Applied Mathematics E-Notes, 22(2022), 345-353 © ISSN 1607-2510
Available free at mirror sites of http://www.math.nthu.edu.tw/∼amen/

Interpolants For Degree-n Approximation Over Convex Polygons*

Pournima Laxmanrao Powar[†], Rishabh Tiwari[‡], Vishnu Narayan Mishra[§]
Lakshmi Narayan Mishra[¶]

Received 11 June 2020

Abstract

Wachspress' coordinates are a generalization of barycentric coordinates to convex polygons with four or more vertices. These coordinates, which were introduced in the context of the finite element method correspond to rational functions in which the polynomial denominator is popularly known as the adjoint. Corresponding to each element of the discretization, a class of Wachspress coordinates is defined such that each class has a unique adjoint depending on the geometry of the element. In this paper, an interesting property of Wachspress coordinates is investigated, viz."two wedge functions which are linear on the common adjacent side of the polygon, attain the same value at the mid point of that side". Applying this property, a more general form of Dasgupta's recursive relation is derived. Moreover, this method is extended to the polynomial approximation of higher degree over a polygon of any order. A Mathematica program is also developed in view of the above assertions which enhance the application of the devised tool.

1 Introduction

Partial differential equations play a key role in solving problems of mathematical physics in particular, related to shapes or physical properties like conductivity, elasticity, stokes flow, etc. Since, in many cases, analytical or exact solutions to these partial differential equations can be impossible or expensive to obtain, the process of simulation is adopted to find approximate solutions [1, 17]. The idea of a test function whose functional properties are already known were initiated by Galerkin [5, 9] for the simulation process. Later with the aid of computers this idea was extensively applied on larger domains such as computation of stress/strain for multi-storied buildings, for ships to study the sustainable conditions in all types of atmospheric variations, bridges, aerospace engineering [6, 14, 2], etc. Nowadays, the finite element method (FEM) is the most widely used simulation technique, and is applied in almost all branches of science and technology, whether it is computer graphics [7], computer vision [20], image processing [18], computational mechanics [7] or prediction of some unknown information [10, 16], to name a few applications.

In order to simplify the process of approximation, several tools were developed by the researchers including the most popular tool of barycentric coordinates initially proposed by Mӧbius [13]. This tool has enormous applications in the filed of computer aided designs [7], but it is restricted to an n-simplex in n-dimensional Euclidean space. It is known that barycentric coordinates possess the following properties:

- Partition of unity.
- Linear reproduction property.
- Non negativity.

*Mathematics Subject Classifications: 41A20, 65M60, 65D05, 65D17.

[†]Department of Mathematics and Computer Science R.D. University, Jabalpur, 482001, (M.P.), India

[‡]Department of Mathematics and Computer Science R.D. University, Jabalpur, 482001, (M.P.), India

[§]Department of Mathematics, Indira Gandhi National Tribal University, Lalpur, Amarkantak, Anuppur, Madhya Pradesh 484 887, India

[¶](Corresponding Author) Department of Mathematics, School of Advanced Sciences, Vellore Institute of Technology, Vellore 632 014, Tamil Nadu, India

- Lagrange property.
- Regularity (i.e. infinitely many times differentiable).

A generalization of barycentric coordinates to a convex [19] polygon with $n > 3$ vertices was introduced by Wachspress and the construction was formally announced in the Proceeding of the Dundee conference [22]. In FEM, the domain under consideration is divided into non-overlapping smaller sub-domains of $\mathbb{R}^2$ and corresponding to each sub-domain rational basis functions are defined [3], which in turn found to be a powerful tool to approximate a given function or data. These basis functions appeared to be quite easy to execute on a computer in comparison to other approximants. It may be noted that inter-element continuity holds by virtue of construction in the case of FEM [21] whereas it has been imposed in the case of bi-variate splines [15].

It is worth mentioning that, the basic problem which was encountered by researchers while dealing with bi-variate splines was to resolve the issue of dimension and hence this tool of approximation could not attract the users, although it was extensively studied theoretically by several mathematicians [15, 12].

