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Some Quantum Estimates Of Hermite-Hadamard Inequalities For φ-Convex Functions*

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Abstract

In this paper, we develop some quantum estimates of Hermite–Hadamard type inequalities for φ-convex functions. In some special cases, these quantum estimates reduce to known results.

1 Introduction

In recent years, the topic of quantum calculus has attracted the attention of several scholars. Quantum calculus stands as a connection between mathematics and physics. It has large applications in many mathematical areas such as number theory, special functions, quantum mechanics and mathematical inequalities. In quantum analysis, we obtain q-analogues of mathematical objects that can be recaptured as $q \to 1^-$. In recent years, many researchers have shown their interest in studying and investigating quantum calculus. Quantum analysis is also very helpful in numerous fields and has large applications in various areas of pure and applied sciences. For some recent developments in quantum calculus, interested readers are referred to [18–20, 22].

Theory of inequalities and theory of convex functions are closely related to each other, thus a rich literature on inequalities. One of the most extensively studied inequality in the literature is the Hermite-Hadamard inequality: if $\mathfrak{F} : \mathcal{I} \subset \mathbb{R} \to \mathbb{R}$ is a convex function defined on the interval $\mathcal{I}$ of real numbers and $\omega_1, \omega_2 \in \mathcal{I}$ with $\omega_1 < \omega_2$, then

$$\mathfrak{F}\left(\frac{\omega_1+\omega_2}{2}\right) \le \frac{1}{\omega_2-\omega_1}\int_{\omega_1}^{\omega_2} \mathfrak{F}(x)dx \le \frac{\mathfrak{F}(\omega_1)+\mathfrak{F}(\omega_2)}{2}. \quad (1)$$

This famous result of Hermite and Hadamard can be considered as necessary and sufficient condition for a function to be convex. Various improvements, generalizations, and variants of this inequality can be found in the papers [1–17, 25, 27].

Inspired by the works of Liu and Zhuang [23] and Liu, Zhuang and Park [24], we aim to develop some quantum estimates of Hermite-Hadamard type for q-differentiable φ-convex functions.

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2 Preliminaries

In this section, we recall some previously known concepts and basic results. Let $\mathcal{I}$ be an interval in real line $\mathbb{R}$ and $\varphi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a bifunction.

Definition 1 ([21]) *A function $\mathfrak{F} : \mathcal{I} \to \mathbb{R}$ is called convex with respect to φ (briefly φ-convex), if*

$$\mathfrak{F}(t\omega_1 + (1-t)\omega_2) \leq \mathfrak{F}(\omega_2) + t\varphi(\mathfrak{F}(\omega_1), \mathfrak{F}(\omega_2))$$

for all $\omega_1, \omega_2 \in \mathcal{I}$ and $t \in [0,1]$.

Remark 1 *If we set $\varphi(A, B) = A - B$ in the above definition, then we recover the classical definition of convex function.*

Clearly, any convex function is φ-convex function. Furthermore, there exists φ-convex functions which are not convex. For example, we consider $\mathfrak{F} : \mathbb{R} \to \mathbb{R}$ as

$$\mathfrak{F}(z) = \begin{cases} -z, & \text{if } z \geq 0, \\ z, & \text{if } z < 0, \end{cases}$$

and $\varphi : [-\infty, 0] \times [-\infty, 0] \to \mathbb{R}$ as

$$\varphi(A, B) = \begin{cases} A, & \text{if } B = 0, \\ -B, & \text{if } A = 0, \\ -A - B, & \text{if } A < 0, B < 0. \end{cases}$$

Then, it is not hard to check that $\mathfrak{F}$ is φ-convex. Also, it is obvious that $\mathfrak{F}$ is not a convex function. On the other hand, let $\mathcal{I} = [\omega_1, \omega_2] \subseteq \mathbb{R}$ be an interval and $0 < q < 1$ be a constant.

Definition 2 ([26]) *Assume $\mathfrak{F} : \mathcal{I} \to \mathbb{R}$ is a continuous function and let $x \in \mathcal{I}$. Then q-derivative on $\mathcal{I}$ of function $\mathfrak{F}$ at x is defined as*

$$_{\omega_1}D_q\mathfrak{F}(x) = \frac{\mathfrak{F}(x) - \mathfrak{F}(qx + (1-q)\omega_1)}{(1-q)(x-\omega_1)}, \ x \neq \omega_1 \qquad _{\omega_1}D_q\mathfrak{F}(\omega_1) = \lim_{x \to \omega_1} {}_{\omega_1}D_q\mathfrak{F}(x). \tag{2}$$

We say that $\mathfrak{F}$ is q-differentiable on $\mathcal{I}$ provided $_{\omega_1}D_q\mathfrak{F}(x)$ exists for all $x \in \mathcal{I}$. Note that if $\omega_1 = 0$ in (2), then $_0D_q\mathfrak{F} = D_q\mathfrak{F}$, where D_q is the well-known q-derivative of the function $\mathfrak{F}(x)$ defined by

$$D_q\mathfrak{F}(x) = \frac{\mathfrak{F}(x) - \mathfrak{F}(qx)}{(1-q)x}.$$

Definition 3 ([26]) *Let $\mathfrak{F} : \mathcal{I} \to \mathbb{R}$ be a continuous function. We define the second-order q-derivative on interval $\mathcal{I}$, which is denoted as $_{\omega_1}D_q^2\mathfrak{F}$, provided $_{\omega_1}D_q\mathfrak{F}$ is q-differentiable on $\mathcal{I}$ with $_{\omega_1}D_q^2\mathfrak{F} = {}_{\omega_1}D_q({}_{\omega_1}D_q\mathfrak{F}) : \mathcal{I} \to \mathbb{R}$. Similarly, we define higher order q-derivative on $\mathcal{I}$, $_{\omega_1}D_q^n : \mathcal{I} \to \mathbb{R}$.*

Definition 4 ([26]) *Let $\mathfrak{F} : \mathcal{I} \subset \mathbb{R} \to \mathbb{R}$ be a continuous function. Then q-integral on $\mathcal{I}$ is defined by*

$$\int_{\omega_1}^{x} \mathfrak{F}(t) \ _{\omega_1}d_qt = (1-q)(x-\omega_1)\sum_{i=0}^{\infty} q^i\mathfrak{F}(q^ix + (1-q^i)\omega_1)$$

for $x \in \mathcal{I}$. Note that if $\omega_1 = 0$, then we have the classical q-integral, which is defined by

$$\int_0^x \mathfrak{F}(t)_0d_qt = (1-q)x\sum_{i=0}^{\infty} q^i\mathfrak{F}(q^ix)$$

for $x \in [0, \infty)$.

Theorem 1 ([26]) *Assume $\mathfrak{F}, \mathfrak{G} : \mathcal{I} \to \mathbb{R}$ are continuous functions, $\alpha \in \mathbb{R}$. Then, for $x \in \mathcal{I}$,*

$$\int_{\omega_1}^{x} [\mathfrak{F}(t) + \mathfrak{G}(t)] \ {}_{\omega_1}d_q t = \int_{\omega_1}^{x} \mathfrak{F}(t)_{\omega_1} d_q t + \int_{\omega_1}^{x} \mathfrak{G}(t)_{\omega_1} d_q t;$$

$$\int_{\omega_1}^{x} (\alpha\mathfrak{F})(t)_{\omega_1} d_q t = \alpha \int_{\omega_1}^{x} \mathfrak{F}(t)_{\omega_1} d_q t.$$

Also, we introduce the q-analogues of ω_1 and $(x-\omega_1)^n$ and the definition of q-Beta function.

Definition 5 ([22]) *For any real number ω_1,*

$$[\omega_1] = \frac{1 - q^{\omega_1}}{1 - q}$$

is called the q-analogue of ω_1. In particular, for $i \in \mathbb{Z}^+$, we denote

$$[i] = \frac{1 - q^i}{1 - q} = q^{i-1} + \cdots + q + 1.$$

Definition 6 ([22]) *If i is an integer, the q-analogue of $(x-\omega_1)^i$ is the polynomial*

$$(x-\omega_1)_q^i = \begin{cases} 1, & \text{if } i = 0, \\ (x-\omega_1)(x-q\omega_1)\cdots(x-q^{i-1}\omega_1), & \text{if } i \geq 1 \end{cases}$$

Definition 7 ([22]) *For any $t, s > 0$,*

$$\beta_q(t,s) = \int_0^1 x^{t-1}(1-qx)_q^{s-1} \ {}_0d_q x \tag{3}$$

is called the q-Beta function. Note that

$$\beta_q(t,1) = \int_0^1 x^{t-1} \ {}_0d_q x = \frac{1}{[t]}, \tag{4}$$

where $[t]$ is the q-analogue of t.

At last, we present the following lemmas [23] that will be used in this paper.

Lemma 1 *(a) Let $\mathfrak{F}(x) = 1$. Then we have*

$$\int_0^1 \ {}_0d_q x = (1-q)\sum_{i=0}^{\infty} q^i = 1.$$

(b) If $\mathfrak{F}(x) = x$ for $x \in [0,1]$, then we have

$$\int_0^1 x \ {}_0d_q x = (1-q)\sum_{i=0}^{\infty} q^{2i} = \frac{1}{1+q}.$$

(c) Let $\mathfrak{F}(x) = x^2$ for $x \in [0,1]$. Then we have

$$\int_0^1 x^2 \ {}_0d_q x = (1-q)\sum_{i=0}^{\infty} q^{3i} = \frac{1}{1+q+q^2}.$$

Lemma 2 *(a) Let $\mathfrak{F}(x) = 1 - qx$ for $x \in [0,1]$, where $0 < q < 1$ is a constant. Then we have*

$$\int_0^1 (1-qx)\ {}_0d_qx = \int_0^1\ {}_0d_qx - q\int_0^1 x\ {}_0d_qx = \frac{1}{1+q}.$$

(b) Let $\mathfrak{F}(x) = x(1-qx)$ for $x \in [0,1]$, where $0 < q < 1$ is a constant. Then we have

$$\int_0^1 x(1-qx)\ {}_0d_qx = \int_0^1 x\ {}_0d_qx - q\int_0^1 x^2\ {}_0d_qx = \frac{1}{(1+q)(1+q+q^2)}.$$

(c) If $\mathfrak{F}(x) = x^2(1-qx)$ for $x \in [0,1]$, where $0 < q < 1$ is a constant, then we have

$$\int_0^1 x^2(1-qx)\ {}_0d_qx = \int_0^1 x^2\ {}_0d_qx - q\int_0^1 x^3\ {}_0d_qx = \frac{1}{(1+q+q^2)(1+q+q^2+q^3)}.$$

3 Hermite-Hadamard Inequalities for φ-Convex Functions

We need the following auxiliary result, which will be useful in proving our main results. This result was also proved by Liu and Zhuang [23].

