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# Spectral Properties Of A Discrete Sturm-Liouville Equation*

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**Abstract**

We deal with a boundary value problem (BVP) for a discrete Sturm-Liouville equation with boundary conditions depending on spectral parameter. Here, we give a polynomial-type Jost solution and determine the Jost function of this BVP. Using the analytical properties and asymptotic behavior of Jost function on unit disc, we examine the Green function, resolvent operator, point spectrum and the set of spectral singularities of given BVP. At the end, we compare our results with other similar works.

## 1 Introduction

It is well-known that there are several studies about the spectral analysis of Sturm-Liouville, Schrödinger and Klein-Gordon differential equations in literature [1]–[7]. Since there is a close parallelism between the spectral theory of these differential equations and the spectral theory of discrete cases of these equations, the interest of boundary value problems for difference equations is growing rapidly. Some problems of the spectral theory of difference equations and operators have been treated by various authors in connection with the classical moment problem [8], [9] and the references therein. Such equations play an important role in modeling of certain problems from economics, engineering, control theory and have many applications in natural sciences. Also, spectral analysis of nonselfadjoint discrete Sturm-Liouville, Schrödinger and Dirac equations with spectral singularities have been studied in [10]–[17]. In these studies, authors have been interested in exponential type Jost solution given on $\overline{\mathbb{C}}_+ := \{z \in \mathbb{C}, \operatorname{Im} z \geq 0\}$ to investigate the spectral analysis of the problem except [17]. In [17], the authors investigate the spectral properties of matrix difference equation and the boundary condition does not consist the spectral parameter in [17]. Differently from these works, we examine the spectral properties of a BVP for discrete Sturm-Liouville equation including polynomial-type Jost solution with boundary conditions depending on spectral parameter. In [15], the boundary condition also consists spectral parameter, but it gives the properties of exponential-type Jost solution on $\overline{\mathbb{C}}_+$ to get the spectral properties of problem. Although there are more theorems for analytic functions given on complex plane than given on unit disc, it is our aim getting similar results which are as strong as in [15]. Note that the papers [20], [21] are about the spectral analysis of Dirac systems with polynomial-type Jost solutions, but they do not consist spectral parameter in boundary conditions.

Let us consider a nonselfadjoint BVP consisting of a discrete Sturm-Liouville equation

$$a_{n-1}y_{n-1} + b_n y_n + a_n y_{n+1} = \lambda y_n,\ n \in \mathbb{N} = \{1, 2, ...\} \tag{1}$$

and the boundary conditions

$$(\gamma_0 + \gamma_1\lambda)\, y_1 + (\beta_0 + \beta_1\lambda)\, y_0 = 0,\ \gamma_0\beta_1 - \gamma_1\beta_0 \neq 0,\ \beta_1 \neq 0, \tag{2}$$

where $\{a_n\}_{n\in\mathbb{N}\cup\{0\}}$ and $\{b_n\}_{n\in\mathbb{N}}$ are complex sequences, $\gamma_i, \beta_i \in \mathbb{C}$ for $i = 0, 1$ and $a_n \neq 0$ for all $n \in \mathbb{N}\cup\{0\}$. The remaining part of this study is organized as follows: In Section 2, we are interested in polynomial-type Jost solution and Jost function of BVP (1)–(2) for $\lambda = z + z^{-1}$. Also, we find analytical properties and asymptotic behavior of the Jost function, Green function and resolvent of BVP (1)–(2) in this section. In Section 3, we investigate the properties of eigenvalues and spectral singularities of this BVP. We present a

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condition on the coefficients in section 4 that guarantees that BVP (1)–(2) has a finite number of eigenvalues and spectral singularities with finite multiplicities. Last section is a conclusion part, we summarize our results and compare them with similar ones in this section.