Wachspress' coordinates are defined in such a way that they satisfy all the conditions of barycentric coordinates initiated by Möbius. Since the denominator (adjoint) of the rational wedge function depends on the geometry of the element, its computation for some elements is not straight forward. Thus, the computational time to determine rational wedge functions becomes quite complex. Meyer et. al. [11] have initiated the formulation of wedge functions for arbitrary convex polygons for linear approximation.

In [2], Dasgupta has developed a simple recursive technique, independent of the geometry of the element, by which the adjoint could be computed merely in few steps if the Cartesian coordinates of the vertices of the element are known. This approach devised for convex polygons was identified by Dasgupta and Wachspress [4] as best from a general-purpose algorithm point of view, and has been designated as GADJ (Gautam's Adjoint) algorithm.

The present paper explores the fact that "**for linear approximation, the wedge functions corresponding to $(i-1)^{th}$ node and i^{th} node, attain the same value at the midpoint of the common adjacent side joining the vertices $(i-1)$ and i of the polygonal element**". This observation leads to derive a more general recurrence relation, which is a generalization of Dasgupta's recurrence relation [2]. The formula derived by Dasgupta is applicable only on those polygons whose sides do not pass through the origin whereas the proposed recurrence relation in this paper is free of all restrictions and easy to execute on any polygonal discretization of the domain.

Moreover, a recursive method for the construction of degree-n approximation over an order-m polygon, is also proposed. Using this method, wedge functions for any degree of approximation can be easily computed if Cartesian coordinates of nodes and side nodes of the convex polygonal element are known. Our technique is applicable to achieve higher degree approximation as well, which was not considered in [11].

A Mathematica program to compute the adjoint of the wedge function for a convex polygon of order n (where n is a parameter defined by the user, $n > 3$) has been developed. Consequently, the linear approximation for the given function or data can be computed with the aid of this program. To demonstrate the performance of our method, two illustrative examples are presented in this paper.

2 Setup and Formulation

Let $\Omega \subseteq \mathbb{R}^2$ be the domain discretized using polygons of order m, P_m be an arbitrary element of the domain Ω with vertices $i = 1, \cdots, m$ and edges s_i joining vertices $i-1$ and i, $\{i_j\}_{j=1}^{n-1}$ be the side nodes on side s_i (such that $i_j \neq i-1$ or i for any j and $i_j \neq i_k$ for $j \neq k$) and interior nodes $\{c_k\}_{k=1}^{\frac{(n-1)(n-2)}{2}}$ (The interior nodes have been chosen in such a way that there exists a unique curve of degree (n-3) passing through $\frac{n(n-3)}{2}$ number of interior nodes). Throughout this work, the i^{th} node (vertex), i_j^{th} node (side node) and c_k^{th} node (interior node) are considered to have Cartesian coordinates (x_i, y_i), (x_{i_j}, y_{i_j}) and (x_{c_k}, y_{c_k}), respectively.

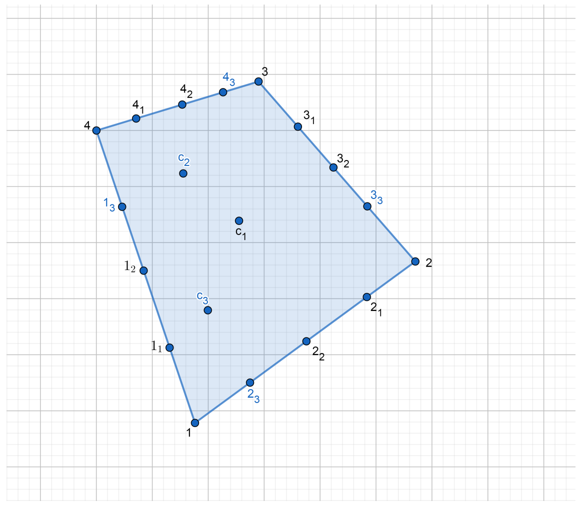

Figure 1: Illustrating nodes, side nodes and interior nodes for $m = 4$ and $n = 3$.

