Applied Mathematics E-Notes, 21(2021), 587-598 © ISSN 1607-2510
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A Three Sequential Fractional Differential Problem Of Duffing Type*

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Received 12 December 2020

Abstract

In this work, using fractional calculus, we study an (α, β, γ) sequential integro-differential problem of Duffing type. The studied problem allows us, in particular, to obtain the standard Duffing problem. The serious difficulty in our problem is the "(CO-SG)-absence"; the absence of commutativity and semi group properties for the left hand side derivatives. By taking into account both the (CO-SG)-absence and the conditions of the problem, we present the integral representation of the problem. Then, by virtue of the integral representation, we prove some existence and uniqueness results. Also, we prove an existence result using Schaefer fixed point theorem. In addition of these, two illustrative examples are discussed.

1 Introduction

It is well known that differential equations of arbitrary order are used for modelling several phenomena of physics and engineering sciences. Fore some applications, we refer the reader to the papers [1, 2, 3, 7, 9, 10, 11, 12, 15, 18, 22, 23, 24, 27, 28, 30]. The Duffing equation is considered as an excellent example of a dynamical system that is used to model certain driven-damped oscillators, see [5, 8, 13, 16, 17, 19, 20, 21, 25, 26, 29]. The standard form of Duffing problem is given by the following differential equation[6]:

$$z''(t) + az'(t) + f(t, z(t)) = h(t), \quad t \in [0,1], \; a > 0,$$

under the conditions:

$$z(0) = A \in \mathbb{R}, \quad z'(0) = B \in \mathbb{R},$$

where the t-function z is the displacement, z' is the velocity, z'' is the acceleration, and f and h are two given functions. Some authors have studied new types of the above Duffing equation. For example in [8], the authors have examined the application of a numerical approach of the forced nonlinear Duffing equation:

$$\begin{cases} D^{\beta}u(t) + \delta D^{\alpha}u(t) + \rho u(t) + \mu u^3(t) = \lambda \sin(\omega t), \\ u(0) = A^* \in \mathbb{R}, \; D^{\alpha}u(0) = B^* \in \mathbb{R}, \\ 0 < \alpha < 1, \; 1 < \beta < 2, \; t \in [0,1], \end{cases}$$

taking into account that D^{β} and D^{α} are the Caputo fractional derivatives, while δ, ρ, μ and λ are positive real numbers.

In [26], the authors have investigated the following problem of Duffing type:

$$\begin{cases} D^{\beta}y(t) + aD^{\alpha}y(t) + f(t, y(t)) = h(t), \\ y(z_0) = y_0, \; y'(z_0) = y_1, \\ 0 < \alpha < 1, \; 1 < \beta < 2, \; a > 0, \; t \in [0,1], \end{cases}$$

*Mathematics Subject Classifications2010: 30C45, 39B72, 39B82.

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where D^β and D^α are the Caputo fractional derivatives and z_0 is an initial value in $[0,1]$. In a very recent work [4], the authors have been concerned with the following Duffing type problem:

$$\begin{cases} D^\beta D^\alpha z(t) + kf(t, D^\alpha z(t)) + g(t, z(t), D^p z(t)) = h(t), \\ z(0) = A^* \in \mathbb{R},\ D^\alpha z(0) = B^* \in \mathbb{R},\ z(1) = C^* \in \mathbb{R}, \\ 0 < p < \alpha < 1,\ 1 < \beta < 2,\ t \in [0,1], \end{cases}$$

where D^α, D^β, D^p are the Caputo derivatives, k is a real constant, the functions f, g and h are continuous. In the present paper, our idea is the investigation of the following fractional problem of Duffing type:

$$\begin{cases} D^\gamma D^\beta D^\alpha z(t) + kf(t, D^\alpha z(t)) + g(t, z(t), D^p z(t)) + h(t, z(t), J^q(z(t))) = L(t), \\ z(0) = A_1 \in \mathbb{R},\ D^\alpha z(0) = A_2 \in \mathbb{R},\ J^\alpha z(1) = A_3 \in \mathbb{R}, \\ 0 \le p < \alpha \le 1,\ 0 \le \beta,\ \gamma \le 1,\ 1 < \alpha + \beta \le 2,\ 1 < \beta + \gamma \le 2,\ t \in I. \end{cases} \tag{1}$$

We suppose that $I := [0,1]$, the derivatives of the problem are in the sense of Caputo, J^q is the Riemann-Liouville integral with $q \ge 0$, $f : I \times \mathbb{R}^2 \to \mathbb{R}$, $g : I \times \mathbb{R}^3 \to \mathbb{R}$, $h : I \times \mathbb{R}^3 \to \mathbb{R}$ and $L : I \to \mathbb{R}$ are four given functions. It is very important to note the following remarks:

1* In the left hand side of the above problem, we consider three parameters of Caputo derivation; this condition allows us to be concerned with a three sequential Duffing problem that does not verify the above (CO-SG) properties.