Lemma 3 *Let $\mathfrak{F} : \mathcal{I} = [\omega_1, \omega_2] \subset \mathbb{R} \to \mathbb{R}$ be a twice q-differentiable function on $\mathcal{I}^0$ with ${}_{\omega_1}D_q^2\mathfrak{F}$ continuous and integrable on $\mathfrak{F}$, where $0 < q < 1$. Then the following identity holds:*

$$\frac{q\mathfrak{F}(\omega_1)+\mathfrak{F}(\omega_2)}{1+q} - \frac{1}{\omega_2-\omega_1}\int_{\omega_1}^{\omega_2} \mathfrak{F}(x)_{\omega_1}d_qx = \frac{q^2(\omega_2-\omega_1)^2}{1+q}\int_0^1 t(1-qt)_a D_q^2\mathfrak{F}((1-t)\omega_1+t\omega_2)_0 d_qt. \quad (5)$$

The next theorem gives some estimates for the left-hand side of the result of (5) through φ-convex functions.

Theorem 2 *Let $\mathfrak{F} : \mathcal{I} = [\omega_1, \omega_2] \subset \mathbb{R} \to \mathbb{R}$ be a twice q-differentiable function on $\mathcal{I}^0$ with ${}_{\omega_1}D_q^2\mathfrak{F}$ continuous and integrable on $\mathcal{I}$, where $0 < q < 1$. If $|{}_{\omega_1}D_q^2\mathfrak{F}|^m$ is φ-convex on $[\omega_1, \omega_2]$ for $m \geq 1$, then the following inequality holds:*

$$\begin{aligned} &\left|\frac{q\mathfrak{F}(\omega_1)+\mathfrak{F}(\omega_2)}{1+q} - \frac{1}{\omega_2-\omega_1}\int_{\omega_1}^{\omega_2} \mathfrak{F}(x)_{\omega_1}d_qx\right| \\ \leq\ &\frac{q^2(\omega_2-\omega_1)^2}{(1+q)^{2-\frac{1}{m}}}\Big(K_1\left|{}_{\omega_1}D_q^2\mathfrak{F}(\omega_1)\right|^m + K_2\varphi\left(\left|{}_{\omega_1}D_q^2\mathfrak{F}(\omega_2)\right|^m, \left|{}_{\omega_1}D_q^2\mathfrak{F}(\omega_1)\right|^m\right)\Big)^{\frac{1}{m}}, \end{aligned} \quad (6)$$

where

$$K_1 = \int_0^1 t(1-qt)^m\ {}_0d_qt = (1-q)\sum_{i=0}^{\infty} q^{2i}(1-q^{i+1})^m$$

and

$$K_2 = \int_0^1 t^2(1-qt)^m\ {}_0d_qt = (1-q)\sum_{i=0}^{\infty} q^{3i}(1-q^{i+1})^m.$$

Proof. Using Lemma 3, Hölder inequality and the fact that $|_{\omega_1}D_q^2\mathfrak{F}|^m$ is a φ-convex function, we have

$$\begin{aligned}
&\left|\frac{q\mathfrak{F}(\omega_1)+\mathfrak{F}(\omega_2)}{1+q}-\frac{1}{\omega_2-\omega_1}\int_{\omega_1}^{\omega_2}\mathfrak{F}(x)_{\omega_1}d_qx\right| \\
=\ &\left|\frac{q^2(\omega_2-\omega_1)^2}{1+q}\int_0^1 t(1-qt)_{\omega_1}D_q^2\mathfrak{F}((1-t)\omega_1+t\omega_2)_0d_qt\right| \\
\le\ &\frac{q^2(\omega_2-\omega_1)^2}{1+q}\int_0^1 t(1-qt)\left|_{\omega_1}D_q^2\mathfrak{F}((1-t)\omega_1+t\omega_2)\right|\ _0d_qt \\
\le\ &\frac{q^2(\omega_2-\omega_1)^2}{1+q}\left(\int_0^1 t\ _0d_qt\right)^{1-\frac{1}{m}}\left(\int_0^1 t(1-qt)^m\left|_{\omega_1}D_q^2\mathfrak{F}((1-t)\omega_1+t\omega_2)\right|^m\ _0d_qt\right)^{\frac{1}{m}} \\
\le\ &\frac{q^2(\omega_2-\omega_1)^2}{1+q}\left(\int_0^1 t\ _0d_qt\right)^{1-\frac{1}{m}}\left(\left|_{\omega_1}D_q^2\mathfrak{F}(\omega_1)\right|^m\int_0^1 t(1-qt)^m\ _0d_qt\right. \\
&\left.+\varphi\left(\left|_{\omega_1}D_q^2\mathfrak{F}(\omega_2)\right|^m,\left|_{\omega_1}D_q^2\mathfrak{F}(\omega_1)\right|^m\right)\int_0^1 t^2(1-qt)^m\ _0d_qt\right)^{\frac{1}{m}}.
\end{aligned}$$

Now, applying Lemma 1(b), we have

$$\begin{aligned}
&\left|\frac{q\mathfrak{F}(\omega_1)+\mathfrak{F}(\omega_2)}{1+q}-\frac{1}{\omega_2-\omega_1}\int_{\omega_1}^{\omega_2}\mathfrak{F}(x)_{\omega_1}d_qx\right| \\
\le\ &\frac{q^2(\omega_2-\omega_1)^2}{1+q}\left(\frac{1}{1+q}\right)^{1-\frac{1}{m}}\left(\left|_{\omega_1}D_q^2\mathfrak{F}(\omega_1)\right|^m K_1+\varphi\left(\left|_{\omega_1}D_q^2\mathfrak{F}(\omega_2)\right|^m,\left|_{\omega_1}D_q^2\mathfrak{F}(\omega_1)\right|^m\right)K_2\right)^{\frac{1}{m}} \\
=\ &\frac{q^2(\omega_2-\omega_1)^2}{(1+q)^{2-\frac{1}{m}}}\left(K_1\left|_{\omega_1}D_q^2\mathfrak{F}(a)\right|^m+K_2\varphi\left(\left|_{\omega_1}D_q^2\mathfrak{F}(\omega_2)\right|^m,\left|_{\omega_1}D_q^2\mathfrak{F}(\omega_1)\right|^m\right)\right)^{\frac{1}{m}}.
\end{aligned}$$

It is easy to check by Definition 4 that

$$K_1=\int_0^1 t(1-qt)^m\ _0d_qt=(1-q)\sum_{i=0}^{\infty}q^{2i}(1-q^{i+1})^m$$

and

$$K_2=\int_0^1 t^2(1-qt)^m\ _0d_qt=(1-q)\sum_{i=0}^{\infty}q^{3i}(1-q^{i+1})^m.$$

Thus, we get (6). ■

Remark 2 *By setting $\varphi(A,B)=A-B$ in Theorem 2, we recapture [23, Theorem 5.1].*

Corollary 1 *In Theorem 2, if $q\to 1^-$, then we get*

$$K_1=\int_0^1 t(1-t)^m dt=\frac{1}{(m+1)(m+2)},\quad K_2=\int_0^1 t^2(1-t)^m dt=\frac{2}{(m+1)(m+2)(m+3)},$$

and (6) reduces to the following inequality:

$$\begin{aligned}
&\left|\frac{\mathfrak{F}(\omega_1)+\mathfrak{F}(\omega_2)}{2}-\frac{1}{\omega_2-\omega_1}\int_{\omega_1}^{\omega_2}\mathfrak{F}(x)dx\right| \\
\le\ &\frac{(\omega_2-\omega_1)^2}{2^{2-\frac{1}{m}}}\left(\frac{1}{(m+1)(m+2)}\left|\mathfrak{F}''(\omega_1)\right|^m+\frac{2}{(m+1)(m+2)(m+3)}\varphi\left(\left|\mathfrak{F}''(\omega_2)\right|^m,\left|\mathfrak{F}''(\omega_1)\right|^m\right)\right)^{\frac{1}{m}}.
\end{aligned}$$

Remark 3 *If $\varphi(A,B) = A - B$ and $q \to 1^-$, then (6) reduces to the following inequality:*

$$\left|\frac{\mathfrak{F}(\omega_1)+\mathfrak{F}(\omega_2)}{2} - \frac{1}{\omega_2-\omega_1}\int_{\omega_1}^{\omega_2}\mathfrak{F}(x)dx\right| \le \frac{(\omega_2-\omega_1)^2}{2^{2-\frac{1}{m}}}\left(\frac{(m+1)|\mathfrak{F}''(\omega_1)|^m + 2|\mathfrak{F}''(\omega_2)|^m}{(m+1)(m+2)(m+3)}\right)^{\frac{1}{m}}.$$

Corollary 2 *If m is a positive integer, then Theorem 2 amounts to:*

$$(1-qt)^m \le (1-qt)_q^m,$$

and (6) reduces to

$$\begin{aligned}&\left|\frac{q\mathfrak{F}(\omega_1)+\mathfrak{F}(\omega_2)}{1+q} - \frac{1}{\omega_2-\omega_1}\int_{\omega_1}^{\omega_2}\mathfrak{F}(x)_{\omega_1}d_qx\right| \\ \le\ & \frac{q^2(\omega_2-\omega_1)^2}{(1+q)^{2-\frac{1}{m}}}\left(\beta_q(2,m+1)\left|{}_{\omega_1}D_q^2\mathfrak{F}(\omega_1)\right|^m + \beta_q(3,m+1)\varphi\left(\left|{}_{\omega_1}D_q^2\mathfrak{F}(\omega_2)\right|^m, \left|{}_{\omega_1}D_q^2\mathfrak{F}(\omega_1)\right|^m\right)\right)^{\frac{1}{m}}.\end{aligned}$$

Theorem 3 *Let $\mathfrak{F} : \mathcal{I} = [\omega_1,\omega_2] \subset \mathbb{R} \to \mathbb{R}$ be a twice q-differentiable function on $\mathcal{I}^0$ and ${}_{\omega_1}D_q^2\mathfrak{F}$ be continuous and integrable on $\mathcal{I}$, where $0 < q < 1$. If $|{}_{\omega_1}D_q^2\mathfrak{F}|^m$ is φ-convex on $[\omega_1,\omega_2]$, where $n, m > 1$, $\frac{1}{n}+\frac{1}{m} = 1$, then*

$$\begin{aligned}&\left|\frac{q\mathfrak{F}(\omega_1)+\mathfrak{F}(\omega_2)}{1+q} - \frac{1}{\omega_2-\omega_1}\int_{\omega_1}^{\omega_2}\mathfrak{F}(x)_{\omega_1}d_qx\right| \\ \le\ & \frac{q^2(\omega_2-\omega_1)^2}{1+q}(L)^{\frac{1}{n}}\left(\frac{\left|{}_{\omega_1}D_q^2\mathfrak{F}(\omega_1)\right|^m}{1+q} + \frac{\varphi\left(\left|{}_{\omega_1}D_q^2\mathfrak{F}(\omega_2)\right|^m, \left|{}_{\omega_1}D_q^2\mathfrak{F}(\omega_1)\right|^m\right)}{1+q+q^2}\right)^{\frac{1}{m}},\end{aligned} \tag{7}$$

where

$$L = \int_0^1 t(1-qt)^n\ {}_0d_qt = (1-q)\sum_{i=0}^{\infty} q^{2i}(1-q^{i+1})^n.$$

