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New Estimates Related To The Ratios Of Gamma Functions*

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Abstract

The aim of this work is to give new estimates to the ratios of gamma functions P_1 and P_2 given by Mortici, Cristea and Lu [Completely monotonic functions and inequalities associated to some ratio of gamma function, *Appl. Math. Comp.* Vol. 240, (2014), 168-174] and to show the lower and upper bounds for P_1 and P_2.

1 Introduction and Motivation

The following ratios

$$P_1 = \frac{1 \cdot 4 \cdot 7 \cdot \dots \cdot (3n-2)}{3 \cdot 6 \cdot \dots \cdot (3n)} = \frac{\Gamma\left(n+\frac{1}{3}\right)}{\Gamma(n+1)\,\Gamma\left(\frac{1}{3}\right)}, \tag{1}$$

and

$$P_2 = \frac{2 \cdot 5 \cdot 8 \cdot \dots \cdot (3n-1)}{3 \cdot 6 \cdot \dots \cdot (3n)} = \frac{\Gamma\left(n+\frac{2}{3}\right)}{\Gamma(n+1)\,\Gamma\left(\frac{2}{3}\right)}, \tag{2}$$

defined for all integers $n = 1, 2, 3 \cdots$, have an important role in pure mathematics or applied statistics, statistical physics and they are closely related to the gamma function given by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\,dt,$$

for all real numbers $x > 0$, see [1]. There are many mathematicians who have obtained results related to P_1 and P_2 and we can mention them: Mortici, Cristea and Lu [19], Lin, Deng and Chen [15], Farhangdoost and Kargar Dolatabadi [13], Bai, Dong and Liu [3], Cao and Wang [4], Cao, Tanigawa and Zhai [5], Chen [6], Zhang and Wang [20], Deng, Tao and Chen [12], Chen and Paris [7]–[9], You, Huang and Liu [23], Crînganu [10]–[11], Yang and Tian [22], You [24].

Mortici, Cristea and Lu [19] obtained, in 2014, the following inequalities, for all integers $n \geq 1$:

$$\frac{\frac{\sqrt{3}}{2\pi}\Gamma\left(\frac{2}{3}\right)}{\sqrt[3]{n^2+\frac{1}{3}n}} \exp\left(-\frac{2}{81n^2}\right) < P_1 < \frac{\frac{\sqrt{3}}{2\pi}\Gamma\left(\frac{2}{3}\right)}{\sqrt[3]{n^2+\frac{1}{3}n}} \exp\left(-\frac{2}{81n^2}+\frac{2}{243n^3}\right),$$

and

$$\frac{\frac{1}{\Gamma\left(\frac{2}{3}\right)}}{\sqrt[3]{n+\frac{1}{3}}} \exp\left(-\frac{1}{81n^2}\right) < P_2 < \frac{\frac{1}{\Gamma\left(\frac{2}{3}\right)}}{\sqrt[3]{n+\frac{1}{3}}} \exp\left(-\frac{1}{81n^2}+\frac{2}{243n^3}\right).$$

Motivated by Mortici, Cristea and Lu, Chen and Paris, Lin, Deng, You, Crînganu, Cao, Wang, Huang, Liu, Bai and Dong, I intend to show a double inequality related to P_1 and P_2 and to propose the following approximations for the ratios P_1 and P_2:

$$P_1 \approx \frac{\frac{\sqrt{3}}{2\pi}\Gamma\left(\frac{2}{3}\right)}{\sqrt[3]{n^2+\frac{1}{3}n+\frac{2}{27}}} \exp\left(\frac{4}{2187n^4}-\frac{11}{6561n^5}\right) \tag{3}$$

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and

$$P_2 \approx \frac{\frac{1}{\Gamma\left(\frac{2}{3}\right)}}{\sqrt[3]{n+\frac{1}{3}+\frac{1}{27n}}} \exp\left(\frac{1}{243n^3}-\frac{17}{2187n^4}\right). \tag{4}$$

In this work, we want to establish the lower and upper bounds for P_1 and P_2 using the above approximations.

