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On A New Application Of Quasi-Power Increasing Sequences*

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Abstract

In this paper, we proved a basic theorem under weaker conditions dealing with the absolute Cesàro summability factors of infinite series by using a quasi-β-power increasing sequence instead of an almost increasing sequence. This new theorem also includes several known and new results on the absolute Cesàro summability factors of infinite series.

1 Introduction

A positive sequence (b_n) is said to be almost increasing sequence if there exists a positive increasing sequence (c_n) and two positive constants M and N such that $Mc_n \leq b_n \leq Nc_n$ (see [2]). A positive sequence (X_n) is said to be quasi-β-power increasing sequence if there exists a constant $K = K(\beta, X) \geq 1$ such that $Kn^\beta X_n \geq m^\beta X_m$ for all $n \geq m \geq 1$. Every almost increasing sequence is a quasi-β-power increasing sequence for any non-negative β, but the converse need not be true as can be seen by taking $X_n = n^{-\beta}$ (see [9]). For any sequence (λ_n) we write that $\Delta^2\lambda_n = \Delta\lambda_n - \Delta\lambda_{n+1}$ and $\Delta\lambda_n = \lambda_n - \lambda_{n+1}$. Let $\sum a_n$ be a given infinite series. By t_n^α we denote the nth Cesàro mean of order α, with $\alpha > -1$, of the sequence (na_n), that is (see [6])

$$t_n^\alpha = \frac{1}{A_n^\alpha}\sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v, \quad ({t_n}^1 = t_n) \tag{1}$$

where

$$A_n^\alpha = \frac{(\alpha+1)(\alpha+2)....(\alpha+n)}{n!} \simeq \frac{n^\alpha}{\Gamma(\alpha+1)}, \quad A_{-n}^\alpha = 0 \quad \texttt{for} \quad n > 0.$$

Let (φ_n) be a sequence of complex numbers. The series $\sum a_n$ is said to be summable $\varphi - |C, \alpha|_k$, $k \geq 1$, if (see [1])

$$\sum_{n=1}^{\infty} \frac{1}{n^k} \left|\varphi_n t_n^\alpha\right|^k < \infty.$$

In the special case when $\varphi_n = n^{1-\frac{1}{k}}$, $\varphi - |C, \alpha|_k$ summability is the same as $|C, \alpha|_k$ summability (see [7]). If we take $\varphi_n = n^{\delta+1-\frac{1}{k}}$, then we obtain $|C, \alpha; \delta|_k$ summability (see [8]).

2 Known Results

The following theorems are known dealing with an application of almost increasing sequences to factored infinite series.

Theorem 1 ([3]) *Let (w_n^α) be a sequence defined by (see [10])*

$$w_n^\alpha = \begin{cases} |t_n^\alpha|, & \alpha = 1 \\ \max_{1 \leq v \leq n} |t_v^\alpha|, & 0 < \alpha < 1. \end{cases} \tag{2}$$

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Let (σ_n) be a positive sequence and let (X_n) be an almost increasing sequence. If the conditions

$$\sum_{n=1}^{\infty} n \left|\Delta^2 \lambda_n\right| X_n < \infty, \tag{3}$$

$$|\lambda_n| X_n = O(1) \text{ as } n \to \infty, \tag{4}$$

$$\sigma_n = O(1) \text{ as } n \to \infty, \tag{5}$$

$$n\Delta\sigma_n = O(1) \text{ as } n \to \infty, \tag{6}$$

$$\sum_{v=1}^{n} \frac{(w_v^\alpha)^k}{v X_v^{k-1}} = O(X_n) \text{ as } n \to \infty \tag{7}$$

hold, then the series $\sum a_n \lambda_n \sigma_n$ is summable $|C, \alpha|_k$, where $0 < \alpha \leq 1$ and $\mathtt{k} \geq 1$.

Theorem 2 ([4]) *Let (φ_n) be a sequence of complex numbers and let (w_n^α) be a sequence defined as in (2). Let (σ_n) be a positive sequence and let (X_n) be an almost increasing sequence. Suppose also that there exists an $\epsilon > 0$ such that the sequence $(n^{\epsilon-k} |\varphi_n|^k)$ is non-increasing. If the conditions (3)–(6) and*

$$\sum_{v=1}^{n} \frac{(|\varphi_v| w_v^\alpha)^k}{v^k X_v{}^{k-1}} = O(X_n) \quad \text{as} \quad n \to \infty \tag{8}$$

hold, then the series $\sum a_n \lambda_n \sigma_n$ is summable $\varphi - |C, \alpha|_k$, where $0 < \alpha \leq 1$, $\epsilon + (\alpha - 1)\mathtt{k} > 0$, and $\mathtt{k} \geq 1$.

