{\tau}\left(\frac{t_2^\rho-s^\rho}{\rho}\right)^{\alpha-1}\left(\frac{\tau^\rho-s^\rho}{\rho}\right)^{\beta-1}s^{\rho-1}ds\\
\leq\ & \frac{\tau^{\rho-1}}{\Gamma(\beta)}\int_0^{t_1}\left(\frac{\tau^\rho-s^\rho}{\rho}\right)^{\beta-1}s^{\rho-1}ds\cdot O((t_2^\rho-t_1^\rho)^\alpha)\\
\leq\ & \tau^{\rho-1}\left[\left(\frac{\tau^\rho}{\rho}\right)^{\beta}-\left(\frac{\tau^\rho-t_1^\rho}{\rho}\right)^{\beta}\right]\cdot O((t_2^\rho-t_1^\rho)^\alpha).
\end{aligned}
$$

■

4 Existence and Stability Results

Now, to prove our results, we give the following conditions

(H_1) $f:[0,1]\times\mathbb{R}\longrightarrow\mathbb{R}$ is a continuous function.

(H_2) There exists a constant $L>0$ such that

$$
|f(t,u)-f(t,v)|\leq L\,|u-v|\,,\ \forall u,v\in\mathbb{R},\ t\in[0,1].
$$

(H_3) There exists a constant $M > 0$ with $|f(t,u)| \leq M$ for each $t \in [0,1]$.

In light of Lemma 3, we define the operator $F : C([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$ as follows

$$\begin{aligned} Fu(t) &= u_0 \left(\frac{t^\rho}{\rho}\right)^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda\left(\frac{t^\rho}{\rho}\right)^\alpha\right) + u_1 \left(\frac{t^\rho}{\rho}\right)^{\alpha} E_{\alpha,\alpha+1}\left(-\lambda\left(\frac{t^\rho}{\rho}\right)^\alpha\right) \\ &\quad + \int_0^1 K(t,\tau)\ f(\tau,u(\tau))d\tau, \end{aligned} \tag{23}$$

where $K(t,\tau)$ is defined by (11).

Lemma 7 *F is a completely continuous operator*

Proof. Firstly, according to (H_1), we note that the operator F is well defined. Next, choosing

$$\eta \geq \frac{|u_0|}{\rho^{\alpha-1}\Gamma(\alpha)} + \frac{|u_1|}{\rho^\alpha\Gamma(\alpha+1)} + M\Lambda_0.$$

We define

$$\Omega_\eta = \{u \in C([0,1],\mathbb{R}) :\ \|u\|_\infty \leq \eta,\ \eta > 0\}. \tag{24}$$

Clearly, Ω_η is a nonempty, bounded, closed and convex subset of $C([0,1],\mathbb{R})$. We show that $F(\Omega_\eta)$ is uniformly bounded. Let $u \in \Omega_\eta$, in fact for any $t \in [0,1]$, by condition (H_3), from Lemma 2(i) and equation (16), we obtain

$$\begin{aligned} |Fu(t)| &\leq \left| u_0 \left(\frac{t^\rho}{\rho}\right)^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda\left(\frac{t^\rho}{\rho}\right)^\alpha\right)\right| \\ &\quad + \left| u_1 \left(\frac{t^\rho}{\rho}\right)^{\alpha} E_{\alpha,\alpha+1}\left(-\lambda\left(\frac{t^\rho}{\rho}\right)^\alpha\right)\right| \\ &\quad + \int_0^1 K(t,\tau)\,|f(\tau,u(\tau))|\,d\tau \\ &\leq \frac{|u_0|\,t^{\rho(\alpha-1)}}{\rho^{\alpha-1}\Gamma(\alpha)} + \frac{|u_1|\,t^{\rho\alpha}}{\rho^\alpha\Gamma(\alpha+1)} + M\int_0^1 K(t,\tau)\,d\tau \\ &\leq \frac{|u_0|\,t^{\rho(\alpha-1)}}{\rho^{\alpha-1}\Gamma(\alpha)} + \frac{|u_1|\,t^{\rho\alpha}}{\rho^\alpha\Gamma(\alpha+1)} + \frac{M(\rho^\alpha\ + \rho^\beta)}{\rho^{\alpha+\beta}\Gamma(\alpha+1)\Gamma(\beta+1)} \\ &\leq \frac{|u_0|}{\rho^{\alpha-1}\Gamma(\alpha)} + \frac{|u_1|}{\rho^\alpha\Gamma(\alpha+1)} + M\Lambda_0 \\ &\leq \eta. \end{aligned}$$