## 2 Jost Solution and Jost Function

Suppose that the complex sequences $\{a_n\}$ and $\{b_n\}$ satisfy

$$\sum_{n=1}^{\infty} n\left(|1-a_n|+|b_n|\right)<\infty. \tag{3}$$

**Theorem 1** *Assume (3). For $n \in \mathbb{N}\cup\{0\}$ and $m\in\mathbb{N}$, define*

$$\alpha_n := \prod_{k=n}^{\infty}\left[a_k\right]^{-1},$$

$$A_{n1} := -\sum_{k=n+1}^{\infty} b_k,$$

$$A_{n2} := \sum_{k=n+1}^{\infty}\left\{1-a_k^2+b_k\sum_{p=k+1}^{\infty} b_p\right\},$$

$$A_{n,m+2} := A_{n+1},m+\sum_{k=n+1}^{\infty}\left\{\left(1-a_k^2\right)A_{k+1,m}-b_kA_{k,m+1}\right\}.$$

*Then*

$$e_n(z) := \alpha_n z^n\left(1+\sum_{m=1}^{\infty}A_{nm}z^m\right),\ z\in D_0 := \{z:|z|=1\} \tag{4}$$

*solves (1) for $\lambda = z+z^{-1}$.*

**Proof.** By the definitions of $\alpha_n$ and $A_{nm}$, we get

$$a_{n-1}\alpha_{n-1}=\alpha_n,$$
$$A_{n1}-A_{n-1,1}=b_n,$$
$$A_{n2}-A_{n-1,2}=a_n^2-1+b_nA_{n1},$$
$$A_{n,m+2}-A_{n-1,m+2}=a_n^2A_{n+1,m}+b_nA_{n,m+1}-A_{nm}.$$

Thus

$$\begin{aligned} a_{n-1}e_{n-1}+b_ne_n+a_ne_{n+1} &= a_{n-1}\alpha_{n-1}z^{n-1}\left(1+\sum_{m=1}^{\infty}A_{n-1,m}z^m\right)\\ &\quad +b_n\alpha_n z^n\left(1+\sum_{m=1}^{\infty}A_{nm}z^m\right)\\ &\quad +a_n\alpha_{n+1}z^{n+1}\left(1+\sum_{m=1}^{\infty}A_{n+1,m}z^m\right)\\ &= \left(z+z^{-1}\right)\alpha_n z^n\left(1+\sum_{m=1}^{\infty}A_{nm}z^m\right)\\ &= \left(z+z^{-1}\right)e_n. \end{aligned}$$

Hence $e$ solves (1) for $\lambda = z + z^{-1}$. ■

Since (4) is the bounded solution of (1), satisfying the condition

$$\lim_{n\to\infty} e_n(z)\, z^{-n} = 1,\ z \in D_0,$$

analogously to the Sturm-Liouville equation, the solution (4) is called Jost solution of (1). Moreover, by using the definitions of $A_{nm}$ and the condition (3), it can be easily shown that

$$|A_{nm}| \le C \sum_{k=n+\left[\left|\frac{m}{2}\right|\right]} \left(|1-a_k| + |b_k|\right), \tag{5}$$

where $\left[\left|\frac{m}{2}\right|\right]$ is the integer part of $\frac{m}{2}$ and $C$ is a positive constant. It follows from (4) and (5) that $e_n(z)$ has analytic continuation from the set $D_0$ to the set $D_1 := \{z : |z| < 1\} \setminus \{0\}$. Using (4) and the boundary condition (2), we define the function $f$ by

$$f(z) = [\gamma_0 + \gamma_1 \left(z + z^{-1}\right)]e_1(z) + \left[\beta_0 + \beta_1\left(z + z^{-1}\right)\right] e_0(z). \tag{6}$$

Since $e_n(z)$ is analytic in $D_1$ and continuos in $D_2 := \{z : |z| \le 1\} \setminus \{0\}$, the function $f$ also is analytic in $D_1$ and continuous in $D_2$. Similarly to the classical Sturm-Liouville equation the function $f$ is called the Jost function of BVP (1)–(2) [5].