Proof. Using Lemma 3, Hölder inequality and the fact that $|{}_{\omega_1}D_q^2\mathfrak{F}|^m$ is a φ-convex function, we have

$$\begin{aligned}&\left|\frac{q\mathfrak{F}(\omega_1)+\mathfrak{F}(\omega_2)}{1+q} - \frac{1}{\omega_2-\omega_1}\int_{\omega_1}^{\omega_2}\mathfrak{F}(x)_{\omega_1}d_qx\right| \\ =\ & \left|\frac{q^2(\omega_2-\omega_1)^2}{1+q}\int_0^1 t(1-qt)_{\omega_1}D_q^2\mathfrak{F}((1-t)\omega_1+t\omega_2)_0d_qt\right| \\ \le\ & \frac{q^2(\omega_2-\omega_1)^2}{1+q}\int_0^1 t(1-qt)\left|{}_{\omega_1}D_q^2\mathfrak{F}((1-t)\omega_1+t\omega_2)\right|\ {}_0d_qt \\ \le\ & \frac{q^2(\omega_2-\omega_1)^2}{1+q}\left(\int_0^1 t(1-qt)^n\ {}_0d_qt\right)^{\frac{1}{n}}\left(\int_0^1 t\left|{}_{\omega_1}D_q^2\mathfrak{F}((1-t)\omega_1+t\omega_2)\right|^m\ {}_0d_qt\right)^{\frac{1}{m}} \\ \le\ & \frac{q^2(\omega_2-\omega_1)^2}{1+q}\left(\int_0^1 t(1-qt)^n\ {}_0d_qt\right)^{\frac{1}{n}}\left(\left|{}_{\omega_1}D_q^2\mathfrak{F}(\omega_1)\right|^m\int_0^1 t\ {}_0d_qt\right. \\ & \left. +\varphi\left(\left|{}_{\omega_1}D_q^2\mathfrak{F}(\omega_2)\right|^m, \left|{}_{\omega_1}D_q^2\mathfrak{F}(\omega_1)\right|^m\right)\int_0^1 t^2\ {}_0d_qt\right)^{\frac{1}{m}}.\end{aligned}$$

Applying Lemmas 1(b) and 1(c), we have

$$\begin{aligned}
&\left|\frac{q\mathfrak{F}(\omega_1)+\mathfrak{F}(\omega_2)}{1+q}-\frac{1}{\omega_2-\omega_1}\int_{\omega_1}^{\omega_2}\mathfrak{F}(x)_{\omega_1}d_qx\right| \\
\leq\;& \frac{q^2(\omega_2-\omega_1)^2}{1+q}\left(\int_0^1 t(1-qt)^n\ {}_0d_qt\right)^{\frac{1}{n}}\left(\frac{\left|{}_{\omega_1}D_q^2\mathfrak{F}(\omega_1)\right|^m}{1+q}+\frac{\varphi\left(\left|{}_{\omega_1}D_q^2\mathfrak{F}(\omega_2)\right|^m,\left|{}_{\omega_1}D_q^2\mathfrak{F}(\omega_1)\right|^m\right)}{1+q+q^2}\right)^{\frac{1}{m}} \\
=\;& \frac{q^2(\omega_2-\omega_1)^2}{1+q}(L)^{\frac{1}{n}}\left(\frac{\left|{}_{\omega_1}D_q^2\mathfrak{F}(\omega_1)\right|^m}{1+q}+\frac{\varphi\left(\left|{}_{\omega_1}D_q^2\mathfrak{F}(\omega_2)\right|^m,\left|{}_{\omega_1}D_q^2\mathfrak{F}(\omega_1)\right|^m\right)}{1+q+q^2}\right)^{\frac{1}{m}}.
\end{aligned}$$

It is easy to check by Definition 4 that

$$L=\int_0^1 t(1-qt)^n\ {}_0d_qt=(1-q)\sum_{i=0}^{\infty}q^{2i}(1-q^{i+1})^n,$$

and thus, we get (7). ■

Remark 4 *By setting $\varphi(A,B)=A-B$ in Theorem 3, we recapture [23, Theorem 5.2].*

Corollary 3 *By letting $q\to 1^-$ in Theorem 3, we get*

$$L=\int_0^1 t(1-t)^n dt=\frac{1}{(n+1)(n+2)},$$

and (7) reduces to the following inequality:

$$\begin{aligned}
&\left|\frac{\mathfrak{F}(\omega_1)+\mathfrak{F}(\omega_2)}{2}-\frac{1}{\omega_2-\omega_1}\int_{\omega_1}^{\omega_2}\mathfrak{F}(x)dx\right| \\
\leq\;& \frac{(\omega_2-\omega_1)^2}{2}\left(\frac{1}{(n+1)(n+2)}\right)^{\frac{1}{n}}\left(\frac{3\left|\mathfrak{F}''(\omega_1)\right|^m+2\varphi\left(\left|\mathfrak{F}''(\omega_2)\right|^m,\left|\mathfrak{F}''(\omega_1)\right|^m\right)}{6}\right)^{\frac{1}{m}}.
\end{aligned}$$

Remark 5 *If $\varphi(A,B)=A-B$ and $q\to 1^-$, then (7) reduces to the following inequality:*

$$\begin{aligned}
&\left|\frac{\mathfrak{F}(\omega_1)+\mathfrak{F}(\omega_2)}{2}-\frac{1}{\omega_2-\omega_1}\int_{\omega_1}^{\omega_2}\mathfrak{F}(x)dx\right| \\
\leq\;& \frac{(\omega_2-\omega_1)^2}{2}\left(\frac{1}{(n+1)(n+2)}\right)^{\frac{1}{n}}\left(\frac{\left|\mathfrak{F}''(\omega_1)\right|^m+2\left|\mathfrak{F}''(\omega_2)\right|^m}{6}\right)^{\frac{1}{m}}.
\end{aligned}$$

Corollary 4 *In Theorem 3, if n is a positive integer , then*

$$(1-qt)^n\leq(1-qt)_q^n,$$

and (7) reduces to

$$\begin{aligned}
&\left|\frac{q\mathfrak{F}(\omega_1)+\mathfrak{F}(\omega_2)}{1+q}-\frac{1}{\omega_2-\omega_1}\int_{\omega_1}^{\omega_2}\mathfrak{F}(x)_{\omega_1}d_qx\right| \\
\leq\;& \frac{q^2(\omega_2-\omega_1)^2}{1+q}(\beta_q(2,n+1))^{\frac{1}{n}}\left(\frac{\left|{}_aD_q^2\mathfrak{F}(\omega_1)\right|^m}{1+q}+\frac{\varphi\left(\left|{}_{\omega_1}D_q^2\mathfrak{F}(\omega_2)\right|^m,\left|{}_{\omega_1}D_q^2\mathfrak{F}(\omega_1)\right|^m\right)}{1+q+q^2}\right)^{\frac{1}{m}}.
\end{aligned}$$

Theorem 4 *Let $\mathfrak{F} : \mathcal{I} = [\omega_1, \omega_2] \subset \mathbb{R} \to \mathbb{R}$ be a twice q-differentiable function on $\mathcal{I}^0$ and ${}_{\omega_1}D_q^2\mathfrak{F}$ be continuous and integrable on $\mathcal{I}$, where $0 < q < 1$. If $|{}_{\omega_1}D_q^2\mathfrak{F}|^m$ is φ-convex on $[\omega_1, \omega_2]$, where $n, m > 1$, $\frac{1}{n} + \frac{1}{m} = 1$, then*

$$\begin{aligned} & \left| \frac{q\mathfrak{F}(\omega_1) + \mathfrak{F}(\omega_2)}{1+q} - \frac{1}{\omega_2 - \omega_1} \int_{\omega_1}^{\omega_2} \mathfrak{F}(x)_{\omega_1} d_q x \right| \\ \leq & \frac{q^2(\omega_2 - \omega_1)^2}{1+q} (R)^{\frac{1}{n}} \left(\left|{}_{\omega_1}D_q^2\mathfrak{F}(\omega_1)\right|^m + \frac{\varphi\left(\left|{}_{\omega_1}D_q^2\mathfrak{F}(\omega_2)\right|^m, \left|{}_{\omega_1}D_q^2\mathfrak{F}(\omega_1)\right|^m\right)}{1+q} \right)^{\frac{1}{m}}, \end{aligned} \tag{8}$$

where

$$R = \int_0^1 t^n (1-qt)^n \ {}_0d_q t = (1-q) \sum_{i=0}^{\infty} \left(q^i\right)^{n+1} (1-q^{i+1})^n.$$

Proof. Using Lemma 3, Hölder inequality and the fact that $|{}_{\omega_1}D_q^2\mathfrak{F}|^m$ is a φ-convex function, we have

$$\begin{aligned} & \left| \frac{q\mathfrak{F}(\omega_1) + \mathfrak{F}(\omega_1)}{1+q} - \frac{1}{\omega_2 - \omega_1} \int_{\omega_1}^{\omega_2} \mathfrak{F}(x)_{\omega_1} d_q x \right| \\ = & \left| \frac{q^2(\omega_2 - \omega_1)^2}{1+q} \int_0^1 t(1-qt)_{\omega_1} D_q^2\mathfrak{F}((1-t)\omega_1 + t\omega_2)_0 d_q t \right| \\ \leq & \frac{q^2(\omega_2 - \omega_1)^2}{1+q} \int_0^1 t(1-qt) \left|{}_{\omega_1}D_q^2\mathfrak{F}((1-t)\omega_1 + t\omega_2)\right| \ {}_0d_q t \\ \leq & \frac{q^2(\omega_2 - \omega_1)^2}{1+q} \left(\int_0^1 t^n(1-qt)^n \ {}_0d_q t \right)^{\frac{1}{n}} \left(\int_0^1 \left|{}_{\omega_1}D_q^2\mathfrak{F}((1-t)\omega_1 + t\omega_2)\right|^m \ {}_0d_q t \right)^{\frac{1}{m}} \\ \leq & \frac{q^2(\omega_2 - \omega_1)^2}{1+q} \left(\int_0^1 t^n(1-qt)^n \ {}_0d_q t \right)^{\frac{1}{n}} \left(\left|{}_{\omega_1}D_q^2\mathfrak{F}(\omega_1)\right|^m \int_0^1 \ {}_0d_q t \right. \\ & \left. + \varphi\left(\left|{}_{\omega_1}D_q^2\mathfrak{F}(\omega_2)\right|^m, \left|{}_{\omega_1}D_q^2\mathfrak{F}(\omega_1)\right|^m\right) \int_0^1 t \ {}_0d_q t \right)^{\frac{1}{m}}. \end{aligned}$$

Employing Lemmas 1(a) and 1(b), we obtain

$$\begin{aligned} & \left| \frac{q\mathfrak{F}(\omega_1) + \mathfrak{F}(\omega_2)}{1+q} - \frac{1}{\omega_2 - \omega_1} \int_{\omega_1}^{\omega_2} \mathfrak{F}(x)_{\omega_1} d_q x \right| \\ \leq & \frac{q^2(\omega_2 - \omega_1)^2}{1+q} (R)^{\frac{1}{n}} \left(\left|{}_{\omega_1}D_q^2\mathfrak{F}(\omega_1)\right|^m + \frac{\varphi\left(\left|{}_{\omega_1}D_q^2\mathfrak{F}(\omega_2)\right|^m, \left|{}_{\omega_1}D_q^2\mathfrak{F}(\omega_1)\right|^m\right)}{1+q} \right)^{\frac{1}{m}}. \end{aligned}$$