2* The proposed problem is more interesting and more general, since on one hand, the classical Duffing equation is of order two, and on the other hand, for some values of α, β, γ applied to our problem, we can obtain the standard form of Duffing equation of [6]; so the problem 1 can be used for better modeling the fractional order case.

To the best of our knowledge and taken into consideration the particular equation of [9], this is the first time in the literature where such three sequential Duffing problem is considered.

2 Basic Concepts

In this section, we recall some auxiliary results on fractional calculus that we need in this paper, see [14].

Definition 1 *The Riemann-Liouville integral operator with order $\alpha \ge 0$, for any continuous function f on $[a,b]$ is*

$$\begin{cases} J_a^\alpha[f(t)] = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau) d\tau & \text{for } \alpha > 0 \text{ and } a \le t \le b, \\ J_a^0[f(t)] = f(t) & \text{for } \alpha = 0 \text{ and } a \le t \le b. \end{cases}$$

Definition 2 *We take $f \in C^m([0,1], \mathbb{R})$, $m \in \mathbb{N}^*$ and $m-1 < \alpha \le m$. So the Caputo derivative is*

$$\begin{aligned} D^\alpha f(z) &= \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^z (z-t)^{m-\alpha-1} f^{(m)}(t) dt, & \text{for } m-1 < \alpha < m, \\ f^{(m)}(z), & \text{for } \alpha = m, \end{cases} \\ &= J^{m-\alpha}[f^{(m)}(z)]. \end{aligned}$$

The following auxiliary lemmas are important to prove some of our results.

Lemma 1 *The set of solutions of $D^\alpha z(t) = 0$, $t \in I$, is given by*

$$z(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1},$$

where $c_i \in \mathbb{R}$, $i = 0, 1, \cdots, n-1$, $n = [\alpha] + 1$ and $\alpha > 0$.

Lemma 2 *We take $\alpha > 0$ and $n \in \mathbb{N}^*$. Thus, we have*

$$J^{\alpha}[D^{\alpha}z(t)] = z(t) + c_0 + c_1 t + c_2 t^2 + ... + c_{n-1}t^{n-1},\ t \in I,$$

where $c_i \in \mathbb{R}$, $i = 0, 1, \cdots, n-1$ and $n = [\alpha] + 1$.

Lemma 3 *In the case where $q_1 > q_2 > 0, f \in L^1(I)$, it yields that*

$$D^{q_2}J^{q_1}[f(t)] = J^{q_1 - q_2}[f(t)].$$

Lemma 4 (Schaefer fixed point theorem) *Let Z be a Banach space and $\Phi : Z \to Z$ be any completely continuous operator. If $W := \{z \in Z : z = \eta\Phi z, \quad 0 < \eta < 1\}$ is bounded, then Φ has at least one fixed point in Z.*

Now, we pass to prove the following lemma which will allow us to establish the unique integral representation for (1):

Lemma 5 *Let $R \in C([0,1])$, $t \in I = [0,1]$, $0 \leq \alpha, \beta, \gamma \leq 1$. Then the representation of the problem*

$$\begin{cases} D^{\gamma}\left(D^{\beta}[D^{\alpha}z(t)]\right) = R(t), \\ z(0) = A_1, \quad D^{\alpha}z(0) = A_2, \quad J^{\alpha}z(1) = A_3, \end{cases} \tag{2}$$

is illustrated as

$$\begin{aligned} z(t) \ = \ & \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \int_0^s \frac{(s-u)^{\beta-1}}{\Gamma(\beta)} \int_0^u \frac{(u-v)^{\gamma-1}}{\Gamma(\gamma)} R(v)dvduds \\ & -B_1 \left[\int_0^1 \frac{(1-s)^{2\alpha-1}}{\Gamma(2\alpha)} \left(\int_0^s \frac{(s-u)^{\beta-1}}{\Gamma(\beta)} \left(\int_0^u \frac{(u-v)^{\gamma-1}}{\Gamma(\gamma)} R(v)dv \right) du \right) ds \right] t^{\alpha+\beta} \\ & - [B_2 A_2 + B_3 A_1 - B_1 A_3]\, t^{\alpha+\beta} + B_4 A_2 t^{\alpha} + A_1, \end{aligned} \tag{3}$$