Now, we will define the other solution of (1) to get the resolvent operator of BVP (1)–(2). Let $\psi(\lambda) = \{\psi_n(\lambda)\}_{n\in\mathbb{N}\cup\{0\}}$ be a solution of (1) satisfying the initial conditions

$$\psi_0(\lambda) = -(\gamma_0 + \gamma_1\lambda) \quad \text{and} \quad \psi_1(\lambda) = (\beta_0 + \beta_1\lambda).$$

If we define

$$\varphi(z) = \psi\left(z + z^{-1}\right) = \left\{\psi_n\left(z + z^{-1}\right)\right\}_{n\in\mathbb{N}\cup\{0\}},$$

then we write

$$W[\varphi(z), e(z)] = a_n\left[\psi_{n+1}\left(z + z^{-1}\right) e_n(z) - e_{n+1}(z)\, \psi_n\left(z + z^{-1}\right)\right].$$

Since the Wronskian is independent from $n$, we find

$$\begin{aligned} W[\varphi(z), e(z)] &= a_0\left[\psi_1\left(z + z^{-1}\right) e_0(z) - e_1(z)\, \psi_0\left(z + z^{-1}\right)\right] \\ &= a_0 f(z). \end{aligned}$$

For all $z \in D_2$ and $f(z) \ne 0$, $\varphi(z)$ and $e(z)$ are fundamental solutions of BVP (1)–(2). To investigate the Green function and resolvent, we will use the following Sturm-Liouville form of the difference equation (1)

$$\nabla(a_n \Delta y_n) + h_n y_n = \lambda y_n,\ n \in \mathbb{N} \tag{7}$$

where $\Delta$ is the forward difference operator defined by $\Delta y_n := y_{n+1} - y_n$, $\nabla$ is the backward difference operator given by $\nabla y_n := y_n - y_{n-1}$ and $h_n = a_{n-1} + a_n + b_n$.

**Theorem 2** *Under the condition (3), the resolvent of the BVP (1)–(2) is given by*

$$(R_\lambda g)_n := \sum_{m=1}^{\infty} G_{nm}(z)\, g_m,\ \ g = \{g_m\}_{m\in\mathbb{N}} \in \ell_2(\mathbb{N}),\ \ n \in \mathbb{N} \cup \{0\} \tag{8}$$

*where $\lambda = z + z^{-1}$ and $G_{nm}(z)$ is the Green function of the BVP (1)–(2) defined by*

$$G_{nm}(z) = \begin{cases} -\dfrac{\varphi_m(z)(z)\, e_n(z)}{a_0 f(z)} & ; \quad m \le n \\ -\dfrac{\varphi_n(z)\, e_m(z)}{a_0 f(z)} & ; \quad m > n \end{cases} \tag{9}$$

*for $z \in D_2$ and $f(z) \ne 0$.*

**Proof.** Since $\varphi(z)$ and $e(z)$ are the fundamental solutions of BVP (1)–(2), it is necessary to solve the equation

$$\nabla(a_n \Delta y_n) + h_n y_n - \lambda y_n = g_n \tag{10}$$

to get the Green function. If $y(z) := \{y_n(z)\}_{n\in\mathbb{N}}$ is the general solution of (10), we can write

$$y_n(z) = c_n e_n(z) + d_n \varphi_n(z), \tag{11}$$

where $c_n \neq 0$ and $d_n \neq 0$. Using the method of variation of parameters, we get the coefficients

$$c_n = -\sum_{m=1}^{\infty} \frac{g_m(z)\varphi_m(z)}{a_{m-1} f(z)} \tag{12}$$

and

$$d_n = -\sum_{m=n+1}^{\infty} \frac{g_m(z) e_m(z)}{a_{m-1} f(z)}, \tag{13}$$

respectively. It follows from (11), (12) and (13) that the Green function of the BVP (1)–(2) is (9) and it is easy to write the resolvent operator by using (9). ■

# 3 The Sets of Eigenvalues and Spectral Singularities

By using (8), (9) and the definitions of eigenvalues and spectral singularities [5], [6], we write the sets of eigenvalues and spectral singularities of BVP (1)–(2) by

$$\sigma_d := \left\{\lambda = z + z^{-1} : z \in D_1,\ f(z) = 0\right\} \tag{14}$$

$$\sigma_{ss} := \left\{\lambda = z + z^{-1} : z \in D_0,\ f(z) = 0\right\} \setminus \{0\}, \tag{15}$$

respectively. It is obvious from (14) and (15) that to obtain the properties of eigenvalues and spectral singularities of the BVP (1)–(2), we investigate the properties of zeros of $f$ in $D_2$. Because of this reason, using the equations (4) and (6), we write