Consequently,

$$\|Fu\|_\infty \leq \frac{|u_0|}{\rho^{\alpha-1}\Gamma(\alpha)} + \frac{|u_1|}{\rho^\alpha\Gamma(\alpha+1)} + M\Lambda_0 < \infty, \text{ for all } u \in \Omega_\eta$$

and hence $F(\Omega_\eta)$ is uniformly bounded.

Now, we show that F is equicontinuous. Let $t_1, t_2 \in [0,1]$, with $t_1 < t_2$, $\forall u \in \Omega_\eta$, by the Mean Value Theorem, we obtain

$$\begin{aligned} &|Fu(t_2) - Fu(t_1)| \\ \leq &\left| u_0 \left[\left(\frac{t_2^\rho}{\rho}\right)^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda\left(\frac{t_2^\rho}{\rho}\right)^\alpha\right) - \left(\frac{t_1^\rho}{\rho}\right)^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda\left(\frac{t_1^\rho}{\rho}\right)^\alpha\right)\right]\right| \\ &+ \left| u_1 \left[\left(\frac{t_2^\rho}{\rho}\right)^{\alpha} E_{\alpha,\alpha+1}\left(-\lambda\left(\frac{t_2^\rho}{\rho}\right)^\alpha\right) - \left(\frac{t_1^\rho}{\rho}\right)^{\alpha} E_{\alpha,\alpha+1}\left(-\lambda\left(\frac{t_1^\rho}{\rho}\right)^\alpha\right)\right]\right| \end{aligned}$$

$$
\begin{aligned}
&+\int_0^1 |K(t_2,\tau)-K(t_1,\tau)| \ |f(\tau,u(\tau))|\, d\tau \\
\leq\ & |u_0|\, O\left((t_2^\rho - t_1^\rho)^\alpha\right) + |u_1|\, O\left((t_2^\rho - t_1^\rho)^\alpha\right) \\
&+M\int_0^1 |K(t_2,\tau)-K(t_1,\tau)|\, d\tau \\
\leq\ & O\left((t_2^\rho - t_1^\rho)^\alpha\right) + M\int_0^1 |K(t_2,\tau)-K(t_1,\tau)|\, d\tau.
\end{aligned}
$$

It remains to show that the right-hand side of the above inequality tends to zero as $t_2 \to t_1$. In what follows, we divide the proof into three cases

Case1: For $\tau < t_1 < t_2$, by (20), we have

$$
\begin{aligned}
|Fu(t_2)-Fu(t_1)| &\leq O\left((t_2^\rho - t_1^\rho)^\alpha\right) + M\int_0^1 |k_1(t_2,\tau)-k_1(t_1,\tau)|\, d\tau \\
&\leq O\left((t_2^\rho - t_1^\rho)^\alpha\right) + M\int_0^1 O((t_2^\rho - t_1^\rho)^\alpha)\tau^{\rho-1} d\tau \\
&= O((t_2^\rho - t_1^\rho)^\alpha).
\end{aligned}
$$