In view of the generalized barycentric coordinates properties [8], for degree n approximation over an m-gon, corresponding to each node (vertex, side node and interior node) i, the rational wedge function [2],

$$W_i^n = \frac{N_i^n}{D} \tag{1}$$

(N_i^n is a polynomial of degree $m + n - 3$ and D is a polynomial of degree $m - 3$) is defined in such a way that the class of wedge functions satisfies the following properties:

1. There is one node at each vertex of the polygon, $(n-1)$ side nodes on every side and $\frac{(n-1)(n-2)}{2}$ interior nodes (cf. [21]) (see Figure 1).

   For each node of the polygon there corresponds a wedge

2. Wedge $W_i^n(x, y)$ corresponding to node i is normalized to unity at i.

3. Wedge $W_i^n(x, y)$ is of degree n on sides adjacent to i.

4. Wedge $W_i^n(x, y)$ vanishes on all nodes $j(\neq i)$ lying on the boundary of P_m.

5. The wedges corresponding to P_m form a basis for degree n approximation over it. For the polygon P_m, there must be at least mn boundary and $\frac{(n-1)(n-2)}{2}$ interior nodes. For these to suffice, we must have:

$$\sum_{k=1}^{M} x_k^i y_k^j W_k^n = x^i y^j, \qquad 0 \leq i + j \leq n,$$

   where $M = mn + \frac{(n-1)(n-2)}{2}$.

6. All the wedge functions are infinitely differentiable over the associated polygon.

3 Gautam's Adjoint (GADJ)

In the traditional method of computing adjoint for the wedge functions corresponding to a convex polygon (see [21]), one has to compute the Exterior Intersection Points (EIPs), which are the points where the extended sides of the considered polygon meet. The curve of desired degree, passing through these EIPs is the required adjoint for the wedge functions. When any of these exterior intersection points lie on the absolute line, the computation of adjoint becomes complicated.

Many of the researchers proposed alternative methods [4] for computing the adjoint function, and ultimately Dasgupta's technique (GADJ) [2] was recognized to be the best [4], which is described briefly as follows:

3.1 Dasgupta's Algorithm

Consider $P_m \in \Omega$. Let l_i^d be the linear form of the edge s_i,

$$l_i^d = 1 + a_i x + b_i y. \tag{2}$$

Here l_i^d stands for the linear forms considered by Dasgupta. The Wachspress coordinates (wedge functions) for linear approximation over P_m, are defined as:

$$W_i^1 = \frac{N_i^1}{D}, \qquad i = 1, ..., m \tag{3}$$

where

$$N_i^1 = k_i \prod_{j \neq i, j \neq i+1}^{m} l_j^d.$$

The wedge functions, defined as in (3), form partition of unity (cf. Property (5) in Section 2 with $i = j = 0$), Dasgupta [2] utilized this property of the wedge functions and identified that the denominator of wedge functions is the sum of the numerators, i.e. $D = \sum_{i=1}^{m} N_i$. Further, he imposed property (3) on these wedge functions, which insisted that the wedge functions W_i^1 and W_{i+1}^1 corresponding to the vertices i and $(i+1)$, respectively, must be linear on the side s_{i+1}, which in turn yields the following recurrence relation:

$$k_{i+1} = k_i \frac{a_{i+2}(x_i - x_{i+1}) + b_{i+2}(y_i - y_{i+1})}{a_i(x_{i+1} - x_i) + b_i(y_{i+1} - y_i)}, \qquad i \in \mathbb{Z}_m. \tag{4}$$

Remark 1 *The restriction on the linear form (cf. (2)) forces the discretization that no side of any polygon should pass through the origin which limits its implementation.*

4 Governing Lemma

Let $n = 1$ and l_i be the linear form of the edges s_i $(i = 1, \cdots, m)$ [21] defined as:

$$\begin{aligned} l_i \;\; = \;\; & (y_i - y_{i-1})x - (x_i - x_{i-1})y - x_{i-1} \\ & (y_i - y_{i-1}) + y_{i-1}(x_i - x_{i-1})(i \in \mathbb{Z}_m). \end{aligned} \tag{5}$$

Then numerators of the wedge basis functions for degree one approximation are defined as [21]:

$$N_i^1 = k_i^1 \prod_{j \neq i, j \neq i+1}^{m} l_j,$$

and the denominator D (same for all the wedges corresponding to an element) is the sum of all the numerators [2].