$$\begin{aligned} f(z) &= \alpha_0\beta_1 z^{-1} + \gamma_1\alpha_1 + \alpha_0\beta_0 + (\gamma_0\alpha_1 + \alpha_0\beta_1) z + \gamma_1\alpha_1 z^2 \\ &+ \sum_{m=1}^{\infty} \alpha_0\beta_1 A_{0m} z^{m-1} + \sum_{m=1}^{\infty} (\gamma_1\alpha_1 A_{1m} + \alpha_0\beta_0 A_{0m}) z^m \\ &+ \sum_{m=1}^{\infty} (\gamma_0\alpha_1 A_{1m} + \alpha_0\beta_1 A_{0m}) z^{m+1} + \sum_{m=1}^{\infty} \gamma_1\alpha_1 A_{1m} z^{m+2}. \end{aligned}$$

If we define

$$F(z) := f(z) z, \tag{16}$$

then we get

$$\begin{aligned} F(z) &= \alpha_0\beta_1 + (\gamma_1\alpha_1 + \alpha_0\beta_0) z + (\gamma_0\alpha_1 + \alpha_0\beta_1) z^2 + \gamma_1\alpha_1 z^3 \\ &+ \sum_{m=1}^{\infty} \alpha_0\beta_1 A_{0m} z^m + \sum_{m=1}^{\infty} (\gamma_1\alpha_1 A_{1m} + \alpha_0\beta_0 A_{0m}) z^{m+1} \\ &+ \sum_{m=1}^{\infty} (\gamma_0\alpha_1 A_{1m} + \alpha_0\beta_1 A_{0m}) z^{m+2} + \sum_{m=1}^{\infty} \gamma_1\alpha_1 A_{1m} z^{m+3}. \end{aligned} \tag{17}$$

It is clear that the function $F$ is also analytic in $D_1$ and continuous in $D_2$. It follows from (14)–(17) that

$$\sigma_d := \left\{\lambda = z + z^{-1} : z \in D_1,\ F(z) = 0\right\} \tag{18}$$

$$\sigma_{ss} := \left\{\lambda = z + z^{-1} : z \in D_0,\ F(z) = 0\right\} \setminus \{0\}. \tag{19}$$

**Definition 1** *The multiplicity of a zero of $F$ in $D_2$ is called the multiplicity of the corresponding eigenvalue or spectral singularity of the BVP (1)–(2).*

It is clear from (18) and (19) that the quantitative properties of eigenvalues and spectral singularities of the BVP (1)–(2) depend on the quantitative properties of the zeros of the function $F$ in $D_2$. Let us define

$$\begin{aligned} M_1 \quad &: \quad = \{z \in D_1 : F(z) = 0\} \\ M_2 \quad &: \quad = \{z \in D_0 : F(z) = 0\}. \end{aligned} \tag{20}$$

We also denote the set of all limit points of $M_1$ by $M_3$ and the set of all zeros of $F$ with infinite multiplicity in $D_2$ by $M_4$. It follows from these definitions, (18) and (19), we get

$$\begin{aligned} \sigma_d \quad &: \quad = \left\{\lambda = z + z^{-1} : z \in M_1\right\} \\ \sigma_{ss} \quad &: \quad = \left\{\lambda = z + z^{-1} : z \in M_2\right\}. \end{aligned} \tag{21}$$

**Theorem 3** *Under the condition (3), we get the following results.*

*(i) $M_1$ is bounded and countable.*

*(ii) $M_1 \cap M_3 = \emptyset$, $M_1 \cap M_4 = \emptyset$.*

*(iii) The set $M_2$ is compact and $\mu(M_2) = 0$, where $\mu$ denotes the Lebesgue measure in real axis.*

*(iv) $M_3 \subset M_2$, $M_4 \subset M_2$, $\mu(M_3) = \mu(M_4) = 0$.*

*(v) $M_3 \subset M_4$.*

**Proof. i)** Using the condition (3) and the definition of the function $F$ given by (17), we get

$$F(z) = \alpha_0\beta_1 + o(1),\ z \in D_2,\ z \to 0.$$

Since $\alpha_0\beta_1 \neq 0$, the last equation shows that the set $M_1$ is bounded. Using the analyticity of $F$ in $D_1$, we find that $M_1$ has at most countable number of elements. Also it is clear from the uniqueness theorems of analytic functions that the zeros of $F$ in $D_1$ are separated. There is no an accumulation point of these zeros. It gives us the function $F$ has at most a countable number of zeros in $D_1$.