It should be noted that if we take $\epsilon = 1$ and $\varphi_n = n^{1-\frac{1}{k}}$, then we obtain Theorem 1. In fact, in this case the condition (8) reduces to condition (7).

3 Main Result

The aim of this paper is to prove Theorem 2 under weaker conditions by using a quasi-β-power increasing sequence instead of an almost increasing sequence. Now, we shall prove the following main theorem.

Theorem 3 *Let (φ_n) be a sequence of complex numbers and let (w_n^α) be a sequence defined as in (2). Let (σ_n) be a positive sequence and let (X_n) be a quasi-β-power increasing sequence. Suppose also that there exists an $\epsilon > 0$ such that the sequence $(n^{\epsilon-k} |\varphi_n|^k)$ is non-increasing. If the conditions (3)–(6) and (8) hold, then the series $\sum a_n \lambda_n \sigma_n$ is summable $\varphi - |C, \alpha|_k$, where $0 < \alpha \leq 1$, $\epsilon + (\alpha - 1)\mathtt{k} > 0$, and $\mathtt{k} \geq 1$.*

We require the following known lemmas for the proof of our new theorem.

Lemma 4 ([5]) *If $0 < \alpha \leq 1$ and $1 \leq v \leq n$, then*

$$\left|\sum_{p=0}^{v} A_{n-p}^{\alpha-1} a_p\right| \leq \max_{1 \leq m \leq v} \left|\sum_{p=0}^{m} A_{m-p}^{\alpha-1} a_p\right|.$$

Lemma 5 ([9]) *Under the conditions on (X_n) and (λ_n) as taken in the statement of the theorem, the following conditions hold*

$$n X_n |\Delta\lambda_n| = O(1) \quad \text{as } n \to \infty,$$

$$\sum_{n=1}^{\infty} X_n |\Delta\lambda_n| < \infty.$$

4 Proof of Theorem 3

Let (T_n^α) be the nth (C,α) mean, with $0<\alpha\leq 1$, of the sequence $(na_n\lambda_n\sigma_n)$. Then by (1), we have that

$$T_n^\alpha = \frac{1}{A_n^\alpha}\sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v \lambda_v \sigma_v.$$

Now, applying Abel's transformation first and then using Lemma 4, we obtain that

$$\begin{aligned} T_n^\alpha &= \frac{1}{A_n^\alpha}\sum_{v=1}^{n-1}\Delta(\lambda_v\sigma_v)\sum_{p=1}^{v} A_{n-p}^{\alpha-1} p a_p + \frac{\lambda_n\sigma_n}{A_n^\alpha}\sum_{p=1}^{n} A_{n-v}^{\alpha-1} v a_v \\ &= \frac{1}{A_n^\alpha}\sum_{v=1}^{n-1}(\lambda_v\Delta\sigma_v + \sigma_{v+1}\Delta\lambda_v)\sum_{p=1}^{v} A_{n-p}^{\alpha-1} p a_p + \frac{\lambda_n\sigma_n}{A_n^\alpha}\sum_{p=1}^{n} A_{n-v}^{\alpha-1} v a_v. \end{aligned}$$

Then

$$\begin{aligned} |T_n^\alpha| &\leq \frac{1}{A_n^\alpha}\sum_{v=1}^{n-1}|\lambda_v\Delta\sigma_v||\sum_{p=1}^{v} A_{n-p}^{\alpha-1} p\ a_p| + \frac{1}{A_n^\alpha}\sum_{v=1}^{n-1}|\sigma_{v+1}\Delta\lambda_v||\sum_{p=1}^{v} A_{n-p}^{\alpha-1} p\ a_p| \\ &\quad + \frac{|\lambda_n\sigma_n|}{A_n^\alpha}|\sum_{v=1}^{v} A_{n-v}^{\alpha-1} v\ a_v| \\ &\leq \frac{1}{A_n^\alpha}\sum_{v=1}^{n-1} A_v^\alpha w_v^\alpha |\lambda_v||\Delta\sigma_v| + \frac{1}{A_n^\alpha}\sum_{v=1}^{n-1} A_v^\alpha w_v^\alpha |\sigma_{v+1}||\Delta\lambda_v| + |\lambda_n||\sigma_n| w_n^\alpha \\ &= T_{n,1}^\alpha + T_{n,2}^\alpha + T_{n,3}^\alpha. \end{aligned}$$