In view of the above notations and the properties of wedge functions, the basic result of this paper is established in the form of the following lemma:

Lemma 1 *Let s_i be an edge of the m-gon P_m, joining the vertices $i-1$ and i, m_i be the mid point of s_i. Let $W_{i-1}^1(x, y)$ and $W_i^1(x, y)$ be the wedges for linear approximation corresponding to the nodes $i-1$ and i respectively then*

$$W_{i-1}^1(x, y)|_{m_i} = W_i^1(x, y)|_{m_i}. \tag{6}$$

Proof. By property (3), $W_{i-1}^1(x,y)$, $W_i^1(x,y)$ are linear functions on the edge s_i. By properties (4) and (2), $W_{i-1}^1(x,y)$ attains value 0 at the node i and 1 at the node $i-1$. Similarly, $W_i^1(x,y)$ attains value 1 at the node i and 0 at the node $i-1$. Thus, it forms a rectangle R with vertices $i-1$, i, i', $(i-1)'$ (see Figure 2) whose diagonals mutually bisect at m_i'. Let Pm_i' be the perpendicular dropped from m_i' on s_i. By trivial geometry it can be seen that $P = m_i$, i.e. the mid point of s_i and hence W_{i-1} and W_i attain the same value at m_i (cf. Figure 2). Thus,

$$W_{i-1}^1(x,y)|_{m_i} = W_i^1(x,y)|_{m_i}.$$

■

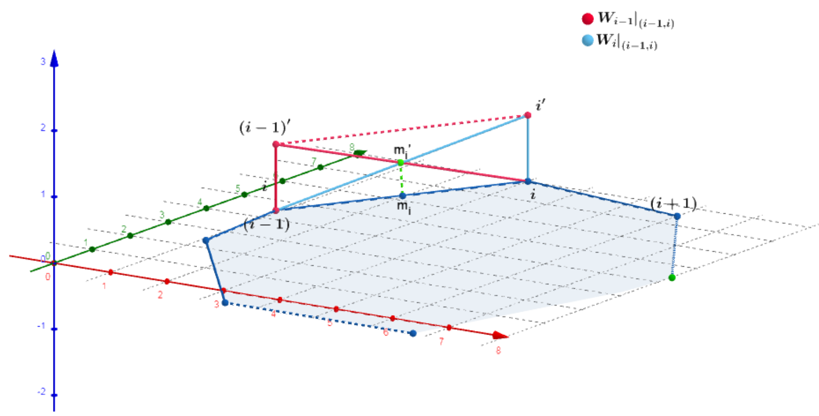

Figure 2: Rectangle R.

5 Explicit Recurrence Relation

Dasgupta's recurrence relation is an eminent result in the field of rational finite elements, as its application makes the computation of adjoint quite smooth in comparison to other techniques. It is interesting to note that merely substituting the Cartesian coordinates of vertices of the polygon in recurrence relation, the value of adjoint can be obtained readily, but with the restriction that no side of the polygon should be a segment of a line through the origin.

In this section, a generalized recurrence relation is derived, which holds for an arbitrary polygonal discretization $\Omega \in \mathbb{R}^2$. In order to establish this recurrence relation, we assume that

$$D = \sum_{i=1}^{m} N_i^1, \tag{7}$$

where N_i^1 is the numerator of the wedge function W_i^1 for the linear approximation over P_m.

