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On Characterizations Of Continuous Distributions By Independence Property Of The Quotient-Type k-th Lower Record Value*

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Abstract

In this paper, we obtain characterizations of a family of continuous probability distributions by independence property of k-th lower record values. Examples of special cases of general classes as power, inverse functions, reflected exponential, Gumbel, and Burr type distributions are discussed.

1 Introduction

Record values are found in many situations of daily life as well as in many statistical applications. Often we are interested in observing new records and in recording them. Many researchers have studied the characterizations of varius probability distributions of upper and lower record values.

Now some notations and definitions. Let $\{X_n, n \geq 1\}$ be a sequence of independent identically distributed (i.i.d.) random variables with cumulative distribution function (cdf) $F(x)$ and probability density function (pdf) $f(x)$. The j-th order statistic of a sample $(X_1, X_2, \cdots, X_n)$ is denoted by $X_{j:n}$. For a fixed positive integer k, Dziubdziela and Kopociński [3] defined the sequences $\{L^{(k)}(n), n \geq 1\}$ of k-th lower record times of $\{X_n, n \geq 1\}$ as follows:

$$L^{(k)}(1) = 1,$$
$$L^{(k)}(n+1) = \min\{j > L^{(k)}(n) : X_{k:L^{(k)}(n)+k-1} > X_{k:j+k-1}\}.$$

The sequences $\{Y_n^{(k)}, n \geq 1\}$ with $Y_n^{(k)} = X_{k:L^{(k)}(n)+k-1}$, $n = 1, 2, \cdots$, are called the sequences of k-th lower record values of $\{X_n, n \geq 1\}$. For convenience, we shall also take $Y_0^{(k)} = 0$. Note that $k = 1$ we have $Y_n^{(1)} = X_{L(n)}$, $n \geq 1$, which are record value of $\{X_n, n \geq 1\}$. Moreover $Y_1^{(k)} = X_{k:k} = \max\{X_1, X_2, \cdots, X_k\}$. For more details and references, see Nevzorov [9] and Ahsanullah et al. [2].

Let $\{Y_n^{(k)}, n \geq 1\}$ be the sequence of k-th lower record values. Then the probability density function of $Y_n^{(k)}, n \geq 1$ is given by

$$f_{Y_n^{(k)}}(x) = \frac{k^n}{\Gamma(n)}(H(x))^{n-1}(F(x))^{k-1}f(x),\ x > 0,$$

and the joint pdf of $Y_m^{(k)}$ and $Y_n^{(k)}$ for $1 \leq m < n$ and $n > 2$ is given by

$$f_{Y_m^{(k)},Y_n^{(k)}}(x, y) = \frac{k^n}{\Gamma(m)\Gamma(n-m)}(H(x))^{m-1} \cdot (H(y) - H(x))^{n-m-1}(F(y))^{k-1}\frac{f(x)}{F(x)}f(y),\ y < x,$$

where $H(x) = -ln(F(x))$.

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The continuous distributions based on record values were extensively studied by many authors and such results are available in Faizan, Haque and Ansari [4], Juhás and Skřivánková [5] and Lee and Jin [6]. Also, Malinowska [8] presented a characterization of the general class of continuous distributions based on independent transforms of k-th lower and upper record values.

In this paper we investigate new characterizations of various continuous distributions by independence property of k-th lower record values. This paper generalized results obtained by Paul [10].

2 Characterization Results

The characterizations of continuous distributions presented in this paper are based on the following lemma.

Lemma 1 *Let $\{X_n, n \geq 1\}$ be a sequence of independent and identically distributed random variables each distributed identically with an absolutely continuous cdf $F(x)$ and pdf $f(x)$ on the support (α, β), where α and β may be finite or infinite. Let $g(x)$ be an increasing and differentiable function with $g(x) \to 0$ as $x \to \alpha^+$ and $g(x) \to 1$ as $x \to \beta^-$ for all $x \in (\alpha, \beta)$. Suppose that*

$$\left(\frac{F(g^{-1}(\frac{u}{v}))}{F(g^{-1}(u))}\right)^k = e^{-k\cdot q(u,v)}$$

and

$$h(u,v) = (k\cdot q(u,v))^r \, e^{-k\cdot q(u,v)} \left(\frac{\partial}{\partial v} k\cdot q(u,v)\right)$$

for $r \geq 0$ and $k \geq 1$, where $h(u,v) \neq 0$ and $\frac{\partial}{\partial v} k\cdot q(u,v) \neq 0$ for any positive u and v. If $h(u,v)$ is independent of u, then $k\cdot q(u,v)$ is a function of v only.