To complete the proof, by Minkowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty}\frac{1}{n^k}|\ \varphi_n T_{n,r}^\alpha\ |^k < \infty, \quad \texttt{for } r=1,2,3.$$

Now, when $\texttt{k}>1$, applying Hölder's inequality with indices k and k', where $\frac{1}{k}+\frac{1}{k'}=1$, we get that

$$\begin{aligned} \sum_{n=2}^{m+1}\frac{1}{n^k}\left|\varphi_n T_{n,1}^\alpha\right|^k &\leq \sum_{n=2}^{m+1} n^{-k}(A_n^\alpha)^{-k}\left|\varphi_n\right|^k\left\{\sum_{v=1}^{n-1} A_v^\alpha w_v^\alpha|\Delta\sigma_v||\lambda_v|\right\}^k \\ &= O(1)\sum_{n=2}^{m+1}\frac{\left|\varphi_n\right|^k}{n^{k+\alpha k}}\sum_{v=1}^{n-1}(v^\alpha)^k(w_v^\alpha)^k|\Delta\sigma_v|^k|\lambda_v|^k\left\{\sum_{v=1}^{n-1}1\right\}^{k-1} \\ &= O(1)\sum_{n=2}^{m+1}\frac{\left|\varphi_n\right|^k}{n^{1+\alpha k}}\sum_{v=1}^{n-1} v^{\alpha k}(w_v^\alpha)^k|\lambda_v|^k\frac{1}{v^k} \\ &= O(1)\sum_{v=1}^{m} v^{\alpha k}(w_v^\alpha)^k v^{-k}|\lambda_v|^k\sum_{n=v+1}^{m+1}\frac{n^{\epsilon-k}\left|\varphi_n\right|^k}{n^{1+\epsilon+(\alpha-1)k}} \\ &= O(1)\sum_{v=1}^{m} v^{\alpha k}(w_v^\alpha)^k v^{-k}|\lambda_v|^k v^{\epsilon-k}\left|\varphi_v\right|^k\sum_{n=v+1}^{m+1}\frac{1}{n^{1+\epsilon+(\alpha-1)k}} \\ &= O(1)\sum_{v=1}^{m} v^{\alpha k}(w_v^\alpha)^k v^{-k}|\lambda_v|^k v^{\epsilon-k}\left|\varphi_v\right|^k\int_v^\infty\frac{dx}{x^{1+\epsilon+(\alpha-1)k}} \end{aligned}$$

$$
\begin{aligned}
&= O(1)\sum_{v=1}^{m}(w_v^{\alpha})^k|\lambda_v||\lambda_v|^{k-1}\frac{|\varphi_v|^k}{v^k}\\
&= O(1)\sum_{v=1}^{m}|\lambda_v|\frac{(w_v^{\alpha}|\varphi_v|)^k}{v^kX_v^{k-1}}\\
&= O(1)\sum_{v=1}^{m-1}\Delta|\lambda_v|\sum_{r=1}^{v}\frac{(w_r^{\alpha}|\varphi_r|)^k}{r^kX_r^{k-1}}+O(1)|\lambda_m|\sum_{v=1}^{m}\frac{(w_v^{\alpha}\mid\varphi_v\mid)^k}{v^kX_v^{k-1}}\\
&= O(1)\sum_{v=1}^{m}|\Delta\lambda_v|X_v+O(1)|\lambda_m|X_m\\
&= O(1)\quad\text{as } m\to\infty,
\end{aligned}
$$

by the hypotheses of Theorem 3 and Lemma 5.