2 The Results

Let us consider P_1 and P_2 given by (1) and (2). We propose the following approximation:

$$P_1 \approx \frac{\frac{\sqrt{3}}{2\pi}\Gamma\left(\frac{2}{3}\right)}{\sqrt[3]{n^2+\frac{1}{3}n+\frac{2}{27}}} \tag{5}$$

and

$$P_2 \approx \frac{\frac{1}{\Gamma\left(\frac{2}{3}\right)}}{\sqrt[3]{n+\frac{1}{3}+\frac{1}{27n}}}. \tag{6}$$

The above approximation (5) is obtained by considering the following classes of approximation

$$P_1 \approx \frac{\frac{\sqrt{3}}{2\pi}\Gamma\left(\frac{2}{3}\right)}{\sqrt[3]{n^2+an+b}}, \tag{7}$$

where a, b are real parameters. To find the best approximation (7), we set the relative error sequence v_n by the following formulas, for every integer $n \geq 1$

$$P_1 \approx \frac{\frac{\sqrt{3}}{2\pi}\Gamma\left(\frac{2}{3}\right)}{\sqrt[3]{n^2+an+b}} \exp\left(v_n\right)$$

and we take into account an approximation (7) better as the speed of convergence of v_n is higher. We obtain:

$$v_n - v_{n+1} = \ln\frac{3n+3}{3n+1} + \frac{1}{3}\ln\frac{n^2+an+b}{(n+1)^2+a(n+1)+b}$$

and using Maple software

$$\begin{aligned} v_n - v_{n+1} \quad = \quad & \left(\frac{1}{3}a-\frac{1}{9}\right)\frac{1}{n^2} + \left(\frac{2}{3}b-\frac{1}{3}a-\frac{1}{3}a^2+\frac{8}{81}\right)\frac{1}{n^3} \\ & + \left(\frac{1}{3}a-b-ab+\frac{1}{2}a^2+\frac{1}{3}a^3-\frac{13}{162}\right)\frac{1}{n^4} \\ & + \left(\frac{4}{3}b-\frac{1}{3}a+2ab-\frac{2}{3}a^2-\frac{2}{3}a^3-\frac{2}{3}b^2-\frac{1}{3}a^4+\frac{4}{3}a^2b+\frac{16}{243}\right)\frac{1}{n^5} \\ & +(\frac{1}{3}a-\frac{5}{3}b-\frac{10}{3}ab+\frac{5}{6}a^2+\frac{10}{9}a^3+\frac{5}{3}b^2+\frac{5}{6}a^4 \\ & +\frac{1}{3}a^5+\frac{5}{3}ab^2-\frac{10}{3}a^2b-\frac{5}{3}a^3b-\frac{121}{2187})\frac{1}{n^6} + O\left(\frac{1}{n^7}\right). \end{aligned} \tag{8}$$

The sequence v_n is the fastest possible when $v_n - v_{n+1}$ is the fastest possible; that is when the first three coefficients in (8) are zero [19]. We obtain $a = \frac{1}{3}$, $b = \frac{2}{27}$. Then, (8) is written in this form

$$v_n - v_{n+1} = \frac{16}{2187n^5} - \frac{160}{6561n^6} + O\left(\frac{1}{n^7}\right).$$

As we discussed, (5) is the best approximation among all approximations (7). Now, we present the bounds for P_1 related to the approximation (3):

Theorem 1 *We have the following double inequality for every integer* $n \geq 1$

$$\left(\frac{\frac{\sqrt{3}}{2\pi}\Gamma\left(\frac{2}{3}\right)}{\sqrt[3]{n^2+\frac{1}{3}n+\frac{2}{27}}}\right)\exp\left(\frac{4}{2187n^4}-\frac{11}{6561n^5}\right) < P_1 < \left(\frac{\frac{\sqrt{3}}{2\pi}\Gamma\left(\frac{2}{3}\right)}{\sqrt[3]{n^2+\frac{1}{3}n+\frac{2}{27}}}\right)\cdot\exp\left(\frac{4}{2187n^4}\right). \tag{9}$$