In view of Lemma 1, it is clear that

$$W_{i-1}^1|_{m_i} = W_i^1|_{m_i} \tag{8}$$

where

$$W_{i-1}^1 = k_{i-1}^1 l_{i+1} l_{i+2} \cdots l_{i+m-2} \quad \text{and} \quad W_i^1 = k_i^1 l_{i+2} l_{i+3} \cdots l_{i+m-1}. \tag{9}$$

Substituting the values of W_{i-1} and W_i from (9) in (8), the following recurrence relation is obtained:

$$k_i^1 = k_{i-1}^1 \frac{l_{i+1}}{l_{i-1}}|_{m_i} \qquad i = 1, ..., m \tag{10}$$

and $l_{i+m-1} = l_{i-1}$ (as $i+m-1 = i-1$ under modulo m).

As described in the technique of Dasgupta [2], on substituting the values k_i^1 $(i = 1, \cdots, m)$ in (7), the adjoint (denominator of wedge function) may be computed.

We have used Mathematica to develop a general program that computes a linear approximation over a convex polygon of order m. The code can be downloaded on Program and executed using the following user defined parameters:

- Order of polygon.
- Cartesian coordinates of vertices of the polygon.
- Function to be approximated.

In support of our assertion, we now consider an example of an element of the pentagonal discretization, whose sides pass through the origin.

Example 1 *Let the domain* $\Omega \subseteq \mathbb{R}^2$ *be discretized by pentagons* $(m = 5)$, $P_5 = (1, 2, 3, 4, 5) \in \Omega$ *such that the Cartesian coordinates of the vertices are (cf. Figure 3):* $1 = (0, 0)$, $2 = (1, 0)$, $3 = (7/5, 3/5)$, $4 = (3/5, 7/5)$, $5 = (0, 1)$.

First, it can be seen that the recurrence relation derived by Dasgupta [2] is not applicable in this case. Indeed, the linear form l_2 *of* s_2 *(see Figure 3) is given by* y*, whereas the linear form considered by Dasgupta is* $1 + a_2x + b_2y$*. In order to apply the recurrence relation obtained in this paper, the following linear forms*

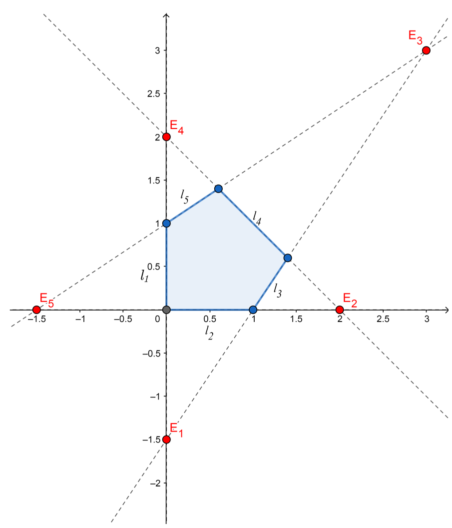

Figure 3: Polygon of order 5.

are needed: $l_1 \cong -x$, $l_2 \cong -y$, $l_3 \cong -\frac{3}{5} + \frac{3}{5}x - \frac{2}{5}y$, $l_4 \cong -\frac{8}{5} + \frac{4}{5}x + \frac{4}{5}y$ *and* $l_5 \cong -\frac{3}{5} - \frac{2}{5}x + \frac{3}{5}y$*. By virtue of (10) the normalizing constants* k_i^1*s are:* $k_1^1 = 1$, $k_2^1 = \frac{3}{5}$, $k_3^1 = \frac{4}{5}$, $k_4^1 = \frac{4}{5}$ *and* $k_5^1 = \frac{3}{5}$*. Hence by (7),*

$$D = \frac{24}{125}x^2 + \frac{24}{125}y^2 - \frac{32}{125}xy - \frac{12}{125}x - \frac{12}{125}y - \frac{72}{125}.$$

It can be easily verified that D is the unique curve passing through EIPs.

6 Extension of Recurrence Relation for Higher Degree Approximation

In this section, a method to compute a higher degree approximation has been formulated. The technique defined previously for computation of wedge functions for higher degree approximation were lengthy and dependent on the Wachspress approach of EIP, for the computation of the unknowns $k_i's$, while the method proposed here is based on the Dasgupta's approach.