**ii)** We know from the uniqueness theorem of analytic functions that the elements of the set $M_4$ is on $D_0$. On the other hand the set $M_1$ is on $D_1$, so we find easily that $M_1 \cap M_4 = \emptyset$. We also get $M_1 \cap M_3 = \emptyset$ by using similar method.

**iii)** We know that the set $M_2$ is bounded from the asymptotic equation of the function $F$. If we show that the set $M_2$ is closed then by using Borel-Lebesgue Theorem, we get that it is a compact set. Let us take $z \in \overline{M_2}$. So, there exists a set $\{z_n\} \subset M_2$ such that $z_n \to z$. Since $\forall\ n \in \mathbb{N}$, $z_n \in M_2$, we write $F(z_n) = 0$. By using the continuity of $F$ on $D_2$, we get $F(z_n) \to F(z)$ and $F(z) = 0$. Also from the uniqueness theorems of analytic functions, we find that $z \in D_0$. It gives us $z \in M_2$ and $M_2$ is a closed set. This result shows that $M_2$ is a compact set from the Borel-Lebesgue Theorem. By using Privalov Theorem [18], we get $\mu(M_2) = 0$.

**iv)** Let us take $z_0 \in M_3$. Using the uniqueness theorems of analytic functions, we find $M_3 \subset D_0$. Since $z_0 \in M_3$, there exists a set $\{z_n\} \subset M_1$ such that $\lim_n z_n = z_0$ and $F(z_n) = 0$. We know that the function $F$ is continuous on $D_2$, so we write

$$F(z_0) = F\left(\lim_{n\to\infty} z_n\right) = \lim_{n\to\infty} F(z_n) = 0.$$

It gives us $z_0 \in M_2$, i.e., $M_3 \subset M_2$. Similarly, we get $M_4 \subset M_2$. Since $\mu(M_2) = 0$, we also find

$$\mu(M_3) = \mu(M_4) = 0.$$

**v)** Assume that $z_0 \in M_3$ but $z_0 \notin M_4$. In that occasion, we can say that $z_0$ is a zero of $F$ with finite multiplicity and there exists $m \in \mathbb{N}$ such that

$$F(z) = (z - z_0)^m g(z), \ g(z_0) \neq 0.$$

Last equation gives us

$$\frac{F(z)}{(z-z_0)^m} = g(z).$$

Since the function $F$ is analytic in $D_1$ and continuous in $D_2$, the function $g$ is also analytic in $D_1$ and continuous in $D_2$. Also, since $z_0 \in M_3$, there exists a set $\{z_n\} \subset M_1$ such that $\lim_n z_n = z_0$. For all $m \in \mathbb{N}$ the function $g$ satisfies

$$g(z_n) = \frac{F(z_n)}{(z_n - z_0)^m} = 0$$

and

$$\lim_{n\to\infty} g(z_n) = g(z_0) = 0.$$

But it gives a contradiction. Because $g(z_0) \neq 0$. It completes the proof of **(v)**. ∎

Now, we can give the following theorem as a result of Theorem 3 and (21).

**Theorem 4** *Assume (3). Then the set $\sigma_d$ is bounded has at most countable number of elements and its limit points can lie only in $[-2,2]$. Also $\sigma_{ss} \subset [-2,2]$ and the Lebesgue measure of the set $\sigma_{ss}$ in the real axis is zero.*

In the following, we will assume that for some $\varepsilon > 0$, the complex sequences $\{a_p\}$ and $\{b_p\}$ satisfy

$$\sum_{p=1}^{\infty} e^{\varepsilon p^{\delta}} \left(|1 - a(p)| + |b(p)|\right) < \infty, \quad \frac{1}{2} \leq \delta \leq 1. \tag{22}$$

It is obvious that (22) implies (3). Hence Theorems 1–4 remain true when the assumption (3) is replaced by (22). We need the condition (22) for the next theorems.