Proof. Double inequality (9) is equivalent to $a_n > 0$ for left side inequality and $b_n < 0$ for the right side inequality, for every integer $n \geq 1$, where

$$a_n = \ln P_1 - \ln\left(\frac{\sqrt{3}}{2\pi}\Gamma\left(\frac{2}{3}\right)\right) + \frac{1}{3}\ln\left(n^2+\frac{1}{3}n+\frac{2}{27}\right) - \frac{4}{2187n^4} + \frac{11}{6561n^5}$$

and

$$b_n = \ln P_1 - \ln\left(\frac{\sqrt{3}}{2\pi}\Gamma\left(\frac{2}{3}\right)\right) + \frac{1}{3}\ln\left(n^2+\frac{1}{3}n+\frac{2}{27}\right) - \frac{4}{2187n^4}.$$

As a_n and b_n converge to 0, it suffices to prove that a_n is strictly decreasing and b_n is strictly increasing, for every integer $n \geq 1$. Then, we get $a_{n+1} - a_n = f(n)$ and $b_{n+1} - b_n = g(n)$, where

$$\begin{aligned} f(x) &= \ln\frac{3x+1}{3x+3} + \frac{1}{3}\ln\frac{(x+1)^2+\frac{1}{3}(x+1)+\frac{2}{27}}{x^2+\frac{1}{3}x+\frac{2}{27}} \\ &\quad -\frac{4}{2187(x+1)^4} + \frac{4}{2187x^4} + \frac{11}{6561(x+1)^5} - \frac{11}{6561x^5} \end{aligned}$$

and

$$g(x) = \ln\frac{3x+1}{3x+3} + \frac{1}{3}\ln\frac{(x+1)^2+\frac{1}{3}(x+1)+\frac{2}{27}}{x^2+\frac{1}{3}x+\frac{2}{27}} - \frac{4}{2187(x+1)^4} + \frac{4}{2187x^4}.$$

Then, we get

$$f'(x) = \frac{P(x)}{6561(x+1)^6(9x+27x^2+2)^1(63x+27x^2+38)^1(3x+1)^1x^6} > 0$$

and

$$g'(x) = -\frac{16Q(x)}{2187(x+1)^5(9x+27x^2+2)^1(63x+27x^2+38)^1(3x+1)^1x^5} < 0,$$

where

$$\begin{aligned} P(x) &= 196\,830x^{10} + 742\,365x^9 + 1776\,735x^8 + 4005\,594x^7 \\ &\quad +6579\,606x^6 + 6703\,512x^5 + 4195\,032x^4 + 1641\,689x^3 \\ &\quad +404\,889x^2 + 59\,712x + 4180 \end{aligned}$$

and

$$\begin{aligned} Q(x) &= 10\,935x^8 + 56\,295x^7 + 116\,910x^6 + 126\,897x^5 \\ &\quad +78\,836x^4 + 29\,860x^3 + 7291x^2 + 1076x + 76 \end{aligned}$$

are two polynomials with all coefficients positive numbers, for all real numbers $x \geq 1$. Thus, f is strictly increasing on $[1, \infty)$ and g is strictly decreasing on $[1, \infty)$, with $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = 0$, so $f < 0$ and $g > 0$ on $[1, \infty)$. The proof is completed. ■

The above approximation (6) is obtained by considering the following classes of approximation

$$P_2 \approx \frac{\frac{1}{\Gamma\left(\frac{2}{3}\right)}}{\sqrt[3]{n+a+\frac{b}{n}}}, \tag{10}$$

where a, b are real parameters. To find the best approximation (10), we set the relative error sequence v_n by the following formulas, for every integer $n \geq 1$:

$$P_2 \approx \frac{\frac{1}{\Gamma\left(\frac{2}{3}\right)}}{\sqrt[3]{n+a+\frac{b}{n}}} \exp\left(v_n\right)$$

and we take into account an approximation (10) better as the speed of convergence of v_n is higher. We obtain:

$$v_n - v_{n+1} = \ln \frac{3n+3}{3n+2} + \frac{1}{3} \ln \frac{n+a+\frac{b}{n}}{(n+1)+a+\frac{b}{(n+1)}}$$