Again, we have that

$$
\begin{aligned}
\sum_{n=2}^{m+1}\frac{1}{n^k}\left|\varphi_nT_{n,2}^{\alpha}\right|^k &\le \sum_{n=2}^{m+1}n^{-k}(A_n^{\alpha})^{-k}\left|\varphi_n\right|^k\left\{\sum_{v=1}^{n-1}A_v^{\alpha}w_v^{\alpha}|\sigma_{v+1}||\Delta\lambda_v|\right\}^k\\
&= O(1)\sum_{n=2}^{m+1}\frac{|\varphi_n|^k}{n^{k+\alpha k}}\left\{\sum_{v=1}^{n}v^{\alpha}(w_v^{\alpha})|\Delta\lambda_v|\right\}^k\\
&= O(1)\sum_{n=2}^{m+1}\frac{|\varphi_n|^k}{n^{k+\alpha k}}\sum_{v=1}^{n-1}v^{\alpha k}(w_v^{\alpha})^k|\Delta\lambda_v|^k\left\{\sum_{v=1}^{n-1}1\right\}^{k-1}\\
&= O(1)\sum_{n=2}^{m+1}\frac{|\varphi_n|^k}{n^{1+\alpha k}}\sum_{v=1}^{n-1}v^{\alpha k}(w_v^{\alpha})^k|\Delta\lambda_v|^k\\
&= O(1)\sum_{v=1}^{m}v^{\alpha k}(w_v^{\alpha})^k|\Delta\lambda_v||\Delta\lambda_v|^{k-1}\sum_{n=v+1}^{m+1}\frac{|\varphi_n|^k}{n^{1+\alpha k}}\\
&= O(1)\sum_{v=1}^{m}v^{\alpha k}(w_v^{\alpha})^k|\Delta\lambda_v||\Delta\lambda_v|^{k-1}\sum_{n=v+1}^{m+1}\frac{n^{\epsilon-k}|\varphi_n|^k}{n^{1+\epsilon+(\alpha-1)k}}\\
&= O(1)\sum_{v=1}^{m}\frac{v^{\alpha k}(w_v^{\alpha})^k|\Delta\lambda_v|}{v^{k-1}X_v^{k-1}}v^{\epsilon-k}\left|\varphi_v\right|^k\int_v^{\infty}\frac{dx}{x^{1+\epsilon+(\alpha-1)k}}\\
&= O(1)\sum_{v=1}^{m}v|\Delta\lambda_v|\frac{(w_v^{\alpha}|\varphi_v|)^k}{v^kX_v^{k-1}}\\
&= O(1)\sum_{v=1}^{m-1}\Delta\left(v|\Delta\lambda_v|\right)\sum_{r=1}^{v}\frac{(w_r^{\alpha}|\varphi_r|)^k}{r^kX_r^{k-1}}+O(1)m|\Delta\lambda_m|\sum_{v=1}^{m}\frac{(w_v^{\alpha}|\varphi_v|)^k}{v^kX_v^{k-1}}\\
&= O(1)\sum_{v=1}^{m-1}v|\Delta^2\lambda_v|X_v+O(1)\sum_{v=1}^{m-1}X_v|\Delta\lambda_v|+O(1)m|\Delta\lambda_m|X_m\\
&= O(1)\text{ as } m\to\infty,
\end{aligned}
$$

by hypotheses of Theorem 3 and Lemma 5.

Finally, as in $T_{n,1}^{\alpha}$, we have that

$$
\sum_{n=1}^{m}\frac{1}{n^k}|T_{n,3}^{\alpha}\varphi_n|^k = \sum_{n=1}^{m}\frac{1}{n^k}|\lambda_n\ \sigma_n\ w_n^{\alpha}\varphi_n|^k
$$

$$
\begin{aligned}
&= O(1)\sum_{n=1}^{m}|\lambda_n|\frac{(w_n^{\alpha}|\ \varphi_n\ |)^k}{n^k X_n^{k-1}} \\
&= O(1) \ \text{ as } m \to \infty,
\end{aligned}
$$

by the hypotheses of Theorem 3 and Lemma 5. This completes the proof of Theorem 3.

5 Conclusions

If we set $\alpha = 1$, $\varphi_n = n^{1-\frac{1}{k}}$, and (X_n) as an almost increasing sequence, then we have a known result of Sulaiman dealing with $\mid C,1\mid_k$ summability factors of infinite series (see [11]). Also, if we take (X_n) as an almost increasing sequence, then we obtain Theorem 2. Furthermore, if we set $\varphi_n = n^{1-\frac{1}{k}}$, and (X_n) as an almost increasing sequence, then we have Theorem 1. Finally, if we take $\varphi_n = n^{\delta+1-\frac{1}{k}}$, then we obtain a new result dealing with the $\mid C,\alpha;\delta\mid_k$ summability factors of infinite series.

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