Case 2: For $t_1 < t_2 < \tau$, by (21), we get

$$
\begin{aligned}
&|Fu(t_2)-Fu(t_1)| \\
\leq\ & O\left((t_2^\rho - t_1^\rho)^\alpha\right) + M\int_0^1 |k_2(t_2,\tau)-k_2(t_1,\tau)|\, d\tau \\
\leq\ & O\left((t_2^\rho - t_1^\rho)^\alpha\right) + M\int_0^1 \tau^{\rho-1}\left(\frac{\tau^\rho - t_2^\rho}{\rho}\right)^{\beta-1} .O((t_2^\rho - t_1^\rho)^\alpha))d\tau \\
=\ & O((t_2^\rho - t_1^\rho)^\alpha).
\end{aligned}
$$

Case 3: In the same way, for $t_1 < \tau < t_2$, by (22), we obtain

$$
\begin{aligned}
&|Fu(t_2)-Fu(t_1)| \\
\leq\ & O\left((t_2^\rho - t_1^\rho)^\alpha\right) + M\int_0^1 |k_1(t_2,\tau)-k_2(t_1,\tau)|\, d\tau \\
\leq\ & O\left((t_2^\rho - t_1^\rho)^\alpha\right) + \frac{M}{\Gamma(\beta+1)}\int_0^1 \tau^{\rho-1}\left[\left(\frac{\tau^\rho}{\rho}\right)^{\beta} - \left(\frac{\tau^\rho - t_1^\rho}{\rho}\right)^{\beta}\right] .O((t_2^\rho - t_1^\rho)^\alpha))d\tau \\
=\ & O((t_2^\rho - t_1^\rho)^\alpha).
\end{aligned}
$$

Consequently, by this three cases, we have $|Fu(t_2) - Fu(t_1)| \to 0$ as $t_2 \to t_1$. Finally, by the Ascoli-Arzela Theorem 3, we have $F : C([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$ is completely continuous. ■

Next, we study the existence and uniqueness of solution for the PVP (1)–(3).

4.1 Existence of At Least One Solution via Schauder's Fixed Point Theorem

Now, we demonstrate the first existence result, by using the fixed point theorem of Schauder

Theorem 7 *Assume that (H_1) and (H_3) are satisfied. Then the problem (1)–(3) has at least one solution on $[0,1]$.*

Proof. Let the operator F defined in (23), then we shall show that F satisfies the assumptions of Schauder's Fixed Point Theorem. It means, we will prove that the operator F is continuous and completely continuous.

Let's first show that F is continuous. Let $\{u_n\}$ be a sequence such that $u_n \to u$ in $C([0,1],\mathbb{R})$, for each $t \in [0,1]$, by (16), we get

$$\begin{aligned}
|Fu_n(t) - Fu(t)| &= \left| \int_0^1 K(t,\tau) f(\tau, u_n(\tau)) d\tau - \int_0^1 K(t,\tau) f(\tau, u(\tau)) d\tau \right| \\
&\leq \int_0^1 K(t,\tau) |f(\tau, u_n(\tau)) - f(\tau, u(\tau))| \, d\tau \\
&\leq \int_0^1 K(t,\tau) \sup |f(\tau, u_n(\tau)) - f(\tau, u(\tau))| \, d\tau \\
&\leq \|f(., u_n) - f(., u)\|_\infty \int_0^1 K(t,\tau) \, d\tau \\
&\leq \frac{(\rho^\alpha + \rho^\beta) \|f(., u_n) - f(., u)\|_\infty}{\rho^{\alpha+\beta}\Gamma(\alpha+1)\Gamma(\beta+1)} \\
&\leq \Lambda_0 \|f(., u_n) - f(., u)\|_\infty .
\end{aligned}$$

Then $\|Fu_n - Fu\|_\infty \leq \Lambda_0 \|f(., u_n) - f(., u)\|_\infty$. Since f is continuous, we see that $\|f(., u_n) - f(., u)\|_\infty \to 0$ as $n \to \infty$. Consequently, F is continuous. From Lemma 7, we know that F is a completely continuous operator. As a consequence of Schauder's Fixed Point Theorem 5, we deduce that F has a fixed point which is a solution of the problem (1)–(3) on $[0,1]$. ■

Next, we demonstrate the second existence result, by using the fixed point theorem of Schaefer.