It is easy to check by Definition 4 that

$$R = \int_0^1 t^n (1-qt)^n \ {}_0d_q t = (1-q) \sum_{i=0}^{\infty} \left(q^i\right)^{n+1} (1-q^{i+1})^n,$$

and thus, we get (8). ■

Remark 6 *By setting $\varphi(A, B) = A - B$ in Theorem 4, we recapture [23, Theorem 5.4].*

Corollary 5 *In Theorem 4, if $q \to 1$, then we have*

$$R = \int_0^1 t^n (1-t)^n dt = \beta(n+1, n+1).$$

Using the properties of Beta function, that is, $\beta(x,x) = 2^{1-2x}\beta(\frac{1}{2},x)$ *and* $\beta(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(xy)}$*, we obtain that*

$$\beta(n+1,n+1) = 2^{1-2(n+1)}\beta\left(\frac{1}{2},n+1\right) = 2^{-2n-1}\frac{\Gamma(\frac{1}{2})\Gamma(n+1)}{\Gamma(\frac{3}{2}+n)},$$

where $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ *and* $\Gamma(t)$ *is Gamma function:*

$$\Gamma(t) = \int_0^\infty x^{t-1}e^{-x}dx, \quad t>0.$$

Thus, inequality (8) reduces to the following inequality:

$$\begin{aligned}
&\left|\frac{\mathfrak{F}(\omega_1)+\mathfrak{F}(\omega_2)}{2} - \frac{1}{\omega_2-\omega_1}\int_{\omega_1}^{\omega_2}\mathfrak{F}(x)dx\right| \\
\leq\ & \frac{(\omega_2-\omega_1)^2}{2}\left(2^{-2n-1}\frac{\Gamma(\frac{1}{2})\Gamma(n+1)}{\Gamma(\frac{3}{2}+n)}\right)^{\frac{1}{n}}\left(|\mathfrak{F}''(\omega_1)|^m + \frac{\varphi\left(|\mathfrak{F}''(\omega_2)|^m, |\mathfrak{F}''(\omega_1)|^m\right)}{2}\right)^{\frac{1}{m}} \\
=\ & \frac{(\omega_2-\omega_1)^2}{8}\left(\frac{\sqrt{\pi}}{2}\right)^{\frac{1}{n}}\left(\frac{\Gamma(n+1)}{\Gamma(\frac{3}{2}+n)}\right)^{\frac{1}{n}}\left(|\mathfrak{F}''(\omega_1)|^m + \frac{\varphi\left(|\mathfrak{F}''(\omega_2)|^m, |\mathfrak{F}''(\omega_1)|^m\right)}{2}\right)^{\frac{1}{m}}.
\end{aligned}$$

Remark 7 *If* $\varphi(A,B) = A-B$ *and* $q \to 1^-$*, then (8) reduces to the following inequality:*

$$\begin{aligned}
&\left|\frac{\mathfrak{F}(\omega_1)+\mathfrak{F}(\omega_2)}{2} - \frac{1}{\omega_2-\omega_1}\int_{\omega_1}^{\omega_2}\mathfrak{F}(x)dx\right| \\
\leq\ & \frac{(\omega_2-\omega_1)^2}{8}\left(\frac{\sqrt{\pi}}{2}\right)^{\frac{1}{n}}\left(\frac{\Gamma(n+1)}{\Gamma(\frac{3}{2}+n)}\right)^{\frac{1}{n}}\left(\frac{|\mathfrak{F}''(\omega_1)|^m + |f''(\omega_2)|^m}{2}\right)^{\frac{1}{m}}.
\end{aligned}$$

Corollary 6 *In Theorem 4, if* n *is a positive integer,* $n>1$*, then*

$$(1-qt)^n \leq (1-qt)_q^n,$$

and (8) reduces to

$$\begin{aligned}
&\left|\frac{q\mathfrak{F}(\omega_1)+\mathfrak{F}(\omega_2)}{1+q} - \frac{1}{\omega_2-\omega_1}\int_{\omega_1}^{\omega_2}\mathfrak{F}(x)_{\omega_1}d_qx\right| \\
\leq\ & \frac{q^2(\omega_2-\omega_1)^2}{1+q}\left(\beta_q(n+1,n+1)\right)^{\frac{1}{n}}\left(\left|{}_{\omega_1}D_q^2\mathfrak{F}(\omega_1)\right|^m + \frac{\varphi\left(\left|{}_{\omega_1}D_q^2\mathfrak{F}(\omega_2)\right|^m, \left|{}_{\omega_1}D_q^2\mathfrak{F}(\omega_2)\right|^m\right)}{1+q}\right)^{\frac{1}{m}}.
\end{aligned}$$

Theorem 5 *Let* $\mathfrak{F}:\mathcal{I}=[\omega_1,\omega_2]\subset\mathbb{R}\to\mathbb{R}$ *be a twice q-differentiable function on* $\mathcal{I}^0$ *with* ${}_{\omega_1}D_q^2\mathfrak{F}$ *be continuous and integrable on* $\mathcal{I}$ *where* $0<q<1$*. If* $|{}_{\omega_1}D_q^2\mathfrak{F}|^m$ *is* φ*-convex on* $[\omega_1,\omega_2]$ *where* $n,m>1$*,* $\frac{1}{n}+\frac{1}{m}=1$*, then*

$$\begin{aligned}
&\left|\frac{q\mathfrak{F}(\omega_1)+\mathfrak{F}(\omega_2)}{1+q} - \frac{1}{\omega_2-\omega_1}\int_{\omega_1}^{\omega_2}\mathfrak{F}(x)_{\omega_1}d_qx\right| \\
\leq\ & \frac{q^2(\omega_2-\omega_1)^2}{1+q}\left(\frac{1}{[n+1]}\right)^{\frac{1}{n}}\left(W_1\left|{}_{\omega_1}D_q^2\mathfrak{F}(\omega_1)\right|^m + W_2\varphi\left(\left|{}_{\omega_1}D_q^2\mathfrak{F}(\omega_2)\right|^m, \left|{}_{\omega_1}D_q^2\mathfrak{F}(\omega_1)\right|^m\right)\right)^{\frac{1}{m}}, \quad (9)
\end{aligned}$$

where

$$W_1 = \int_0^1 (1-qt)^m \ {}_0d_qt = (1-q)\sum_{i=0}^\infty q^i(1-q^{i+1})^m$$

and

$$W_2=\int_0^1 t(1-qt)^m\ {}_0d_qt=(1-q)\sum_{i=0}^{\infty}q^{2i}(1-q^{i+1})^m.$$

Proof. Using Lemma 3, Hölder inequality and the fact that $|_{\omega_1}D_q^2\mathfrak{F}|^m$ is a φ-convex function, we have

$$\begin{aligned}
&\left|\frac{q\mathfrak{F}(\omega_1)+\mathfrak{F}(\omega_2)}{1+q}-\frac{1}{\omega_2-\omega_1}\int_{\omega_1}^{\omega_2}\mathfrak{F}(x)_{\omega_1}d_qx\right|\\
=&\left|\frac{q^2(\omega_2-\omega_1)^2}{1+q}\int_0^1 t(1-qt)_{\omega_1}D_q^2\mathfrak{F}((1-t)\omega_1+t\omega_2)_0d_qt\right|\\
\leq&\ \frac{q^2(\omega_2-\omega_1)^2}{1+q}\int_0^1 t(1-qt)\left|_{\omega_1}D_q^2\mathfrak{F}((1-t)\omega_1+t\omega_2)\right|\ {}_0d_qt\\
\leq&\ \frac{q^2(\omega_2-\omega_1)^2}{1+q}\left(\int_0^1 t^n\ {}_0d_qt\right)^{\frac{1}{n}}\left(\int_0^1(1-qt)^m\left|_{\omega_1}D_q^2\mathfrak{F}((1-t)\omega_1+t\omega_2)\right|^m\ {}_0d_qt\right)^{\frac{1}{m}}\\
\leq&\ \frac{q^2(\omega_2-\omega_1)^2}{1+q}\left(\int_0^1 t^n\ {}_0d_qt\right)^{\frac{1}{n}}\left(\left|_{\omega_1}D_q^2\mathfrak{F}(\omega_1)\right|^m\int_0^1(1-qt)^m\ {}_0d_qt\right.\\
&\left.+\varphi\left(\left|_{\omega_1}D_q^2\mathfrak{F}(\omega_2)\right|^m,\left|_{\omega_1}D_q^2\mathfrak{F}(\omega_1)\right|^m\right)\int_0^1 t(1-qt)^m\ {}_0d_qt\right)^{\frac{1}{m}},
\end{aligned}$$

and applying (4) in Definition 7, we have

$$\begin{aligned}
&\left|\frac{q\mathfrak{F}(\omega_1)+\mathfrak{F}(\omega_2)}{1+q}-\frac{1}{\omega_2-\omega_1}\int_{\omega_1}^{\omega_2}\mathfrak{F}(x)_{\omega_1}d_qx\right|\\
\leq&\ \frac{q^2(\omega_2-\omega_1)^2}{1+q}\left(\frac{1}{[n+1]}\right)^{\frac{1}{n}}\left(\left|_{\omega_1}D_q^2\mathfrak{F}(\omega_1)\right|^m\int_0^1(1-qt)^m\ {}_0d_qt\right.\\
&\left.+\varphi\left(\left|_{\omega_1}D_q^2\mathfrak{F}(\omega_2)\right|^m,\left|_{\omega_1}D_q^2\mathfrak{F}(\omega_1)\right|^m\right)\int_0^1 t(1-qt)^m\ {}_0d_qt\right)^{\frac{1}{m}}\\
=&\ \frac{q^2(\omega_2-\omega_1)^2}{1+q}\left(\frac{1}{[n+1]}\right)^{\frac{1}{n}}\left(W_1\left|_{\omega_1}D_q^2\mathfrak{F}(\omega_1)\right|^m+W_2\varphi\left(\left|_{\omega_1}D_q^2\mathfrak{F}(\omega_2)\right|^m,\left|_{\omega_1}D_q^2\mathfrak{F}(\omega_1)\right|^m\right)\right)^{\frac{1}{m}}.
\end{aligned}$$

It is easy to check by Definition 4 that

$$W_1=\int_0^1(1-qt)^m\ {}_0d_qt=(1-q)\sum_{i=0}^{\infty}q^{i}(1-q^{i+1})^m$$

and

$$W_2=\int_0^1 t(1-qt)^m\ {}_0d_qt=(1-q)\sum_{i=0}^{\infty}q^{2i}(1-q^{i+1})^m,$$

thus, we get (9). ■

Remark 8 *By setting $\varphi(A,B)=A-B$ in Theorem 5, we recapture [23, Theorem 5.5].*