## 4 Main Properties of Eigenvalues and Spectral Singularities

Let us suppose condition (22) for $\delta = 1$,

$$\sum_{p=1}^{\infty} e^{\varepsilon p} \left(|1 - a(p)| + |b(p)|\right) < \infty, \quad \varepsilon > 0. \tag{23}$$

**Theorem 5** *Under the condition (23), the BVP (1)–(2) has a finite number of eigenvalues and spectral singularities and each of them is of finite multiplicity.*

**Proof.** It follows from (5) and (23) that

$$|A_{nm}| \leq C_1 \exp\left(-\frac{\varepsilon}{4}(n+m)\right), \; n=0,1. \; m \in \mathbb{N}, \tag{24}$$

where $C_1$ is a constant. By using (24), we find that the function $F$ has an analytic continuation to the set

$$\widetilde{D} := \left\{ z : |z| < e^{\frac{\varepsilon}{4}}, \; \epsilon > 0 \right\}.$$

In that case, the limit points of the zeros of $F$ in $D_2$ cannot lie in $D_0$. From Theorem 3, we obtain that the bounded sets $M_1$ and $M_2$ have no limit points. It gives us the sets $M_1$ and $M_2$ have a finite number of elements. Using the analyticity of $F$ in $\widetilde{D}$, we find that all zeros of $F$ in $D_2$ have finite multiplicity. It shows that the BVP (1)-(2) has a finite number of eigenvalues and spectral singularities and each of them is of finite multiplicity. ■

Now, let us take the condition (22) for $\delta \neq 1$

$$\sum_{p=1}^{\infty} e^{\varepsilon p^{\delta}} \left(|1-a(p)| + |b(p)|\right) < \infty, \quad \varepsilon > 0, \quad \frac{1}{2} \leq \delta < 1. \tag{25}$$

It is clear that this new condition is weaker than (23). We will investigate the results of Theorem 5 under the condition (25). But we need to use different method to get the same result. Because the function $F$ does not have analytic continuation to the set $\widetilde{D}$ under the condition (25). In the following, we will give some auxiliary lemmas to give the next theorem.

**Lemma 1** *Under the condition (25), the inequality*

$$\left|F^{(k)}(z)\right| \leq \eta_k, \; k \in \mathbb{N} \cup \{0\} \tag{26}$$

*holds, where*

$$\eta_k \leq B b^k k! k^{k\left(\frac{1}{\delta}-1\right)}, \tag{27}$$

*and $B$, $b$ are positive constants depending on $\varepsilon$ and $\delta$.*

**Proof.** From (6) and (25), we get

$$|A_{nm}| \leq C e^{\left(-\frac{\varepsilon}{4} m^{\delta}\right)}, \; n=0,1 \;\; m \in \mathbb{N}. \tag{28}$$

Using (17) and (28), we can write

$$\left|F^{(k)}(z)\right| \leq C_1 \sum_{m=k}^{\infty} m^k e^{\left(-\frac{\varepsilon}{4} m^{\delta}\right)}, \; z \in D_2, \; k \in \mathbb{N}.$$

We can also write for $\sum_{m=k}^{\infty} m^k e^{\left(-\frac{\varepsilon}{4} m^{\delta}\right)}$

$$\sum_{m=k}^{\infty} m^k e^{\left(-\frac{\varepsilon}{4} m^{\delta}\right)} \leq \sum_{m=1}^{\infty} m^k e^{\left(-\frac{\varepsilon}{4} m^{\delta}\right)} = \int_0^n t^k e^{-\frac{\varepsilon}{4}} t^{\delta} dt \leq \int_0^{\infty} t^k e^{-\frac{\varepsilon}{4}} t^{\delta} dt.$$