4.2 Existence of At Least One Solution via Schaefer's Fixed Point Theorem

Theorem 8 *Assume that* (H_1) *and* (H_3) *hold. Then the problem (1)–(3) has at least one solution.*

Proof. Consider F as in (23). Clearly, F is a continuous and completely continuous operator.

Now, it remains to show that the set

$$\mathcal{E} = \{u \in C([0,1],\mathbb{R}) : u = \lambda u, \ \lambda \in (0,1)\}$$

is bounded. Let $u \in \mathcal{E}$. Then, $u = \lambda F u$ for some $\lambda \in (0,1)$. For each $t \in [0,1]$, by (H_3), Lemma 2 and equation (16), we obtain

$$\begin{aligned}
|u(t)| &= \lambda \left| u_0 \left(\frac{t^\rho}{\rho}\right)^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda \left(\frac{t^\rho}{\rho}\right)^\alpha\right) + u_1 \left(\frac{t^\rho}{\rho}\right)^\alpha E_{\alpha,\alpha+1}\left(-\lambda \left(\frac{t^\rho}{\rho}\right)^\alpha\right) \right. \\
&\quad \left. + \int_0^1 K(t,\tau) \ f(\tau, u(\tau)) d\tau \right| \\
&\leq \left| u_0 \left(\frac{t^\rho}{\rho}\right)^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda \left(\frac{t^\rho}{\rho}\right)^\alpha\right) \right| + \left| u_1 \left(\frac{t^\rho}{\rho}\right)^\alpha E_{\alpha,\alpha+1}\left(-\lambda \left(\frac{t^\rho}{\rho}\right)^\alpha\right) \right| \\
&\quad + \int_0^1 K(t,\tau) \ |f(\tau, u(\tau))| \, d\tau \\
&\leq \frac{|u_0|}{\rho^{\alpha-1}\Gamma(\alpha)} + \frac{|u_1|}{\rho^\alpha \Gamma(\alpha+1)} + M\Lambda_0 < \infty.
\end{aligned}$$

This shows that the set $\mathcal{E}$ is bounded. Hence the fixed point theorem of Schaefer guarantees that F has a fixed point, which is a solution of (1)–(3). ■

Our third existence result for (1)–(3) is based on the non-linear alternative of Leray-Schauder type.

4.3 Existence of At Least One Solution via the Non-Linear Alternative of Leray-Schauder Type

Theorem 9 *Assume that hypotheses* (H_1) *and* (H_3) *hold, and that there exists* $\theta > 0$, *such that*

$$\frac{1}{\theta}\left(\frac{|u_0|}{\rho^{\alpha-1}\Gamma(\alpha)} + \frac{|u_1|}{\rho^{\alpha}\Gamma(\alpha+1)} + M\Lambda_0\right) < 1. \tag{25}$$

Then the problem (1)–(3) has at least one solution on $[0, 1]$.

Proof. Consider the operator F defined in (23), then we shall show that all assumption of Leray-Schauder Fixed Point Theorem 6 are satisfied by the operator F. The proof will be given in several claims.

Claim 1: Clearly F is continuous.

Claim 2: F maps bounded sets into bounded sets in $C([0, 1], \mathbb{R})$.

Actually, it is enough to show that for any $\theta > 0$, there exists $l > 0$ such that for each $u \in D_\theta = \{u \in C([0, 1], \mathbb{R}) : \|u\|_\infty \leq \theta\}$, we have $\|Fu\|_\infty \leq l$.