Corollary 7 *In Theorem 5, if $q\to 1$, then we have*

$$W_1=\int_0^1(1-t)^m dt=\frac{1}{m+1},\quad W_2=\int_0^1 t(1-t)^m dt=\frac{1}{(m+1)(m+2)},$$

and (9) *reduces to the following inequality:*

$$\begin{aligned}&\left|\frac{\mathfrak{F}(\omega_1)+\mathfrak{F}(\omega_2)}{2}-\frac{1}{\omega_2-\omega_1}\int_{\omega_1}^{\omega_2}\mathfrak{F}(x)dx\right|\\ \leq\ &\frac{(\omega_2-\omega_1)^2}{2}\left(\frac{1}{n+1}\right)^{\frac{1}{n}}\left(\frac{(m+2)\left|\mathfrak{F}''(\omega_1)\right|^m+\varphi\left(\left|\mathfrak{F}''(\omega_2)\right|^m,\left|\mathfrak{F}''(\omega_1)\right|^m\right)}{(m+1)(m+2)}\right)^{\frac{1}{m}}. \qquad (10)\end{aligned}$$

Remark 9 *If* $\varphi(A,B)=A-B$ *and* $q\to 1$, *then* (9) *reduces to the following inequality:*

$$\begin{aligned}&\left|\frac{\mathfrak{F}(\omega_1)+\mathfrak{F}(\omega_2)}{2}-\frac{1}{\omega_2-\omega_1}\int_{\omega_1}^{\omega_2}\mathfrak{F}(x)dx\right|\\ \leq\ &\frac{(\omega_2-\omega_1)^2}{2}\left(\frac{1}{n+1}\right)^{\frac{1}{n}}\left(\frac{(m+1)\left|\mathfrak{F}''(\omega_1)\right|^m+\left|\mathfrak{F}''(\omega_2)\right|^m}{(m+1)(m+2)}\right)^{\frac{1}{m}}. \qquad (11)\end{aligned}$$

Corollary 8 *In Theorem* 5, *if* m *is a positive integer,* $m>1$, *then*

$$(1-qt)^m\leq(1-qt)_q^m,$$

and (9) *reduces to*

$$\begin{aligned}&\left|\frac{q\mathfrak{F}(\omega_1)+\mathfrak{F}(\omega_2)}{1+q}-\frac{1}{\omega_2-\omega_1}\int_{\omega_1}^{\omega_2}\mathfrak{F}(x)_{\omega_1}d_qx\right|\\ \leq\ &\frac{q^2(\omega_2-\omega_1)^2}{1+q}\left(\frac{1}{[n+1]}\right)^{\frac{1}{n}}\Big(\beta_q(1,m+1)\left|{}_{\omega_1}D_q^2\mathfrak{F}(\omega_1)\right|^m\\ &+\beta_q(2,m+1)\varphi\left(\left|{}_{\omega_1}D_q^2\mathfrak{F}(\omega_2)\right|^m,\left|{}_{\omega_1}D_q^2\mathfrak{F}(\omega_1)\right|^m\right)\Big)^{\frac{1}{m}}\end{aligned}$$

Theorem 6 *Let* $\mathfrak{F}:\mathcal{I}=[\omega_1,\omega_2]\subset\mathbb{R}\to\mathbb{R}$ *be a twice* q*-differentiable function on* $\mathcal{I}^0$ *with* ${}_{\omega_1}D_q^2\mathfrak{F}$ *be continuous and integrable on* $\mathcal{I}$ *where* $0<q<1$. *If* $|{}_{\omega_1}D_q^2\mathfrak{F}|^m$ *is* φ*-convex on* $[\omega_1,\omega_2]$ *where* $n,m>1$, $\frac{1}{n}+\frac{1}{m}=1$, *then*

$$\begin{aligned}&\left|\frac{q\mathfrak{F}(\omega_1)+\mathfrak{F}(\omega_2)}{1+q}-\frac{1}{\omega_2-\omega_1}\int_{\omega_1}^{\omega_2}\mathfrak{F}(x)_{\omega_1}d_qx\right|\\ \leq\ &\frac{q^2(\omega_2-\omega_1)^2}{1+q}(M)^{\frac{1}{n}}\left(\frac{\left|{}_{\omega_1}D_q^2\mathfrak{F}(\omega_1)\right|^m}{[m+1]}+\frac{\varphi\left(\left|{}_{\omega_1}D_q^2\mathfrak{F}(\omega_2)\right|^m,\left|{}_{\omega_1}D_q^2\mathfrak{F}(\omega_1)\right|^m\right)}{[m+2]}\right)^{\frac{1}{m}} \qquad (12)\end{aligned}$$

where

$$M=\int_0^1(1-qt)^n\ {}_0d_qt=(1-q)\sum_{i=0}^{\infty}q^i(1-q^{i+1})^n.$$

Proof. Using Lemma 3, Hölder inequality and the fact that $|_{\omega_1}D_q^2\mathfrak{F}|^m$ is a φ-convex function, we have

$$\begin{aligned}
&\left|\frac{q\mathfrak{F}(\omega_1)+\mathfrak{F}(\omega_2)}{1+q}-\frac{1}{\omega_2-\omega_1}\int_{\omega_1}^{\omega_2}\mathfrak{F}(x)_{\omega_1}d_qx\right| \\
=&\left|\frac{q^2(\omega_2-\omega_1)^2}{1+q}\int_0^1 t(1-qt)_{\omega_1}D_q^2\mathfrak{F}((1-t)\omega_1+t\omega_2)_0d_qt\right| \\
\leq&\frac{q^2(\omega_2-\omega_1)^2}{1+q}\int_0^1 t(1-qt)\left|_{\omega_1}D_q^2\mathfrak{F}((1-t)\omega_1+t\omega_2)\right| {}_0d_qt \\
\leq&\frac{q^2(\omega_2-\omega_1)^2}{1+q}\left(\int_0^1(1-qt)^n {}_0d_qt\right)^{\frac{1}{n}}\left(\int_0^1 t^m\left|_{\omega_1}D_q^2\mathfrak{F}((1-t)\omega_1+t\omega_2)\right|^m {}_0d_qt\right)^{\frac{1}{m}} \\
\leq&\frac{q^2(\omega_2-\omega_1)^2}{1+q}\left(\int_0^1(1-qt)^n {}_0d_qt\right)^{\frac{1}{n}}\left(\left|_{\omega_1}D_q^2\mathfrak{F}(\omega_1)\right|^m\int_0^1 t^m {}_0d_qt\right. \\
&\left.+\varphi\left(\left|_{\omega_1}D_q^2\mathfrak{F}(\omega_2)\right|^m,\left|_{\omega_1}D_q^2\mathfrak{F}(\omega_1)\right|^m\right)\int_0^1 t^{m+1} {}_0d_qt\right)^{\frac{1}{m}},
\end{aligned}$$

and applying (4) in Definition 7, we have

$$\begin{aligned}
&\left|\frac{q\mathfrak{F}(\omega_1)+\mathfrak{F}(\omega_2)}{1+q}-\frac{1}{\omega_2-\omega_1}\int_{\omega_1}^{\omega_2}\mathfrak{F}(x)_{\omega_1}d_qx\right| \\
\leq&\frac{q^2(\omega_2-\omega_1)^2}{1+q}\left(\int_0^1(1-qt)^n {}_0d_qt\right)^{\frac{1}{n}}\left(\frac{\left|_{\omega_1}D_q^2\mathfrak{F}(\omega_1)\right|^m}{[m+1]}+\frac{\varphi\left(\left|_{\omega_1}D_q^2\mathfrak{F}(\omega_2)\right|^m,\left|_{\omega_1}D_q^2\mathfrak{F}(\omega_1)\right|^m\right)}{[m+2]}\right)^{\frac{1}{m}} \\
=&\frac{q^2(\omega_2-\omega_1)^2}{1+q}(M)^{\frac{1}{n}}\left(\frac{\left|_{\omega_1}D_q^2\mathfrak{F}(\omega_1)\right|^m}{[m+1]}+\frac{\varphi\left(\left|_{\omega_1}D_q^2\mathfrak{F}(\omega_2)\right|^m,\left|_{\omega_1}D_q^2\mathfrak{F}(\omega_1)\right|^m\right)}{[m+2]}\right)^{\frac{1}{m}}.
\end{aligned}$$

It is easy to check by Definition 4 that

$$M=\int_0^1(1-qt)^n {}_0d_qt=(1-q)\sum_{i=0}^{\infty}q^i(1-q^{i+1})^n,$$

thus, we get (10). ■

Remark 10 *By setting $\varphi(A,B)=A-B$ in Theorem 6, we recapture [23, Theorem 5.6].*

Corollary 9 *In Theorem 6, if $q\to 1$, then we have*

$$M=\int_0^1(1-t)^n dt=\frac{1}{n+1}$$

and (12) reduces to (10) in Corollary 7.

Remark 11 *If $\varphi(A,B)=A-B$ and $q\to 1$, then (12) reduces to (11) in Remark 9.*

Corollary 10 *In Theorem 6, if n is a positive integer, $n>1$, then*

$$(1-qt)^n\leq(1-qt)_q^n,$$

and (12) reduces to

$$\begin{aligned}
&\left|\frac{q\mathfrak{F}(\omega_1)+\mathfrak{F}(\omega_2)}{1+q}-\frac{1}{\omega_2-\omega_1}\int_{\omega_1}^{\omega_2}\mathfrak{F}(x)_{\omega_1}d_qx\right| \\
\leq&\frac{q^2(\omega_2-\omega_1)^2}{1+q}\left(\beta_q(1,n+1)\right)^{\frac{1}{n}}\left(\frac{\left|_{\omega_1}D_q^2\mathfrak{F}(\omega_1)\right|^m}{[m+1]}+\frac{\varphi\left(\left|_{\omega_1}D_q^2\mathfrak{F}(\omega_2)\right|^m,\left|_{\omega_1}D_q^2\mathfrak{F}(\omega_1)\right|^m\right)}{[m+2]}\right)^{\frac{1}{m}}.
\end{aligned}$$

Theorem 7 *Let $\mathfrak{F} : \mathcal{I} = [\omega_1, \omega_2] \subset \mathbb{R} \to \mathbb{R}$ be a twice q-differentiable function on $\mathcal{I}^0$ with ${}_{\omega_1}D_q^2\mathfrak{F}$ be continuous and integrable on $\mathcal{I}$ where $0 < q < 1$. If $|{}_{\omega_1}D_q^2\mathfrak{F}|^m$ is φ-convex on $[\omega_1, \omega_2]$ for $m \geq 1$, then*

$$\begin{aligned}
&\left|\frac{q\mathfrak{F}(\omega_1)+\mathfrak{F}(\omega_2)}{1+q} - \frac{1}{\omega_2-\omega_1}\int_{\omega_1}^{\omega_2}\mathfrak{F}(x)_{\omega_1}d_qx\right| \\
\leq\ & \frac{q^2(\omega_2-\omega_1)^2}{1+q}\left(V_1\left|{}_{\omega_1}D_q^2\mathfrak{F}(\omega_1)\right|^m + V_2\varphi\left(\left|{}_{\omega_1}D_q^2\mathfrak{F}(\omega_2)\right|^m, \left|{}_{\omega_1}D_q^2\mathfrak{F}(\omega_1)\right|^m\right)\right)^{\frac{1}{m}}, \qquad (13)
\end{aligned}$$

where

$$V_1 = \int_0^1 t^m(1-qt)^m\ {}_0d_qt = (1-q)\sum_{i=0}^{\infty}(q^i)^{m+1}(1-q^{i+1})^m$$

and

$$V_2 = \int_0^1 t^{m+1}(1-qt)^m\ {}_0d_qt = (1-q)\sum_{i=0}^{\infty}(q^i)^{m+2}(1-q^{i+1})^m.$$