Proof. Let

$$\begin{aligned} h(u,v) &= \big(k\cdot q(u,v)\big)^r e^{-k\cdot q(u,v)} \big(\frac{\partial}{\partial v} k\cdot q(u,v)\big) \\ &= \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \big(k\cdot q(u,v)\big)^{j+r} \big(\frac{\partial}{\partial v} k\cdot q(u,v)\big). \end{aligned}$$

Since $h(u,v)$ is independent of u, we obtain

$$h(u,v) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \big(k\cdot q(u,v)\big)^{j+r} \big(\frac{\partial}{\partial v} k\cdot q(u,v)\big) = l(v). \tag{1}$$

Integrating (1) with respect to v, we get

$$\int h(u,v)dv = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \frac{1}{(j+r+1)} \big(k\cdot q(u,v)\big)^{j+r+1} = \int l(v)dv + c = L(v). \tag{2}$$

Here L is a function of v only and c is independent of v but may depend on u.

Now taking $v \to 1$, $k\cdot q(u,v) \to 0$, we have c independently of u from (2). Differentiating (2) with respect to u, we get

$$\begin{aligned} \frac{\partial}{\partial u} L(v) &= \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \big(k\cdot q(u,v)\big)^{j+r} \big(\frac{\partial}{\partial u} k\cdot q(u,v)\big) \\ &= l(v) \big(\frac{\partial}{\partial v} k\cdot q(u,v)\big)^{-1} \big(\frac{\partial}{\partial u} k\cdot q(u,v)\big) = 0. \end{aligned}$$

Now we know $l(v) \neq 0$ and $\frac{\partial}{\partial v} k \cdot q(u,v) \neq 0$, so we must have

$$\frac{\partial}{\partial u} k \cdot q(u,v) = 0.$$

Hence $k \cdot q(u,v)$ is a function of v only. This completes the proof. ■

Theorem 1 *Let $\{X_n, n \geq 1\}$ be a sequence of independent and identically distributed random variables each distributed identically with an absolutely continuous cdf $F(x)$ and pdf $f(x)$ on the support (α, β), where α and β may be finite or infinite. Let $g(x)$ be an increasing and differentiable function with $g(x) \to 0$ as $x \to \alpha^+$ and $g(x) \to 1$ as $x \to \beta^-$ for all $x \in (\alpha, \beta)$. Then $F(x) = (g(x))^\lambda$ for all $0 < g(x) < 1, \lambda > 0$, if and only if $\frac{g(X^{(k)}_{L(m)})}{g(X^{(k)}_{L(n)})}$ and $g(X^{(k)}_{L(m)})$ are independent for $1 \leq m < n$.*

Proof. If $F(x) = (g(x))^\lambda$, then it is easy to see that $\frac{g(X^{(k)}_{L(m)})}{g(X^{(k)}_{L(n)})}$ and $g(X^{(k)}_{L(m)})$ are independent for $1 \leq m < n$.