Let $u \in D_\theta$, for each $t \in [0, 1]$, we have

$$\begin{aligned}
|Fu(t)| \leq& \left|u_0\left(\frac{t^\rho}{\rho}\right)^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda\left(\frac{t^\rho}{\rho}\right)^{\alpha}\right)\right| \\
&+\left|u_1\left(\frac{t^\rho}{\rho}\right)^{\alpha} E_{\alpha,\alpha+1}\left(-\lambda\left(\frac{t^\rho}{\rho}\right)^{\alpha}\right)\right| \\
&+\int_0^1 K(t,\tau)\left|f(\tau,u(\tau))\right| d\tau \\
\leq& \frac{|u_0|\, t^{\rho(\alpha-1)}}{\rho^{\alpha-1}\Gamma(\alpha)} + \frac{|u_1|\, t^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)} + \frac{M(\rho^\alpha + \rho^\beta)}{\rho^{\alpha+\beta}\Gamma(\alpha+1)\Gamma(\beta+1)} \\
\leq& \frac{|u_0|}{\rho^{\alpha-1}\Gamma(\alpha)} + \frac{|u_1|}{\rho^{\alpha}\Gamma(\alpha+1)} + M\Lambda_0.
\end{aligned}$$

Thus

$$\|Fu\|_\infty \leq \frac{|u_0|}{\rho^{\alpha-1}\Gamma(\alpha)} + \frac{|u_1|}{\rho^{\alpha}\Gamma(\alpha+1)} + M\Lambda_0 := l < \infty. \tag{26}$$

Claim 3: It is Clear that F maps bounded sets into equicontinuous sets of $C([0, 1], \mathbb{R})$. From Claim 1 –Claim 3, we conclude that $F : C([0, 1], \mathbb{R}) \to C([0, 1], \mathbb{R})$ is continuous and complectly continuous.

Claim 4: A priori bounds.

Let $u \in \partial D_\theta$, such that $u = \mu F u$, for some $0 < \mu < 1$. From (26), we obtain

$$\begin{aligned}
\|u\|_\infty &= \mu\|Fu\|_\infty \leq \|Fu\|_\infty, \\
&\leq \frac{|u_0|}{\rho^{\alpha-1}\Gamma(\alpha)} + \frac{|u_1|}{\rho^{\alpha}\Gamma(\alpha+1)} + M\Lambda_0,
\end{aligned}$$

and thus

$$\theta \leq \frac{|u_0|}{\rho^{\alpha-1}\Gamma(\alpha)} + \frac{|u_1|}{\rho^{\alpha}\Gamma(\alpha+1)} + M\Lambda_0,$$

hence,

$$\frac{1}{\theta}\left[\frac{|u_0|}{\rho^{\alpha-1}\Gamma(\alpha)} + \frac{|u_1|}{\rho^{\alpha}\Gamma(\alpha+1)} + M\Lambda_0\right] \geq 1,$$

which contradicts (25). Consequently, by the nonlinear alternative of Leray-Schauder Fixed Point Theorem 6, the problem (1)–(3) has at least one solution on $[0, 1]$. ∎

Finally, we will prove the existence and uniqueness result of the solution for the problem (1)–(3), which is based on Banach's Fixed Point Theorem.

4.4 Uniqueness of Solution

In this subsection, we show the last existence result which is based on the Banach's contraction principle.

Theorem 10 *Assume that* (H_1)–(H_2) *hold for* $\alpha, \beta \in (0,1)$, ρ, $\lambda > 0$. *Then the problem (1)–(3) has a unique solution on* $[0,1]$, *provided that*

$$0 < L\Lambda_0 < 1. \tag{27}$$

Proof. Consider the operator $F : C([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$ given by (23), we shall show that F is a contraction mapping. Let $u, v \in C([0,1],\mathbb{R})$, for any $t \in [0,1]$, according to (H_2) and by equation (16), we obtain

$$\begin{aligned} |Fu(t) - Fv(t)| &= \left| \int_0^1 K(t,\tau)\ f(\tau, u(\tau))d\tau - \int_0^1 K(t,\tau)\ f(\tau, v(\tau))d\tau \right| \\ &\leq \int_0^1 K(t,\tau) \left| f(\tau, u(\tau) - \ f(\tau, v(\tau)) \right| d\tau \\ &\leq L \int_0^1 K(t,\tau) \left| u(\tau) - \ v(\tau) \right| d\tau. \end{aligned}$$