Proof. Using Lemma 3, Hölder inequality and the fact that $|{}_{\omega_1}D_q^2\mathfrak{F}|^m$ is a φ-convex function, we have

$$\begin{aligned}
&\left|\frac{q\mathfrak{F}(\omega_1)+\mathfrak{F}(\omega_2)}{1+q} - \frac{1}{\omega_2-\omega_1}\int_{\omega_1}^{\omega_2}\mathfrak{F}(x)_{\omega_1}d_qx\right| \\
=\ & \left|\frac{q^2(\omega_2-\omega_1)^2}{1+q}\int_0^1 t(1-qt)_{\omega_1}D_q^2\mathfrak{F}((1-t)\omega_1+t\omega_2)_0d_qt\right| \\
\leq\ & \frac{q^2(\omega_2-\omega_1)^2}{1+q}\int_0^1 t(1-qt)\left|{}_{\omega_1}D_q^2\mathfrak{F}((1-t)\omega_1+t\omega_2)\right|\ {}_0d_qt \\
\leq\ & \frac{q^2(\omega_2-\omega_1)^2}{1+q}\left(\int_0^1\ {}_0d_qt\right)^{1-\frac{1}{m}}\left(\int_0^1 t^m(1-qt)^m\left|{}_{\omega_1}D_q^2\mathfrak{F}((1-t)\omega_1+t\omega_2)\right|^m\ {}_0d_qt\right)^{\frac{1}{m}} \\
\leq\ & \frac{q^2(\omega_2-\omega_1)^2}{1+q}\left(\int_0^1\ {}_0d_qt\right)^{1-\frac{1}{m}}\Bigg(\left|{}_{\omega_1}D_q^2\mathfrak{F}(\omega_1)\right|^m\int_0^1 t^m(1-qt)^m\ {}_0d_qt \\
&+\varphi\left(\left|{}_{\omega_1}D_q^2\mathfrak{F}(\omega_2)\right|^m, \left|{}_{\omega_1}D_q^2\mathfrak{F}(\omega_1)\right|^m\right)\int_0^1 t^{m+1}(1-qt)^m\ {}_0d_qt\Bigg)^{\frac{1}{m}},
\end{aligned}$$

and applying Lemma 1(a), we have

$$\begin{aligned}
&\left|\frac{q\mathfrak{F}(\omega_1)+\mathfrak{F}(\omega_2)}{1+q} - \frac{1}{\omega_2-\omega_1}\int_{\omega_1}^{\omega_2}\mathfrak{F}(x)_{\omega_1}d_qx\right| \\
\leq\ & \frac{q^2(\omega_2-\omega_1)^2}{1+q}\Bigg(\left|{}_{\omega_1}D_q^2\mathfrak{F}(\omega_1)\right|^m\int_0^1 t^m(1-qt)^m\ {}_0d_qt \\
&+\varphi\left(\left|{}_{\omega_1}D_q^2\mathfrak{F}(\omega_2)\right|^m, \left|{}_{\omega_1}D_q^2\mathfrak{F}(\omega_1)\right|^m\right)\int_0^1 t^{m+1}(1-qt)^m\ {}_0d_qt\Bigg)^{\frac{1}{m}} \\
=\ & \frac{q^2(\omega_2-\omega_1)^2}{1+q}\left(V_1\left|{}_{\omega_1}D_q^2\mathfrak{F}(\omega_1)\right|^m + V_2\varphi\left(\left|{}_{\omega_1}D_q^2\mathfrak{F}(\omega_2)\right|^m, \left|{}_{\omega_1}D_q^2\mathfrak{F}(\omega_1)\right|^m\right)\right)^{\frac{1}{m}}.
\end{aligned}$$

It is easy to check by Definition 4 that

$$V_1 = \int_0^1 t^m(1-qt)^m\ {}_0d_qt = (1-q)\sum_{i=0}^{\infty}(q^i)^{m+1}(1-q^{i+1})^m$$

and

$$V_2 = \int_0^1 t^{m+1}(1-qt)^m\ {}_0d_qt = (1-q)\sum_{i=0}^{\infty}(q^i)^{m+2}(1-q^{i+1})^m,$$

thus, we get (13). ■

Remark 12 *By setting $\varphi(A,B) = A - B$ in Theorem 7, we recapture [23, Theorem 5.3].*

Corollary 11 *In Theorem 7, if $q \to 1$, then we have*

$$V_1 = \int_0^1 t^m (1-t)^m dt = \beta(m+1, m+1), \quad V_2 = \int_0^1 t^{m+1}(1-t)^m dt = \beta(m+2, m+1),$$

and (13) reduces to the following inequality:

$$\begin{aligned} &\left| \frac{\mathfrak{F}(\omega_1) + \mathfrak{F}(\omega_2)}{2} - \frac{1}{\omega_2 - \omega_1} \int_{\omega_1}^{\omega_2} \mathfrak{F}(x)dx \right| \\ \leq\ & \frac{(\omega_2 - \omega_1)^2}{2} \Big(\beta(m+1, m+1) \left|\mathfrak{F}''(\omega_1)\right|^m + \beta(m+2, m+1)\varphi\left(\left|\mathfrak{F}''(\omega_2)\right|^m, \left|\mathfrak{F}''(\omega_1)\right|^m \right) \Big)^{\frac{1}{m}}. \end{aligned}$$

Remark 13 *If $\varphi(A,B) = A - B$ and $q \to 1$, then (13) reduces to the following inequality:*

$$\begin{aligned} &\left| \frac{\mathfrak{F}(\omega_1) + \mathfrak{F}(\omega_2)}{2} - \frac{1}{\omega_2 - \omega_1} \int_{\omega_1}^{\omega_2} \mathfrak{F}(x)dx \right| \\ \leq\ & \frac{(\omega_2 - \omega_1)^2}{2} \Big(\beta(m+1, m+2) \left|\mathfrak{F}''(\omega_1)\right|^m + \beta(m+2, m+1) \left|\mathfrak{F}''(\omega_2)\right|^m \Big)^{\frac{1}{m}}. \end{aligned}$$

Corollary 12 *In Theorem 7, if m is a positive integer, $m > 1$, then*

$$(1 - qt)^m \leq (1 - qt)_q^m,$$

and (13) reduces to

$$\begin{aligned} &\left| \frac{q\mathfrak{F}(\omega_1) + \mathfrak{F}(\omega_2)}{1+q} - \frac{1}{\omega_2 - \omega_1} \int_{\omega_1}^{\omega_2} \mathfrak{F}(x)_{\omega_1} d_q x \right| \\ \leq\ & \frac{q^2(\omega_2 - \omega_1)^2}{1+q} \Big(\beta_q(m+1, m+1) \left|{}_{\omega_1}D_q^2 \mathfrak{F}(\omega_1)\right|^m \\ & + \beta_q(m+2, m+1)\varphi\Big(\left|{}_{\omega_1}D_q^2 \mathfrak{F}(\omega_2)\right|^m, \left|{}_{\omega_1}D_q^2 \mathfrak{F}(\omega_1)\right|^m \Big) \Big)^{\frac{1}{m}}. \end{aligned}$$

Theorem 8 *Let $\mathfrak{F} : \mathcal{I} = [\omega_1, \omega_2] \subset \mathbb{R} \to \mathbb{R}$ be a twice q-differentiable function on $\mathcal{I}^0$ with ${}_{\omega_1}D_q^2 \mathfrak{F}$ be continuous and integrable on $\mathcal{I}$ where $0 < q < 1$. If $|{}_{\omega_1}D_q^2 \mathfrak{F}|^m$ is φ-convex on $[\omega_1, \omega_2]$ for $m \geq 1$, then*

$$\begin{aligned} &\left| \frac{q\mathfrak{F}(\omega_1) + \mathfrak{F}(\omega_2)}{1+q} - \frac{1}{\omega_2 - \omega_1} \int_{\omega_1}^{\omega_2} \mathfrak{F}(x)_{\omega_1} d_q x \right| \\ \leq\ & \frac{q^2(\omega_2 - \omega_1)^2}{1+q} \left(\frac{1}{1+q} \right)^{1-\frac{1}{m}} \Bigg(\beta_q(m+1, 2) \left|{}_{\omega_1}D_q^2 \mathfrak{F}(\omega_1)\right|^m \\ & + \beta_q(m+2, 2)\varphi\left(\left|{}_{\omega_1}D_q^2 \mathfrak{F}(\omega_2)\right|^m, \left|{}_{\omega_1}D_q^2 \mathfrak{F}(\omega_1)\right|^m \right) \Bigg)^{\frac{1}{m}}. \end{aligned} \tag{14}$$

Proof. Using Lemma 3, Hölder inequality and the fact that $|_{\omega_1}D_q^2\mathfrak{F}|^m$ is a φ-convex function, we have

$$\begin{aligned}
&\left|\frac{q\mathfrak{F}(\omega_1)+\mathfrak{F}(\omega_2)}{1+q}-\frac{1}{\omega_2-\omega_1}\int_{\omega_1}^{\omega_2}\mathfrak{F}(x)_{\omega_1}d_qx\right| \\
=& \left|\frac{q^2(\omega_2-\omega_1)^2}{1+q}\int_0^1 t(1-qt)_{\omega_1}D_q^2\mathfrak{F}((1-t)\omega_1+t\omega_2)_0d_qt\right| \\
\leq& \frac{q^2(\omega_2-\omega_1)^2}{1+q}\int_0^1 t(1-qt)\left|_{\omega_1}D_q^2\mathfrak{F}((1-t)\omega_1+t\omega_2)\right| {}_0d_qt \\
\leq& \frac{q^2(\omega_2-\omega_1)^2}{1+q}\left(\int_0^1(1-qt)\ {}_0d_qt\right)^{1-\frac{1}{m}}\left(\int_0^1 t^m(1-qt)\left|_{\omega_1}D_q^2\mathfrak{F}((1-t)\omega_1+t\omega_2)\right|^m {}_0d_qt\right)^{\frac{1}{m}} \\
\leq& \frac{q^2(\omega_2-\omega_1)^2}{1+q}\left(\int_0^1(1-qt)\ {}_0d_qt\right)^{1-\frac{1}{m}}\Bigg(\left|_{\omega_1}D_q^2\mathfrak{F}(\omega_1)\right|^m\int_0^1 t^m(1-qt)\ {}_0d_qt \\
&+\varphi\left(\left|_{\omega_1}D_q^2\mathfrak{F}(\omega_2)\right|^m,\left|_{\omega_1}D_q^2\mathfrak{F}(\omega_1)\right|^m\right)\int_0^1 t^{m+1}(1-qt)\ {}_0d_qt\Bigg)^{\frac{1}{m}},
\end{aligned}$$

and applying Lemma 2(a) and the fact that $(1-qt)=(1-qt)_q^1$, we have

$$\begin{aligned}
&\left|\frac{q\mathfrak{F}(\omega_1)+\mathfrak{F}(\omega_2)}{1+q}-\frac{1}{\omega_2-\omega_1}\int_{\omega_1}^{\omega_2}\mathfrak{F}(x)_{\omega_1}d_qx\right| \\
\leq& \frac{q^2(\omega_2-\omega_1)^2}{1+q}\left(\frac{1}{1+q}\right)^{1-\frac{1}{m}}\Big(\beta_q(m+1,2)\left|_{\omega_1}D_q^2\mathfrak{F}(\omega_1)\right|^m \\
&+\beta_q(m+2,2)\varphi\left(\left|_{\omega_1}D_q^2\mathfrak{F}(\omega_2)\right|^m,\left|_{\omega_1}D_q^2\mathfrak{F}(\omega_1)\right|^m\right)\Big)^{\frac{1}{m}},
\end{aligned}$$

thus, we get (14). ■

Remark 14 *By setting $\varphi(A,B)=A-B$ in Theorem 8, we recapture [23, Theorem 5.9].*