and using Maple software

$$\begin{aligned} v_n - v_{n+1} &= \left(\frac{1}{3}a - \frac{1}{9}\right)\frac{1}{n^2} + \left(\frac{2}{3}b - \frac{1}{3}a - \frac{1}{3}a^2 + \frac{10}{81}\right)\frac{1}{n^3} \\ &+ \left(\frac{1}{3}a - b - ab + \frac{1}{2}a^2 + \frac{1}{3}a^3 - \frac{19}{162}\right)\frac{1}{n^4} \\ &+ \left(\frac{4}{3}b - \frac{1}{3}a + 2ab - \frac{2}{3}a^2 - \frac{2}{3}a^3 - \frac{2}{3}b^2 - \frac{1}{3}a^4 + \frac{4}{3}a^2 b + \frac{26}{243}\right)\frac{1}{n^5} \\ &+ O\left(\frac{1}{n^6}\right). \end{aligned} \tag{11}$$

The sequence v_n is the fastest possible when $v_n - v_{n+1}$ is the fastest possible; that is when the first two coefficients in (11) are zero [16]. We obtain $a = \frac{1}{3}$, $b = \frac{1}{27}$. Then, (11) is written in this form

$$v_n - v_{n+1} = \frac{1}{81n^4} - \frac{62}{2187n^5} + O\left(\frac{1}{n^6}\right).$$

As we argued, (6) is the best approximation among all approximations (10). Now, we introduce the bounds for P_2 related to this approximation (4):

Theorem 2 *We have the following double inequality for every integer $n \geq 1$*

$$\left(\frac{\frac{1}{\Gamma\left(\frac{2}{3}\right)}}{\sqrt[3]{n+\frac{1}{3}+\frac{1}{27n}}}\right) \exp\left(\frac{1}{243n^3} - \frac{17}{2187n^4}\right) < P_2 < \left(\frac{\frac{1}{\Gamma\left(\frac{2}{3}\right)}}{\sqrt[3]{n+\frac{1}{3}+\frac{1}{27n}}}\right) \cdot \exp\left(\frac{1}{243n^3}\right). \tag{12}$$

Proof. Double inequality (12) is equivalent to $a_n > 0$ for left side inequality and $b_n < 0$ for the right side inequality, for every integer $n \geq 1$, where

$$a_n = \ln P_2 - \ln\left(\frac{1}{\Gamma\left(\frac{2}{3}\right)}\right) + \frac{1}{3}\ln\left(n + \frac{1}{3} + \frac{1}{27n}\right) - \frac{1}{243n^3} + \frac{17}{2187n^4}$$

and

$$b_n = \ln P_2 - \ln\left(\frac{1}{\Gamma\left(\frac{2}{3}\right)}\right) + \frac{1}{3}\ln\left(n + \frac{1}{3} + \frac{1}{27n}\right) - \frac{1}{243n^3}.$$

As a_n and b_n converge to 0, it suffices to prove that a_n is strictly decreasing and b_n is strictly increasing, for every integer $n \geq 1$. Then, we get $a_{n+1} - a_n = f(n)$ and $b_{n+1} - b_n = g(n)$, where

$$\begin{aligned} f(x) &= \ln\frac{3x+2}{3x+3} + \frac{1}{3}\ln\frac{(x+1)+\frac{1}{3}+\frac{1}{27(x+1)}}{x+\frac{1}{3}+\frac{1}{27x}} \\ &- \frac{1}{243\left(x+1\right)^3} + \frac{1}{243x^3} + \frac{17}{2187\left(x+1\right)^4} - \frac{17}{2187x^4} \end{aligned}$$

and

$$g(x) = \ln \frac{3x+2}{3x+3} + \frac{1}{3} \ln \frac{(x+1) + \frac{1}{3} + \frac{1}{27(x+1)}}{x + \frac{1}{3} + \frac{1}{27x}} - \frac{1}{243\,(x+1)^3} + \frac{1}{243x^3}.$$