Then

$$\|Fu - Fv\|_\infty \leq L\Lambda_0 \|u - v\|_\infty .$$

By the condition (27), F is a contraction mapping, using the principle of Banach Fixed Point Theorem 4, we deduce that there exists a unique solution of the problem (1)–(3) on $[0,1]$. ∎

5 Ulam-Hyers Stability

In this section, we discuss the Ulam-Hyers stability of problem (1)–(3).

Let $\widetilde{\varepsilon} > 0$ and $f : [0,1] \times \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function.

For the mixed fractional BVP (1)-(3), we emphasize on the following inequality

$$\left| {}^{CK}D_{1^-}^{\beta,\rho} \left({}^{K}D_{0^+}^{\alpha,\rho} + \lambda \right) w(t) - f(t, w(t)) \right| \leq \widetilde{\varepsilon}, \quad t \in [0,1]. \tag{28}$$

In a similar way as in [17, 18, 19], we introduce the following definition and remark.

Definition 7 *The mixed fractional boundary value problem (1)–(3) is Ulam-Hyers stable if there exists a constant* $\delta_0 > 0$ *such that for each* $\widetilde{\varepsilon} > 0$ *and for each solution* $w \in C([0,1],\mathbb{R})$ *of inequality (28) there exists a solution* $u \in C([0,1],\mathbb{R})$ *of (1)–(3) with*

$$|w(t) - u(t)| \leq \delta_0 \widetilde{\varepsilon}, \quad t \in [0,1].$$

Remark 1 *A function* $w \in C([0,1],\mathbb{R})$ *is a solution of inequality (28) if and only if there exists a function* $\varphi \in C([0,1],\mathbb{R})$ *such that*

(i) $|\varphi(t)| \leq \widetilde{\varepsilon}$, $t \in [0,1]$,

(ii) ${}^{CK}D_{1^-}^{\beta,\rho} \left({}^{K}D_{0^+}^{\alpha,\rho} + \lambda \right) w(t) = f(t, w(t)) + \varphi(t)$, $t \in [0,1]$.

Theorem 11 *Suppose that* (H_1) *and* (H_2) *hold, then the mixed fractional BVP (1)–(3) is Ulam-Hyers stable if* $L\Lambda_0 < 1$.

Proof. Let $0<\alpha,\beta<1$ and let $w\in C([0,1],\mathbb{R})$ be a solution of inequality (28) with $\left({}^{K}I_{0^{+}}^{1-\alpha,\rho}w\right)(0)=u_0$, $\left({}^{K}D_{0^{+}}^{\alpha,\rho}+\lambda\right)w(1)=u_1$. Then by Remark 1, we have

$$ {}^{CK}D_{1^{-}}^{\beta,\rho}\left({}^{K}D_{0^{+}}^{\alpha,\rho}+\lambda\right)w(t)=f(t,w(t))+\varphi(t),\ t\in[0,1]. $$

By adopting the same arguments as in the proof of Lemma 3, we can write

$$ \begin{aligned} w(t) &= u_0\left(\frac{t^{\rho}}{\rho}\right)^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda\left(\frac{t^{\rho}}{\rho}\right)^{\alpha}\right)+u_1\left(\frac{t^{\rho}}{\rho}\right)^{\alpha}E_{\alpha,\alpha+1}\left(-\lambda\left(\frac{t^{\rho}}{\rho}\right)^{\alpha}\right)\\ &\quad+\int_0^1 K(t,\tau)\,f(\tau,w(\tau))d\tau+\int_0^1 K(t,\tau)\ \varphi(\tau)d\tau. \end{aligned} $$