Corollary 13 *In Theorem 8, if $q\to 1$,*

$$\beta(m+1,2)=\int_0^1 t^m(1-t)dt=\frac{1}{(m+1)(m+2)}$$

and

$$\beta(m+2,2)=\int_0^1 t^{m+1}(1-t)dt=\frac{1}{(m+2)(m+3)},$$

then (14) reduces to

$$\begin{aligned}
&\left|\frac{\mathfrak{F}(\omega_1)+\mathfrak{F}(\omega_2)}{2}-\frac{1}{\omega_2-\omega_1}\int_{\omega_1}^{\omega_2}\mathfrak{F}(x)dx\right| \\
\leq& \frac{(\omega_2-\omega_1)^2}{4}\left(\frac{2}{(m+1)(m+2)(m+3)}\right)^{\frac{1}{m}}\Big((m+3)\left|\mathfrak{F}''(\omega_1)\right|^m \\
&+(m+1)\varphi\left(\left|\mathfrak{F}''(\omega_2)\right|^m,\left|\mathfrak{F}''(\omega_1)\right|^m\right)\Big)^{\frac{1}{m}}.
\end{aligned}$$

Remark 15 *If $\varphi(A,B)=A-B$ and $q\to 1$, then (14) reduces to the following inequality:*

$$\begin{aligned}
&\left|\frac{\mathfrak{F}(\omega_1)+\mathfrak{F}(\omega_2)}{2}-\frac{1}{\omega_2-\omega_1}\int_{\omega_1}^{\omega_2}\mathfrak{F}(x)dx\right| \\
\leq& \frac{(\omega_2-\omega_1)^2}{4}\left(\frac{2}{(m+1)(m+2)(m+3)}\right)^{\frac{1}{m}}\left(2\left|\mathfrak{F}''(\omega_1)\right|^m+(m+1)\left|\mathfrak{F}''(\omega_2)\right|^m\right)^{\frac{1}{m}}.
\end{aligned}$$

Theorem 9 *Let $\mathfrak{F}:\mathcal{I}=[\omega_1,\omega_2]\subset\mathbb{R}\to\mathbb{R}$ be a twice q-differentiable function on $\mathcal{I}^0$ with ${}_{\omega_1}D_q^2\mathfrak{F}$ be continuous and integrable on $\mathcal{I}$ where $0<q<1$. If $|{}_{\omega_1}D_q^2\mathfrak{F}|^m$ is φ-convex on $[\omega_1,\omega_2]$ where $n,m>1$, $\frac{1}{n}+\frac{1}{m}=1$, then*

$$\begin{aligned}&\left|\frac{q\mathfrak{F}(\omega_1)+\mathfrak{F}(\omega_2)}{1+q}-\frac{1}{\omega_2-\omega_1}\int_{\omega_1}^{\omega_2}\mathfrak{F}(x)_{\omega_1}d_qx\right|\\ \leq\;&\frac{q^2(\omega_2-\omega_1)^2}{1+q}\left(\beta_q(n+1,2)\right)^{\frac{1}{n}}\left(\frac{\left|{}_{\omega_1}D_q^2\mathfrak{F}(\omega_1)\right|^m}{1+q}+\frac{\varphi\left(\left|{}_{\omega_1}D_q^2\mathfrak{F}(\omega_2)\right|^m,\left|{}_{\omega_1}D_q^2\mathfrak{F}(\omega_1)\right|^m\right)}{(1+q)(1+q+q^2)}\right)^{\frac{1}{m}}.\end{aligned}\tag{15}$$

Proof. Using Lemma 3, Hölder inequality and the fact that $|{}_{\omega_1}D_q^2\mathfrak{F}|^m$ is a φ-convex function, we have

$$\begin{aligned}&\left|\frac{q\mathfrak{F}(\omega_1)+\mathfrak{F}(\omega_2)}{1+q}-\frac{1}{\omega_2-\omega_1}\int_{\omega_1}^{\omega_2}\mathfrak{F}(x)_{\omega_1}d_qx\right|\\ =\;&\left|\frac{q^2(\omega_2-\omega_1)^2}{1+q}\int_0^1 t(1-qt)_{\omega_1}D_q^2\mathfrak{F}((1-t)\omega_1+t\omega_2)_0d_qt\right|\\ \leq\;&\frac{q^2(\omega_2-\omega_1)^2}{1+q}\int_0^1 t(1-qt)\left|{}_{\omega_1}D_q^2\mathfrak{F}((1-t)\omega_1+t\omega_2)\right|\;{}_0d_qt\\ \leq\;&\frac{q^2(\omega_2-\omega_1)^2}{1+q}\left(\int_0^1 t^n(1-qt)\;{}_0d_qt\right)^{\frac{1}{n}}\left(\int_0^1(1-qt)\left|{}_{\omega_1}D_q^2\mathfrak{F}((1-t)\omega_1+t\omega_2)\right|^m\;{}_0d_qt\right)^{\frac{1}{m}}\\ \leq\;&\frac{q^2(\omega_2-\omega_1)^2}{1+q}\left(\int_0^1 t^n(1-qt)\;{}_0d_qt\right)^{\frac{1}{n}}\left(\left|{}_{\omega_1}D_q^2\mathfrak{F}(\omega_1)\right|^m\int_0^1(1-qt)\;{}_0d_qt\right.\\ &\left.+\varphi\left(\left|{}_{\omega_1}D_q^2\mathfrak{F}(\omega_2)\right|^m,\left|{}_{\omega_1}D_q^2\mathfrak{F}(\omega_1)\right|^m\right)\int_0^1 t(1-qt)\;{}_0d_qt\right)^{\frac{1}{m}},\end{aligned}$$

and applying Lemma 2(a) and (b) and the fact that $(1-qt)=(1-qt)_q^1$, we have

$$\begin{aligned}&\left|\frac{q\mathfrak{F}(\omega_1)+\mathfrak{F}(\omega_2)}{1+q}-\frac{1}{\omega_2-\omega_1}\int_{\omega_1}^{\omega_2}\mathfrak{F}(x)_{\omega_1}d_qx\right|\\ \leq\;&\frac{q^2(\omega_2-\omega_1)^2}{1+q}\left(\int_0^1 t^n(1-qt)\;{}_0d_qt\right)^{\frac{1}{n}}\left(\frac{\left|{}_{\omega_1}D_q^2\mathfrak{F}(\omega_1)\right|^m}{1+q}+\frac{\varphi\left(\left|{}_{\omega_1}D_q^2\mathfrak{F}(\omega_2)\right|^m,\left|{}_{\omega_1}D_q^2\mathfrak{F}(\omega_1)\right|^m\right)}{(1+q)(1+q+q^2)}\right)^{\frac{1}{m}}\\ =\;&\frac{q^2(\omega_2-\omega_1)^2}{1+q}\left(\beta_q(n+1,2)\right)^{\frac{1}{n}}\left(\frac{\left|{}_{\omega_1}D_q^2\mathfrak{F}(\omega_1)\right|^m}{1+q}+\frac{\varphi\left(\left|{}_{\omega_1}D_q^2\mathfrak{F}(\omega_2)\right|^m,\left|{}_{\omega_1}D_q^2\mathfrak{F}(\omega_1)\right|^m\right)}{(1+q)(1+q+q^2)}\right)^{\frac{1}{m}}.\end{aligned}$$

Hence, we get (15). ■

Remark 16 *By setting $\varphi(A,B)=A-B$ in Theorem 9, we recapture [23, Theorem 5.10].*

Corollary 14 *In Theorem 9, if $q\to 1$,*

$$\beta(n+1,2)=\int_0^1 t^n(1-t)dt=\frac{1}{(n+1)(n+2)},$$

and (15) reduces to

$$\begin{aligned}&\left|\frac{\mathfrak{F}(\omega_1)+\mathfrak{F}(\omega_2)}{2}-\frac{1}{\omega_2-\omega_1}\int_{\omega_1}^{\omega_2}\mathfrak{F}(x)dx\right|\\ \leq\;&\frac{(\omega_2-\omega_1)^2}{2}\left(\frac{1}{(n+1)(n+2)}\right)^{\frac{1}{n}}\left(\frac{3\left|\mathfrak{F}''(\omega_1)\right|^m+\varphi\left(\left|\mathfrak{F}''(\omega_2)\right|^m,\left|\mathfrak{F}''(\omega_1)\right|^m\right)}{6}\right)^{\frac{1}{m}}.\end{aligned}$$

Remark 17 *If $\varphi(A,B) = A - B$ and $q \to 1$, then (15) reduces to the following inequality:*

$$\left|\frac{\mathfrak{F}(\omega_1)+\mathfrak{F}(\omega_2)}{2} - \frac{1}{\omega_2-\omega_1}\int_{\omega_1}^{\omega_2}\mathfrak{F}(x)dx\right| \leq \frac{(\omega_2-\omega_1)^2}{2}\left(\frac{1}{(n+1)(n+2)}\right)^{\frac{1}{n}}\left(\frac{2\left|\mathfrak{F}''(\omega_1)\right|^m + \left|\mathfrak{F}''(\omega_2)\right|^m}{6}\right)^{\frac{1}{m}}.$$

Theorem 10 *Let $\mathfrak{F} : \mathcal{I} = [\omega_1, \omega_2] \subset \mathbb{R} \to \mathbb{R}$ be a twice q-differentiable function on $\mathcal{I}^0$ with ${}_{\omega_1}D_q^2\mathfrak{F}$ be continuous and integrable on $\mathcal{I}$ where $0 < q < 1$. If $|{}_{\omega_1}D_q^2\mathfrak{F}|$ is φ-convex on $[\omega_1, \omega_2]$, then*

$$\left|\frac{q\mathfrak{F}(\omega_1)+\mathfrak{F}(\omega_2)}{1+q} - \frac{1}{\omega_2-\omega_1}\int_{\omega_1}^{\omega_2}\mathfrak{F}(x)_{\omega_1}d_qx\right| \leq \frac{q^2(\omega_2-\omega_1)^2}{1+q}\left(\frac{(1+q+q^2+q^3)\left|{}_{\omega_1}D_q^2\mathfrak{F}(\omega_1)\right| + (1+q)\varphi\left(\left|{}_{\omega_1}D_q^2\mathfrak{F}(\omega_2)\right|, \left|{}_{\omega_1}D_q^2\mathfrak{F}(\omega_1)\right|\right)}{(1+q)(1+q+q^2)(1+q+q^2+q^3)}\right). \quad (16)$$