such that

$$\begin{aligned} &B_1 = \tfrac{\Gamma(2\alpha+\beta+1)}{\Gamma(\alpha+\beta+1)}, && B_2 = \tfrac{\Gamma(2\alpha+\beta+1)}{\Gamma(2\alpha+1)\Gamma(\alpha+\beta+1)}, && B_3 = \tfrac{\Gamma(2\alpha+\beta+1)}{\Gamma(\alpha+1)\Gamma(\alpha+\beta+1)}, \\ &B_4 = \tfrac{1}{\Gamma(\alpha+1)}, && B_5 = \tfrac{\Gamma(2\alpha+\beta+1)}{\Gamma(\beta+1)}, && B_6 = \tfrac{\Gamma(2\alpha+\beta+1)}{\Gamma(2\alpha+1)\Gamma(\beta+1)}, \\ &B_7 = \tfrac{\Gamma(2\alpha+\beta+1)}{\Gamma(\alpha+1)\Gamma(\beta+1)}, && B_8 = \tfrac{\Gamma(p+1)}{\Gamma(\alpha+\beta-p+1)}, && B_9 = \tfrac{\Gamma(p+1)}{\Gamma(\alpha-p+1)}. \end{aligned}$$

are non zero parameters.

Proof. Note that, with due attention to Lemma 2, a general solution of (2) can be written by the following formula:

$$\begin{aligned} z(t) \ = \ & \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \int_0^s \frac{(s-u)^{\beta-1}}{\Gamma(\beta)} \int_0^u \frac{(u-v)^{\gamma-1}}{\Gamma(\gamma)} R(v)dvduds \\ & -c_0 \frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} - c_1 \frac{t^{\alpha}}{\Gamma(\alpha+1)} - c_2. \end{aligned} \tag{4}$$

We can easily compute the three constants c_0, c_1 and c_2. We reach

$$\begin{cases} c_0 = \Gamma(2\alpha+\beta+1) \int_0^1 \frac{(1-s)^{2\alpha-1}}{\Gamma(2\alpha)} \left(\int_0^s \frac{(s-u)^{\beta-1}}{\Gamma(\beta)} \left(\int_0^u \frac{(u-v)^{\gamma-1}}{\Gamma(\gamma)} R(v)dv \right) du \right) ds \\ \qquad + \left(\frac{\Gamma(2\alpha+\beta+1)A_2}{\Gamma(2\alpha+1)} + \frac{\Gamma(2\alpha+\beta+1)A_1}{\Gamma(\alpha+1)} - \Gamma(2\alpha+\beta+1) A_3 \right), \\ c_1 = -A_2, \\ c_2 = -A_1. \end{cases}$$

If we insert the above quantities of c_0, c_1 and c_2 in (4), then we obtain (3). ■

3 Main Results

Before presenting to the reader the main results, the following space wit its norm needs to be introduced:

$$Z := \{z \in C(I,\mathbb{R}),\ D^{\alpha}z \in C(I,\mathbb{R}),\ D^{p}z \in C(I,\mathbb{R})\}$$

and

$$\|z\|_Z = \max\{\|z\|_\infty,\ \|D^{\alpha}z\|_\infty,\ \|D^{p}z\|_\infty\},$$

where

$$\|z\|_\infty = \sup_{t\in I}|z(t)|,\quad \|D^{\alpha}z\|_\infty = \sup_{t\in I}|D^{\alpha}z(t)| \quad\text{and}\quad \|D^{p}z\|_\infty = \sup_{t\in I}|D^{p}z(t)|.$$

Then, we define $\Phi : Z \to Z$ by

$$\begin{aligned}\Phi z(t) &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\int_0^s \frac{(s-u)^{\beta-1}}{\Gamma(\beta)}\int_0^u \frac{(u-v)^{\gamma-1}}{\Gamma(\gamma)}\Big[L(v) - kf(v,D^{\alpha}z(v)) - g(v,z(v),D^{p}z(v))\\ &\quad -h(v,z(v),J^{q}(z(v)))\Big]dvduds - B_1\Bigg[\int_0^1 \frac{(1-s)^{2\alpha-1}}{\Gamma(2\alpha)}\Bigg(\int_0^s \frac{(s-u)^{\beta-1}}{\Gamma(\beta)}\Bigg(\int_0^u \frac{(u-v)^{\gamma-1}}{\Gamma(\gamma)}\Big[L(v)\\ &\quad -kf(v,D^{\alpha}z(v)) - g(v,z(v),D^{p}z(v)) - h(v,z(v),J^{q}(z(v)))\Big]dv\Bigg)du\Bigg)ds\Bigg]t^{\alpha+\beta}\\ &\quad -\left[B_2A_2 + B_3A_1 - B_1A_3\right]t^{\alpha+\beta} + B_4A_2t^{\alpha} + A_1. \end{aligned} \tag{5}$$