From this equation and by (16), it follows that

$$ \begin{aligned} &\Bigg|w(t)-u_0\left(\frac{t^{\rho}}{\rho}\right)^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda\left(\frac{t^{\rho}}{\rho}\right)^{\alpha}\right)\\ &\quad-u_1\left(\frac{t^{\rho}}{\rho}\right)^{\alpha}E_{\alpha,\alpha+1}\left(-\lambda\left(\frac{t^{\rho}}{\rho}\right)^{\alpha}\right)-\int_0^1 K(t,\tau)\,f(\tau,w(\tau))d\tau\Bigg|\\ &= \left|\int_0^1 K(t,\tau)\ \varphi(\tau)d\tau\right|\\ &\le \int_0^1 K(t,\tau)\,|\ \varphi(\tau)|\,d\tau\\ &\le \Lambda_0\widetilde{\varepsilon}. \end{aligned} \tag{29} $$

Now, let $u\in C([0,1],\mathbb{R})$ be a unique solution of (1)–(3). Then for each $t\in[0,1]$, we have

$$ \begin{aligned} |w(t)-u(t)| &= \Bigg|w(t)-u_0\left(\frac{t^{\rho}}{\rho}\right)^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda\left(\frac{t^{\rho}}{\rho}\right)^{\alpha}\right)\\ &\quad-u_1\left(\frac{t^{\rho}}{\rho}\right)^{\alpha}E_{\alpha,\alpha+1}\left(-\lambda\left(\frac{t^{\rho}}{\rho}\right)^{\alpha}\right)-\int_0^1 K(t,\tau)\ f(\tau,u(\tau))d\tau\Bigg|\\ &\le \Bigg|w(t)-u_0\left(\frac{t^{\rho}}{\rho}\right)^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda\left(\frac{t^{\rho}}{\rho}\right)^{\alpha}\right)\\ &\quad-u_1\left(\frac{t^{\rho}}{\rho}\right)^{\alpha}E_{\alpha,\alpha+1}\left(-\lambda\left(\frac{t^{\rho}}{\rho}\right)^{\alpha}\right)-\int_0^1 K(t,\tau)\ f(\tau,w(\tau))d\tau\\ &\quad+\int_0^1 K(t,\tau)\ f(\tau,w(\tau))d\tau-\int_0^1 K(t,\tau)\ f(\tau,u(\tau))\ d\tau\Bigg|\\ &\le \Bigg|w(t)-u_0\left(\frac{t^{\rho}}{\rho}\right)^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda\left(\frac{t^{\rho}}{\rho}\right)^{\alpha}\right)\\ &\quad-u_1\left(\frac{t^{\rho}}{\rho}\right)^{\alpha}E_{\alpha,\alpha+1}\left(-\lambda\left(\frac{t^{\rho}}{\rho}\right)^{\alpha}\right)-\int_0^1 K(t,\tau)\ f(\tau,w(\tau))d\tau\Bigg|\\ &\quad+\int_0^1 K(t,\tau)\ \ |f(\tau,w(\tau))-f(\tau,u(\tau))|\ \ d\tau. \end{aligned} $$

From (H_2), (29) and (16), we obtain

$$ \|w-u\|\le\Lambda_0\widetilde{\varepsilon}+L\Lambda_0\|w-u\|, $$

which implies

$$ \|w-u\|\le\delta_0\widetilde{\varepsilon}, $$

where $\delta_0=\frac{\Lambda_0}{1-L\Lambda_0}>0$. Then the mixed fractional boundary value problem (1)–(3) is Ulam-Hyers stable. ■

6 Examples

In this section, we present two examples to explain the applicability of our main result.