Proof. Using Lemma 3, Hölder inequality and the fact that $|{}_{\omega_1}D_q^2\mathfrak{F}|$ is a φ-convex function, we have

$$\begin{aligned}
&\left|\frac{q\mathfrak{F}(\omega_1)+\mathfrak{F}(\omega_2)}{1+q} - \frac{1}{\omega_2-\omega_1}\int_{\omega_1}^{\omega_2}\mathfrak{F}(x)_{\omega_1}d_qx\right| \\
= &\left|\frac{q^2(\omega_2-\omega_2)^2}{1+q}\int_0^1 t(1-qt)_{\omega_1}D_q^2\mathfrak{F}((1-t)\omega_1+t\omega_2)_0d_qt\right| \\
\leq &\frac{q^2(\omega_2-\omega_1)^2}{1+q}\int_0^1 t(1-qt)\left|{}_{\omega_1}D_q^2\mathfrak{F}((1-t)\omega_1+t\omega_2)\right| {}_0d_qt \\
\leq &\frac{q^2(\omega_2-\omega_1)^2}{1+q}\left(\left|{}_{\omega_1}D_q^2\mathfrak{F}(\omega_1)\right|\int_0^1 t(1-qt)\ {}_0d_qt + \varphi\left(\left|{}_{\omega_1}D_q^2\mathfrak{F}(\omega_2)\right|, \left|{}_{\omega_1}D_q^2\mathfrak{F}(\omega_1)\right|\right)\int_0^1 t^2(1-qt)\ {}_0d_qt\right),
\end{aligned}$$

applying Lemma 2(b) and (c), we have

$$\left|\frac{q\mathfrak{F}(\omega_1)+\mathfrak{F}(\omega_2)}{1+q} - \frac{1}{\omega_2-\omega_1}\int_{\omega_1}^{\omega_2}\mathfrak{F}(x)_{\omega_1}d_qx\right| \leq \frac{q^2(\omega_2-\omega_1)^2}{1+q}\left(\frac{(1+q+q^2+q^3)\left|{}_{\omega_1}D_q^2\mathfrak{F}(\omega_1)\right| + (1+q)\varphi\left(\left|{}_{\omega_1}D_q^2\mathfrak{F}(\omega_2)\right|, \left|{}_{\omega_1}D_q^2\mathfrak{F}(\omega_1)\right|\right)}{(1+q)(1+q+q^2)(1+q+q^2+q^3)}\right),$$

thus, we get (16). ■

Remark 18 *By setting $\varphi(A,B) = A - B$ in Theorem 10, we recapture [23, Theorem 5.7].*

Corollary 15 *In Theorem 10, if $q \to 1$, then (16) reduces to the following inequality:*

$$\left|\frac{\mathfrak{F}(\omega_1)+\mathfrak{F}(\omega_2)}{2} - \frac{1}{\omega_2-\omega_1}\int_{\omega_1}^{\omega_2}\mathfrak{F}(x)dx\right| \leq \frac{(\omega_2-\omega_1)^2\left(2\left|\mathfrak{F}''(\omega_1)\right| + \varphi\left(\left|\mathfrak{F}''(\omega_2)\right|, \left|\mathfrak{F}''(\omega_1)\right|\right)\right)}{24}.$$

Remark 19 *If $\varphi(A,B) = A - B$ and $q \to 1$, then (16) reduces to the following inequality:*

$$\left|\frac{\mathfrak{F}(\omega_1)+\mathfrak{F}(\omega_2)}{2} - \frac{1}{\omega_2-\omega_1}\int_{\omega_1}^{\omega_2}\mathfrak{F}(x)dx\right| \leq \frac{(\omega_2-\omega_1)^2\left(\left|\mathfrak{F}''(\omega_1)\right| + \left|\mathfrak{F}''(\omega_2)\right|\right)}{24}.$$

Theorem 11 *Let $\mathfrak{F} : \mathcal{I} = [\omega_1, \omega_2] \subset \mathbb{R} \to \mathbb{R}$ be a twice q-differentiable function on $\mathcal{I}^0$ with ${}_{\omega_1}D_q^2\mathfrak{F}$ be continuous and integrable on $\mathcal{I}$ where $0 < q < 1$. If $|{}_{\omega_1}D_q^2\mathfrak{F}|^m$ is φ-convex on $[\omega_1, \omega_2]$ for $m \geq 1$, then*

$$\left|\frac{q\mathfrak{F}(\omega_1)+\mathfrak{F}(\omega_2)}{1+q} - \frac{1}{\omega_2-\omega_1}\int_{\omega_1}^{\omega_2}\mathfrak{F}(x)_{\omega_1}d_qx\right| \leq \frac{q^2(\omega_2-\omega_1)^2}{(1+q)^2(1+q+q^2)}\left(\frac{(1+q+q^2+q^3)\left|{}_{\omega_1}D_q^2\mathfrak{F}(\omega_1)\right|^m}{1+q+q^2+q^3} + \frac{(1+q)\varphi\left(\left|{}_{\omega_1}D_q^2\mathfrak{F}(\omega_2)\right|^m, \left|{}_{\omega_1}D_q^2\mathfrak{F}(\omega_1)\right|^m\right)}{1+q+q^2+q^3}\right)^{\frac{1}{m}}. \quad (17)$$

Proof. Using Lemma 3, Hölder inequality and the fact that $|{}_{\omega_1}D_q^2\mathfrak{F}|^m$ is a φ-convex function, we have

$$\begin{aligned}
&\left|\frac{q\mathfrak{F}(\omega_1)+\mathfrak{F}(\omega_2)}{1+q} - \frac{1}{\omega_2-\omega_1}\int_{\omega_1}^{\omega_2}\mathfrak{F}(x)_{\omega_1}d_qx\right| \\
=& \left|\frac{q^2(\omega_2-\omega_1)^2}{1+q}\int_0^1 t(1-qt)_{\omega_1}D_q^2\mathfrak{F}((1-t)\omega_1+t\omega_2)_0d_qt\right| \\
\leq& \frac{q^2(\omega_2-\omega_1)^2}{1+q}\int_0^1 t(1-qt)\left|{}_{\omega_1}D_q^2\mathfrak{F}((1-t)\omega_1+t\omega_2)\right| {}_0d_qt \\
\leq& \frac{q^2(\omega_2-\omega_1)^2}{1+q}\left(\int_0^1 t(1-qt)\ {}_0d_qt\right)^{1-\frac{1}{m}}\left(\int_0^1 t(1-qt)\left|{}_{\omega_1}D_q^2\mathfrak{F}((1-t)\omega_1+t\omega_2)\right|^m\ {}_0d_qt\right)^{\frac{1}{m}} \\
\leq& \frac{q^2(\omega_2-\omega_1)^2}{1+q}\left(\int_0^1 t(1-qt)\ {}_0d_qt\right)^{1-\frac{1}{m}}\left(\left|{}_{\omega_1}D_q^2\mathfrak{F}(\omega_1)\right|^m\int_0^1 t(1-qt)\ {}_0d_qt\right. \\
&\left.+\varphi\left(\left|{}_{\omega_1}D_q^2\mathfrak{F}(\omega_2)\right|^m, \left|{}_{\omega_1}D_q^2\mathfrak{F}(\omega_1)\right|^m\right)\int_0^1 t^2(1-qt)\ {}_0d_qt\right)^{\frac{1}{m}}
\end{aligned}$$

and applying Lemma 2(b) and (c), we have

$$\begin{aligned}
&\left|\frac{q\mathfrak{F}(\omega_1)+\mathfrak{F}(\omega_2)}{1+q} - \frac{1}{\omega_2-\omega_1}\int_{\omega_1}^{\omega_2}\mathfrak{F}(x)_{\omega_1}d_qx\right| \\
\leq& \frac{q^2(\omega_2-\omega_1)^2}{1+q}\left(\frac{1}{(1+q)(1+q+q^2)}\right)^{1-\frac{1}{m}}\left(\frac{\left|{}_{\omega_1}D_q^2\mathfrak{F}(\omega_1)\right|^m}{(1+q)(1+q+q^2)} + \frac{\varphi\left(\left|{}_{\omega_1}D_q^2\mathfrak{F}(\omega_2)\right|^m, \left|{}_{\omega_1}D_q^2\mathfrak{F}(\omega_1)\right|^m\right)}{(1+q+q^2)(1+q+q^2+q^3)}\right)^{\frac{1}{m}} \\
=& \frac{q^2(\omega_2-\omega_1)^2}{(1+q)^2(1+q+q^2)}\left(\frac{(1+q+q^2+q^3)\left|{}_{\omega_1}D_q^2\mathfrak{F}(\omega_1)\right|^m + (1+q)\varphi\left(\left|{}_{\omega_1}D_q^2\mathfrak{F}(\omega_2)\right|^m, \left|{}_{\omega_1}D_q^2\mathfrak{F}(\omega_1)\right|^m\right)}{1+q+q^2+q^3}\right)^{\frac{1}{m}}.
\end{aligned}$$

Thus, we get (17). ■

Remark 20 *By setting $\varphi(A, B) = A - B$ in Theorem 11, we recapture [23, Theorem 5.8].*

Corollary 16 *In Theorem 11, if $q \to 1$, then (17) reduces to the following inequality:*

$$\left|\frac{\mathfrak{F}(\omega_1)+\mathfrak{F}(\omega_2)}{2} - \frac{1}{\omega_2-\omega_1}\int_{\omega_1}^{\omega_2}\mathfrak{F}(x)dx\right| \leq \frac{(\omega_2-\omega_1)^2}{12}\left(\frac{2\left|\mathfrak{F}''(\omega_1)\right|^m + \varphi\left(\left|\mathfrak{F}''(\omega_2)\right|^m, \left|\mathfrak{F}''(\omega_1)\right|^m\right)}{2}\right)^{\frac{1}{m}}.$$

Remark 21 *If $\varphi(A, B) = A - B$ and $q \to 1$, then (17) reduces to the following inequality:*

$$\left|\frac{\mathfrak{F}(\omega_1)+\mathfrak{F}(\omega_2)}{2} - \frac{1}{\omega_2-\omega_1}\int_{\omega_1}^{\omega_2}\mathfrak{F}(x)dx\right| \leq \frac{(\omega_2-\omega_1)^2}{12}\left(\frac{\left|\mathfrak{F}''(\omega_1)\right|^m + \left|\mathfrak{F}''(\omega_2)\right|^m}{2}\right)^{\frac{1}{m}}.$$

4 Conclusion

Quantum calculus has large applications in many mathematical areas such as number theory, special functions, quantum mechanics and mathematical inequalities. In this paper, develop some quantum estimates of Hermite-Hadamard type inequalities for φ-convex functions. Theses results in some special cases recapture the known results. We hope that our results may be helpful for further study.

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