Let us use the transformation $U = g(X^{(k)}_{L(m)})$ and $V = \frac{g(X^{(k)}_{L(m)})}{g(X^{(k)}_{L(n)})}$. The Jacobian of the transformation is $J = \frac{\partial}{\partial u}(g^{-1}(u)) \cdot \frac{\partial}{\partial v}(g^{-1}(\frac{u}{v}))$. Since $g(x)$ is an increasing and differentiable function, both $\frac{\partial}{\partial u}(g^{-1}(u))$ and $\frac{\partial}{\partial v}(g^{-1}(\frac{u}{v}))$ are positive. Thus we can write the joint pdf $f_{U,V}(u,v)$ of U and V as

$$\begin{aligned} f_{U,V}(u,v) &= \frac{k^n}{\Gamma(m)\Gamma(n-m)}(H(g^{-1}(u)))^{m-1} h(g^{-1}(u)) f(g^{-1}(\frac{u}{v})) \\ &\cdot [F\big(g^{-1}(\frac{u}{v})\big)]^{k-1}\big(H(g^{-1}(\frac{u}{v})) - H(g^{-1}(u))\big)^{n-m-1} \\ &\cdot \frac{\partial}{\partial u}(g^{-1}(u)) \frac{\partial}{\partial v}(g^{-1}(\frac{u}{v})) \end{aligned} \tag{3}$$

for all $0 < u < 1$ and $v > 1$, where $H(x) = -lnF(x)$ and $h(x) = -\frac{d}{dx}H(x)$. The pdf $f_U(u)$ of U is given by

$$f_U(u) = \frac{k^m H(g^{-1}(u))^{m-1}}{\Gamma(m)} f(g^{-1}(u)) \frac{\partial}{\partial u}(g^{-1}(u))[F\big(g^{-1}(u)\big)]^{k-1} \tag{4}$$

for all $0 < u < 1$ and $m \geq 1$.

From (3) and (4), we can get the conditional pdf of $f_V(v|u)$ as

$$\begin{aligned} f_V(v|u) &= \frac{k^{n-m}\big(H(g^{-1}(\frac{u}{v})) - H(g^{-1}(u))\big)^{n-m-1}}{\Gamma(n-m)[F(g^{-1}(u))]^k [F\big(g^{-1}(\frac{u}{v})\big)]^{1-k}} f(g^{-1}(\frac{u}{v})) \frac{\partial}{\partial v}(g^{-1}(\frac{u}{v})) \\ &= C\Big(-k \cdot ln\frac{F(g^{-1}(\frac{u}{v}))}{F(g^{-1}(u))}\Big)^{n-m-1}\Big(\frac{F(g^{-1}(\frac{u}{v}))}{F(g^{-1}(u))}\Big)^k \Big(-\frac{\partial}{\partial v}\big(-k \cdot ln\frac{F(g^{-1}(\frac{u}{v}))}{F(g^{-1}(u))}\big)\Big), \end{aligned}$$

for all $0 < u < 1$ and $v > 1$, where $C = \frac{1}{\Gamma(n-m)}$. Since U and V are independent and using Lemma 1, we see that

$$k \cdot q(u,v) = -ln\Big(\frac{F(g^{-1}(\frac{u}{v}))}{F(g^{-1}(u))}\Big)^k$$

is a function of v only. Thus

$$\Big(\frac{F(g^{-1}(\frac{u}{v}))}{F(g^{-1}(u))}\Big)^k = L(v)$$

where $L(v)$ is a function of v only. Letting $u \to 1^-$, so $g^{-1}(u) \to \beta$ and $F(g^{-1}(u)) \to 1$, we get $L(v) = \big(F(g^{-1}(\frac{1}{v}))\big)^k$ from $F(\beta) = 1$. Then, we get

$$\big(F(g^{-1}(\frac{u}{v}))\big)^k = \big(F(g^{-1}(u))\big)^k \big(F(g^{-1}(\frac{1}{v}))\big)^k \tag{5}$$

for all $0 < u < 1$ and $v > 1$. The equation (5) is equivalent to

$$F(g^{-1}(\frac{u}{v})) = F(g^{-1}(u))F(g^{-1}(\frac{1}{v})) \tag{6}$$

for all $0 < u < 1$ and $v > 1$. By the theory of functional equation (see Aczel [1]), the only continuous solution of (6) with boundary conditions $F(\alpha) = 0$ and $F(\beta) = 1$ is