Referring the notations described in Section 2, the linear forms l_{i_j} $(j = 1, 2, ..., n-1)$ joining nodes $(i+1)_{n-j}$ and $(i)_j$ are now considered. Also, let γ_{c_k} $(k = 1, \cdots, \frac{(n-1)(n-2)}{2})$ be the unique curve of degree $(n-3)$ passing through the $\frac{n(n-3)}{2}$ interior nodes $\{c_\nu\}_{\nu \neq k}$. Then, the wedge functions for degree n approximation over P_m, are defined using properties mentioned in Section 2 as follows:
For $i = 1, 2, ..., m$

$$W_i^n = k_i^n \frac{\Big(\prod_{j\neq i,i+1}^{m} l_j\Big)\Big(\prod_{j=1}^{n-1} l_{i_j}\Big)}{D}. \tag{11}$$

For i_j, $(j = 1, \cdots, (n-1))$ and $i = 1, 2, ..., m$

$$W_{i_j}^n = k_{i_j}^n \frac{\Big(\prod_{\nu\neq i}^{m} l_\nu\Big)\Big(\prod_{\nu\neq j}^{n-1} l_{i_\nu}\Big)}{D}. \tag{12}$$

For c_k, $(k = 1, \cdots, \frac{(n-1)(n-2)}{2})$

$$W_{c_k}^n = k_{c_k}^n \frac{\Big(\prod_{\nu=1}^{m} l_\nu\Big)\gamma_{c_k}}{D}. \tag{13}$$

It may be easily verified that even if we increase the degree of approximation, the degree of the denominator remains unchanged for fixed order of polygon. Hence, it has been considered that for degree n approximation over a convex polygon of order m, the degree adjoint is $m-3$ and for fixed polygon P_m, D is invariant. Thus, the adjoint D will be the same, while k_r's $(r = i, i_j, c_k)$ will change. With the help of equation (8) and the property (2) that $W_r^n(x, y) = 1$ on the r^{th} node, these new k_r^n's can be computed:

$$k_i^n = \frac{k_i^1}{\Big(\prod_{j=1}^{n-1} l_{i_j}\Big)}\Bigg|_i, \tag{14}$$

$$k_{i_j}^n = \frac{k_i^1 l_{i-1} + k_{i-1}^1 l_{i+1}}{l_{i-1}l_{i+1}\Big(\prod_{\nu\neq j}^{n-1} l_{i_\nu}\Big)}\Bigg|_{i_j} \quad \forall i \text{ and } j, \tag{15}$$

$$k_{c_k}^n = \frac{1}{\gamma_{c_k}} \sum_{\nu=1}^{m} \frac{k_\nu^1}{l_\nu l_{\nu+1}}\Bigg|_{c_k}. \tag{16}$$

The application of (14)–(16) has been elaborated through the following example.

Example 2 *In order to obtain degree 3 approximation the pentagon has been referred as described in Example 1 with additional side nodes* $\{i_j\}_{j=1}^{n-1}$ $(i = 1, ..., m)$*, and interior node* c_1 *with Cartesian coordinates* $1_1 = (0, \frac{1}{3})$, $1_2 = (0, \frac{2}{3})$, $2_1 = (\frac{2}{3}, 0)$, $2_2 = (\frac{1}{3}, 0)$, $3_1 = (\frac{19}{15}, \frac{2}{5})$, $3_2 = (\frac{17}{15}, \frac{1}{5})$, $4_1 = (\frac{13}{15}, \frac{17}{15})$, $4_2 = (\frac{17}{15}, \frac{13}{15})$, $5_1 = (\frac{1}{5}, \frac{17}{15})$, $5_2 = (\frac{2}{5}, \frac{19}{15})$, *and* $c_1 = (\frac{1}{2}, \frac{1}{2})$ *(see Figure 4). By using Equations (11)–(13), the wedge functions corresponding to the nodes 1,* 1_1 *and* c_1 *have been defined as:*

$$W_1^3 = \frac{k_1^3 l_3 l_4 l_5 l_{1_1} l_{1_2}}{D}, \quad W_{1_1}^3 = \frac{k_{1_1}^3 l_2 l_3 l_4 l_5 l_{1_2}}{D} \quad \textit{and} \quad W_{c_1} = \frac{k_{c_1}^3 l_1 l_2 l_3 l_4 l_5}{D},$$