Using the Gamma function and the inequalities $\left(1+\frac{1}{k}\right)^{\frac{k}{\delta}} < e^{\frac{1}{\delta}}$, $(k+1)^{\frac{1}{\delta}-1} < e^{\frac{k}{\delta}}$ and $k^k < k! e^k$, we have

$$\sum_{m=k}^{\infty} e^{\left(-\frac{\varepsilon}{4} m^{\delta}\right)} \leq B b^k k! k^{k\left(\frac{1}{\delta}-1\right)}, \; k \in \mathbb{N},$$

where $B$ and $b$ are positive constants depending on $\varepsilon$ and $\delta$. ■

**Lemma 2** **([19])** *Assume that the function $F \neq 0$ is analytic in unit disc and all derivatives of $F$ belongs to unit disc. Then*

$$\int_0^{2\pi} \ln T(s)\, d\mu(G_s) > -\infty,$$

*where*

$$T(s) := \inf_{k \geq 0} \frac{\eta_k(F)\, s^k}{k!}, \qquad \eta_k(g) := \max_{z \in D_2} \left| F^{(k)}(z) \right|$$

*and $\mu(G_s)$ is the Lebesgues measure of the $s$-neighborhood of $G$, $G \subset [0, 2\pi]$.*

**Lemma 3** *If (25) holds, then $M_4 = \emptyset$.*

**Proof.** By using Lemma 2, we can write

$$\int_0^{2\pi} \ln T(s)\, d\mu(G_s) > -\infty, \tag{29}$$

where

$$T(s) := \inf_{k \geq 0} \frac{\eta_k(F)\, s^k}{k!}$$

and $\mu(M_{4,s})$ is the Lebesgue measure of $s$-neighborhood of $M_4$, and $\eta_k$ defined by (27). Substituting (27) in $T(s)$, we find

$$T(s) = B \exp\left\{ -\frac{1-\delta}{\delta} e^{-1} (bs)^{-\frac{\delta}{1-\delta}} \right\}. \tag{30}$$

It follows from (29) and (30) that

$$\int_0^{2\pi} s^{-\frac{\delta}{1-\delta}}\, d\mu(M_{4,s}) < \infty.$$

Since $\frac{\delta}{1-\delta} \geq 1$, from last inequality, we get that $\mu(M_{4,s}) = 0$ for arbitrary $s$. It gives $M_4 = \emptyset$ and this completes the proof of Lemma. ■

**Theorem 6** *Under the condition (25), the BVP (1)–(2) has a finite number eigenvalues and spectral singularities, and each of them is of finite multiplicity.*

**Proof.** It is enough to show that the function $F$ has a finite number of zeros with finite multiplicities in $D_2$ to prove the theorem. Using Theorem 3 and Lemma 2, we obtain that $M_3 = \emptyset$. It follows from that the function $F$ has only finite number of zeros in $D_2$. Since $M_4 = \emptyset$ from Lemma 3, these zeros are of the finite multiplicity. ■

# 5 Conclusion

In this paper, we investigate the properties of eigenvalues and spectral singularities of a nonselfadjoint BVP which consists a discrete Sturm-Liouville equation and boundary condition depending on spectral parameter. Before presenting main results, we find polynomial-type Jost solution, Green function and resolvent operator of this BVP. Discussing the properties of discrete spectrum and Jost function of this BVP, we prove that it has a finite number of eigenvalues and spectral singularities with finite multiplicities. In literature, there are some similar studies [15], [17]. In [15], the authors get our main results for a discrete Sturm-Liouville equation by using similar method. But they use exponential-type solutions defined on upper complex half plane. In this paper, we give polynomial-type Jost solution on unit disc and give our main results on unit disc. Both of these works use the uniqueness theorems for analytic functions to reach the main results. Although there are more uniqueness theorems for analytic functions given on complex plane than given on unit disc, we find at least as good results as previous works. Because of this, our study is more valuable. In [17], authors also find the spectral properties of a difference equation by using polynomial-type Jost solution. But it is matrix case and does not consist a spectral parameter in boundary condition. Moreover, in this paper, we use a weak condition than the condition given in [17] to get the same result. Because of these reasons this work is new and will be a resource for mathematicians working in this area. As you know, there are a lot of applications of the spectral analysis of difference equations in several disciplines. Since our BVP consists a spectral parameter in boundary condition, it also satisfies extra contribution on the applications in different disciplines. In the future, it is possible to extend this study on quantum difference equations.

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