Example 1 *Consider the following boundary value problem with two different fractional derivatives*

$$\begin{cases} {}^{CK}D_{1^-}^{\frac{1}{5},1}\left({}^{K}D_{0^+}^{\frac{1}{3},1}+\frac{5}{4}\right)u(t)=\frac{\cos(t)}{1+u^2}, \quad t\in J=[0,1], \\ \left({}^{K}I_{0^+}^{\frac{2}{3},1}u\right)(0)=\frac{1}{2}, \ \left({}^{K}D_{0^+}^{\frac{1}{3},1}+\frac{5}{4}\right)u(1)=1. \end{cases} \tag{30}$$

Here, $f(t,u(t))=\frac{\cos(t)}{1+u^2}$, $\alpha=\frac{1}{3}$, $\beta=\frac{1}{5}$, $\alpha+\beta=\frac{8}{15}<1$, $\lambda=\frac{5}{4}$, $\rho=1$, $u_0=\frac{1}{2}$ *and* $u_1=1$. *The function* f *is continuous for any* $t\in[0,1]$ *and we have* $|f(t,u)|\le M=1$, $\forall(t,u)\in[0,1]\times\mathbb{R}$. *Hence the condition* (H_3) *holds. It follows from Theorem 7 and Theorem 8, that the problem (30) has at least one solution.*

Example 2 *Consider the following mixed fractional boundary value problem*

$$\begin{cases} {}^{CK}D_{1^-}^{\frac{1}{4},1}\left({}^{K}D_{0^+}^{\frac{1}{2},1}+2\right)u(t)=t+\frac{u(t)}{5e^t(1+u(t))}, \quad t\in J=[0,1], \\ \left({}^{K}I_{0^+}^{\frac{1}{2},1}u\right)(0)=u_0, \ \left({}^{K}D_{0^+}^{\frac{1}{2},1}+2\right)u(1)=u_1. \end{cases} \tag{31}$$

Here, $\alpha=\frac{1}{2}$, $\beta=\frac{1}{4}$, $\alpha+\beta=\frac{6}{8}<1$, $\lambda=2$, $\rho=1$ *and* $f(t,u(t))=t+\frac{u(t)}{5e^t(1+u(t))}$. *The function* f *is continuous for any* $t\in[0,1]$, *then we have* $|f(t,u)-f(t,v)|\le\frac{1}{5}|u-v|$, $L=\frac{1}{5}$, $\Lambda_0=\frac{2}{\Gamma(\frac{1}{4}+1)\Gamma(\frac{1}{2}+1)}\approx 2.49$ *and* $L\Lambda_0\approx 0.50<1$. *By Theorem 10, the problem (31) has a unique solution* u *on* $[0,1]$.

Now, let $w\in C([0,1],\mathbb{R})$ *be a solution of the inequality*

$$\left|{}^{CK}D_{1^-}^{\frac{1}{4},1}\left({}^{K}D_{0^+}^{\frac{1}{2},1}+2\right)w(t)-\left(t+\frac{w(t)}{5e^t\left(1+w(t)\right)}\right)\right|\le\widetilde{\varepsilon},\ \widetilde{\varepsilon}>o,\ t\in J=[0,1].$$

Then, from Theorem 11, the mixed fractional BVP (31) is Ulam-Hyers stable with

$$\delta_0=\frac{\Lambda_0}{1-L\Lambda_0}=\frac{2.49}{0.5}=4.98>0.$$

7 Conclusion

In this paper, we have discussed the existence and uniqueness of solution for the nonlinear mixed-type FDEs with boundary conditions by applying some fixed point theorems (Banach's contraction principle, Schauder's Fixed Point Theorem, the nonlinear alternative of Leray-Schauder type and Schaefer's Fixed Point Theorem). We have also studied the Ulam-Hyers stability of our problem. The differential operators we have considered are the Katugampola and Caputo-Katugampola, so the Reimann-Lioville, Hadamard, Caputo and Caputo-Hadamard operators can be considered as special cases from our generalized problem. This study serves as a new way for the researchers to discuss interesting problems in fractional differential and integral calculus.

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