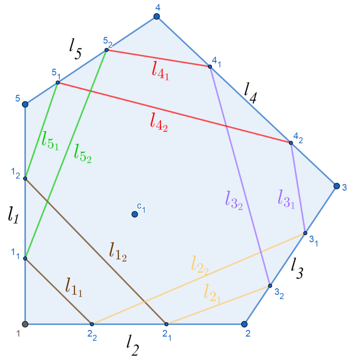

Figure 4: Degree 3 approximation over polygon of order 5.

where $l_{1_1} \cong -3x-3y+1$, $l_{1_2} \cong -3x-3y+2$, $l_{2_1} \cong -\frac{7}{15}y+\frac{1}{5}x-\frac{2}{15}$, $l_{2_2} \cong \frac{2}{5}x-\frac{14}{15}y-\frac{2}{15}$, $l_{3_1} \cong \frac{7}{15}x+\frac{2}{15}y-\frac{29}{45}$, $l_{3_2} \cong \frac{4}{15}y+\frac{14}{15}x-\frac{10}{9}$, $l_{4_1} \cong \frac{7}{15}y+\frac{2}{15}x-\frac{29}{15}$, $l_{4_2} \cong \frac{4}{15}x+\frac{14}{15}y-\frac{10}{9}$, $l_{5_1} \cong -\frac{7}{15}x+\frac{1}{5}y-\frac{2}{15}$, $l_{5_2} \cong \frac{2}{5}y-\frac{14}{15}x-\frac{2}{15}$, *and* l_i*'s have been already calculated in Example 1. Similarly, wedges corresponding to other nodes can be defined.*

Considering the value of k_i^1*'s and adjoint computed in Example 1, the other normalizing constants* k_r^3*'s have been obtained by using the recurrence relations (14)–(16):*
$k_1^3 = \frac{1}{2}$, $k_2^3 = \frac{135}{4}$, $k_3^3 = \frac{405}{16}$, $k_4^3 = \frac{405}{16}$, $k_5^3 = \frac{135}{4}$, $k_{1_1}^3 = \frac{-9}{2}$, $k_{1_2}^3 = \frac{9}{2}$, $k_{2_1}^3 = \frac{-135}{4}$, $k_{2_2}^3 = \frac{135}{2}$, $k_{3_1}^3 = \frac{-405}{16}$, $k_{3_2}^3 = \frac{405}{8}$, $k_{4_1}^3 = \frac{-405}{16}$, $k_{4_2}^3 = \frac{405}{8}$, $k_{5_1}^3 = \frac{-135}{4}$, $k_{5_2}^3 = \frac{135}{2}$, *and* $k_{c_1}^3 = \frac{199}{50}$.

With the aid of these k_r^3*'s the wedge functions for degree three approximation over* P_5 *can be readily defined.*

7 Conclusion

GADJ is one of the ingenious way to compute the adjoint function of Wachspress' coordinates for the polygon of order $m (m > 3)$. A more general form of GADJ is established in this paper, which enhances GADJ algorithm. The significant property of Wachspress coordinates explored in this paper, makes the derivation of recurrence relation more elegant, which is a generalization of GADJ. Moreover, with the help of this new approach, the process of computation of normalizing constants for the wedge functions of degree n approximation over the m-gon has also been demonstrated.

This method of getting wedge functions for higher degree approximation can be adopted in the case of higher dimensional elements (polyhedra), to approximate a function or data dependent on multiple variables with more accuracy.