Then, we get

$$f'(x) = \frac{P(x)}{2187\,(x+1)^5\,(9x+27x^2+1)^1\,(63x+27x^2+37)^1\,(3x+2)^1\,x^5} > 0$$

and

$$g'(x) = -\frac{Q(x)}{81\,(x+1)^4\,(9x+27x^2+1)^1\,(63x+27x^2+37)^1\,(3x+2)^1\,x^4} < 0,$$

where

$$\begin{aligned} P(x) &= 656\,100x^9 + 3324\,240x^8 + 7548\,957x^7 + 10\,361\,763x^6 + 9563\,889x^5 \\ &\quad +6021\,677x^4 + 2480\,893x^3 + 620\,401x^2 + 84\,566x + 5032 \end{aligned}$$

and

$$Q(x) = 3240x^7 + 20\,520x^6 + 47\,709x^5 + 53\,022x^4 + 29\,879x^3 + 8430x^2 + 1199x + 74$$

are two polynomials with all coefficients positive numbers, for all real numbers $x \geq 1$. Thus, f is strictly increasing on $[1, \infty)$ and g is strictly decreasing on $[1, \infty)$, with $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = 0$, so $f < 0$ and $g > 0$ on $[1, \infty)$. The proof is completed. ■

3 Comparison Test

For P_1, let us consider the following sequences given by Mortici, Cristea and Lu

$$r_n^1 = \frac{\frac{\sqrt{3}}{2\pi}\Gamma\left(\frac{2}{3}\right)}{\sqrt[3]{n^2 + \frac{1}{3}n}} \exp\left(-\frac{2}{81n^2} + \frac{2}{243n^3}\right)$$

and given by Cristea

$$c_n^1 = \frac{\frac{\sqrt{3}}{2\pi}\Gamma\left(\frac{2}{3}\right)}{\sqrt[3]{n^2 + \frac{1}{3}n + \frac{2}{27}}} \exp\left(\frac{4}{2187n^4} - \frac{11}{6561n^5}\right).$$

We give the following table with the above sequences:

n	$\lvert P_1 - r_n^1 \rvert$	$\lvert P_1 - c_n^1 \rvert$
1	$2.790\,21 \times 10^{-4}$	$1.907\,37 \times 10^{-4}$
100	$1.593\,11 \times 10^{-15}$	$7.817\,44 \times 10^{-16}$
1000	$3.415\,54 \times 10^{-21}$	$1.704\,53 \times 10^{-21}$
10000	$7.865\,07 \times 10^{-27}$	$3.166\,62 \times 10^{-27}$

Using the values from the above table, we conclude the superiority of the Cristea's sequence $\left(c_n^1\right)_{n\geq 1}$ over Mortici, Cristea and Lu's sequence $\left(r_n^1\right)_{n\geq 1}$.

For P_2, let us consider the following sequences given by Mortici, Cristea and Lu

$$r_n^2 = \frac{\frac{1}{\Gamma\left(\frac{2}{3}\right)}}{\sqrt[3]{n + \frac{1}{3}}} \exp\left(-\frac{1}{81n^2} + \frac{2}{243n^3}\right)$$

and given by Cristea

$$c_n^2 = \left(\frac{\frac{1}{\Gamma\left(\frac{2}{3}\right)}}{\sqrt[3]{n + \frac{1}{3} + \frac{1}{27n}}} \right) \exp\left(\frac{1}{243n^3} - \frac{17}{2187n^4} \right).$$

We give the following table with the above sequences:

n	$\lvert P_2 - r_n^2 \rvert$	$\lvert P_2 - c_n^2 \rvert$
1	$6.659\,91 \times 10^{-3}$	$4.233\,28 \times 10^{-3}$
100	$1.962\,01 \times 10^{-7}$	$1.088\,10 \times 10^{-11}$
1000	$9.116\,12 \times 10^{-10}$	$5.063\,61 \times 10^{-16}$
10000	$4.231\,75 \times 10^{-12}$	$2.350\,93 \times 10^{-20}$

Using the values from the above table, we conclude the superiority of the Cristea's sequence $\left(c_n^2\right)_{n \geq 1}$ over Mortici, Cristea and Lu's sequence $\left(r_n^2\right)_{n \geq 1}$.

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