$$F(x) = (g(x))^{\lambda}$$

for all $0 < g(x) < 1$ and $\lambda > 0$. This completes the proof. ■

Theorem 2 *Let $\{X_n, n \geq 1\}$ be a sequence of independent and identically distributed random variables each distributed identically with an absolutely continuous cdf $F(x)$ and pdf $f(x)$ on the support (α, β), where α and β may be finite or infinite. Let $g(x)$ be an increasing and differentiable function with $g(x) \to 0$ as $x \to \alpha^+$ and $g(x) \to 1$ as $x \to \beta^-$ for all $x \in (\alpha, \beta)$. Then $F(x) = (g(x))^{\lambda}$ for all $0 < g(x) < 1, \lambda > 0$, if and only if $\frac{g(X^{(k)}_{L(m)})}{g(X^{(k)}_{L(m)}) \pm g(X^{(k)}_{L(n)})}$ and $g(X^{(k)}_{L(m)})$ are independent for $1 \leq m < n$.*

Proof. The proof can be done in exactly the same way as that of Theorem 2. ■

Remark 1 *The case when $m = k$, $n = k+1$ and $g(x) = x$ which is the special case of Theorem 2 and Theorem 3 have been treated in [10].*

Remark 2 *If we set $g(x) = x$, $x > 1$ and $k = 1$, then $F(x) = x^{\lambda}$ for all $x > 1$, $\lambda > 0$, if and only if $\frac{X_{L(m)}}{X_{L(n)}}$ and $X_{L(m)}$ are independently distributed in [7].*

Remark 3 *With suitable choice of $g(x)$, various distributions may be characterized as given in Table 1.*

Table 1: Examples based on the distribution function $F(x) = (g(x))^{\lambda}$.

Distribution	**g(x)**	**F(x)**
Power	x	$x^{\lambda}, \;\; x > 0$
Inverse power	$\frac{x-\alpha}{\beta-\alpha}$	$(\frac{x-\alpha}{\beta-\alpha})^{\lambda}, \;\; \alpha < x < \beta$
Reflected exponential	$e^{(x-\beta)}$	$e^{\lambda(x-\beta)}, \;\; -\infty < x < \beta$
Inverse Weibull	$e^{-x^{\theta}}$	$e^{-\lambda x^{\theta}}, \;\; x > 0$
Inverse exponential	$e^{-\frac{1}{x}}$	$e^{-\frac{\lambda}{x}}, \;\; x > 0$
Gumbel	$e^{-e^{-x}}$	$e^{-\lambda e^{-x}}, \;\; -\infty < x < \infty$
Burr type II	$(1+e^{-x})^{-1}$	$(1+e^{-x})^{-\lambda}, \;\; -\infty < x < \infty$
Burr type III	$(1+x^{-p})^{-1}$	$(1+x^{-p})^{-\lambda}, \;\; x > 0$
Burr type IV	$\big(1+(\frac{\beta-x}{x})^{\frac{1}{\beta}}\big)^{-1}$	$\big(1+(\frac{c-x}{x})^{\frac{1}{c}}\big)^{-\lambda}, \;\; 0 < x < \beta$
Burr type V	$(1+ce^{-tanx})^{-1}$	$(1+ce^{-tanx})^{-\lambda}, \;\; -\frac{\pi}{2} < x < \frac{\pi}{2}$
Burr type VI	$(1+ce^{-\lambda sinhx})^{-1}$	$(1+ce^{-\lambda sinhx})^{-\lambda}, \;\; -\infty < x < \infty$
Burr type VII	$(\frac{1+tanhx}{2})$	$(\frac{1+tanhx}{2})^{\lambda}, \;\; -\infty < x < \infty$
Burr type VIII	$(\frac{2}{\pi}tan^{-1}e^{x})$	$(\frac{2}{\pi}tan^{-1}e^{x})^{\lambda}, \;\; -\infty < x < \infty$
Burr type X	$(1-e^{-\theta x^2})$	$(1-e^{-\theta x^2})^{\lambda}, \;\; x > 0$
Burr type XI	$(x-\frac{1}{2\pi}sin2\pi x)$	$(x-\frac{1}{2\pi}sin2\pi x)^{\lambda}, \;\; 0 < x < 1$

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