References

[1] S. Bruvoll and M. S. Floater, Transfinite mean value interpolation in general dimension, J. Comput. Appl. Math., 233(2010), 1631–1639.

[2] G. Dasgupta, Interpolants within convex polygons: Wachspress' shape functions, J. Aerosp. Eng., 16(2003), 1–8.

[3] G. Dasgupta and E. A. Malsch, Boundary element color interpolation for instrumentation, imaging and internet graphics industries, Eng. Anal. Bound. Elem., 26(2002), 379–389.

[4] G. Dasgupta and E. Wachspress, The adjoint for an algebraic finite element, Comput. Math. with Appl., 55(2008), 1988–1997.

[5] C. Fletcher, Comparison of Finite-Difference, Finite-Element, and Spectral Methods, Computational Galerkin Methods, Springer, Berlin, 1984.

[6] B. Haro, S. SilvaNeto, L. Vaz, A. Troyman, U. Monteiro and R. Gutiérrez, Use of added mass and shear flow in thin-walled sections of ships for modal analysis using finite element method, Mar. Syst. Ocean Technol., 13(2018), 43–54

[7] K. Hormann and N. Sukumar, Generalized Barycentric Coordinates in Computer Graphics and Computational Mechanics, CRC press, NewYork, 2017.

[8] J. Kalman, Continuity and convexity of projections and barycentric coordinates in convex polyhedra, Pac. J. Appl. Math., 11(1961), 1017–1022.

[9] I. Lasiecka, Galerkin approximations of abstract parabolic boundary value problems with rough boundary data-L_p theory, Math. Comp., 47(1986), 55–75

[10] S. G. Meshram, P. L. Powar, V. P. Singh and C. Meshram, Application of cubic spline in soil erosion modeling from narmada watersheds, India, Arab. J. Geosci., 11(2018), 1–20.

[11] M. Meyer, A. Barr, H. Lee and M. Desbrun, Generalized barycentric coordinates on irregular polygons, J. Graph. Tools, 7(2002), 13–22.

[12] C. A. Micchelli, Interpolation of scattered data: distance matrices and conditionally positive definite functions, Constr. Approx., 2(1986), 11–22.

[13] A. F. Möbius, Der barycentrische Calcul, ein H´ulfsmittel zur analytischen Behandlung der Geometrie (etc.), Barth, Leipzig, 1827.

[14] A. Navarro-Manso, J. J. del Coz Díaz, M. Alonso-Martínez, D. Castro-Fresno and F. P. A. Rabanal, Patch loading in slender and high depth steel panels: Fem-doe analyses and bridge launching application, Eng. Struct., 83(2015), 74–85.

[15] G. Nürnberger and F. Zeilfelder, Developments in bivariate spline interpolation, J. Comput. Appl. Math., 121(2000), 125–152.

[16] P. L. Powar, R. Tiwari and V. N. Mishra, Interpolants for linear approximation over convex polyhedron, Inf. Sci. Lett., 9(2020), 181–188.

[17] P. L. Powar, R. Tiwari and V. N. Mishra, Wedge functions for degree-n approximation over pentagonal discretization, Ganita, 70(2020), 217–226.

[18] T. Preußer and M. Rumpf, An adaptive finite element method for large scale image processing, J. Vis. Commun. Image Represent., 11(2000), 183–195.

[19] N. Sharma, L. Mishra, V. Mishra and H. Almusawa, End point approximation of standard three-step multivalued iteration algorithm for nonexpansive mappings, Appl. Math. Inf. Sci, 15(2021), 73–81.

[20] P. Upchurch, J. Gardner, G. Pleiss, R. Pless, N. Snavely, K. Bala and K. Weinberger, Deep feature interpolation for image content changes, In Proceedings of the IEEE conference on computer vision and pattern recognition, pages (2017), 7064–7073.

[21] E. Wachspress, Rational Bases and Generalized Barycentrics, Springer, Cham, 2016.

[22] E. L. Wachspress, A rational basis for function approximation, In Conference on Applications of Numerical Analysis, Springer, (1971), 223–252.