The following hypotheses need also to be take into consideration:

(H1): There exist constants $W_i > 0$, $i = 1...5$, such that for each $t \in I$ and for all $a_1, a_2, b_1, b_2 \in \mathbb{R}$, we have

(i) $|f(t,a_1) - f(t,b_1)| \le W_1|a_1 - b_1|$,

(ii) $|g(t,a_1,a_2) - g(t,b_1,b_2))| \le W_2|a_1 - b_1| + W_3|a_2 - b_2|$,

(iii) $|h(t,a_1,a_2) - h(t,b_1,b_2)| \le W_4|a_1 - b_1| + W_5\,|a_2 - b_2|$.

(H2): The four functions $f : I\times\mathbb{R} \to \mathbb{R}$, $g : I\times\mathbb{R}^2 \to \mathbb{R}$, $h : I\times\mathbb{R}^2 \to \mathbb{R}$ and $L : I \to \mathbb{R}$ are continuous.

(H3): There exist constants $E_f > 0$, $E_g > 0$, $E_h > 0$ and $E_L > 0$ such that for each $t \in I$ and all $a_1, a_2 \in \mathbb{R}$, we have

$$|f(t,a_1)| \le E_f,\quad |g(t,a_1,a_2)| \le E_g,\quad |h(t,a_1,a_2)| \le E_h \quad\text{and}\quad |L(t)| \le E_L.$$

In order to facilitate for the reader the proof of the main results, we consider the quantities:

$$\begin{aligned}T_1 \ :&= (|k|\,W_1 + W_2 + W_3 + W_4)\left[\frac{1}{\Gamma(\alpha+\beta+\gamma+1)} + |B_1|\,\frac{1}{\Gamma(2\alpha+\beta+\gamma+1)}\right]\\ &\quad +W_5\left[\frac{1}{\Gamma(\alpha+\beta+\gamma+q+1)} + |B_1|\,\frac{1}{\Gamma(2\alpha+\beta+\gamma+q+1)}\right],\end{aligned}$$

$$\begin{aligned}T_2 \ :&= (|k|\,W_1 + W_2 + W_3 + W_4)\left[\frac{1}{\Gamma(\beta+\gamma+1)} + \frac{|B_5|}{\Gamma(2\alpha+\beta+\gamma+1)}\right]\\ &\quad +W_5\left[\frac{1}{\Gamma(\beta+\gamma+q+1)} + \frac{|B_5|}{\Gamma(2\alpha+\beta+\gamma+q+1)}\right]\end{aligned}$$

and

$$T_3 \quad : \quad = (|k| W_1 + W_2 + W_3 + W_4) \left[\frac{1}{\Gamma(\alpha+\beta+\gamma-p+1)} + \frac{|B_1 B_8|}{\Gamma(2\alpha+\beta+\gamma+1)}\right] + W_5 \left[\frac{1}{\Gamma(\alpha+\beta+\gamma+q-p+1)} + \frac{|B_1 B_8|}{\Gamma(2\alpha+\beta+\gamma+q+1)}\right].$$

At this moment, we are ready to express and verify the main results.

3.1 Existence and Uniqueness Criteria

Theorem 1 *If (H1) holds and $0 < T < 1$ where $T := max(T_1, T_2, T_3)$, then the problem (1) has a unique solution on I.*

Proof. It is sufficient for us to show that Φ is a contractive operator.

A: Let $x, y \in Z$. Then, thanks to (H1), we have

$$\begin{aligned}
&|\Phi y(t) - \Phi x(t)| \\
\leq\ & |k| W_1 D^{\alpha} \|y-x\|_{\infty} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \int_0^s \frac{(s-u)^{\beta-1}}{\Gamma(\beta)} \int_0^u \frac{(u-v)^{\gamma-1}}{\Gamma(\gamma)} dvduds \\
&+ |k| W_1 D^{\alpha} \|y-x\|_{\infty} |B_1| t^{\alpha+\beta} \int_0^1 \frac{(t-s)^{2\alpha-1}}{\Gamma(2\alpha)} \int_0^s \frac{(s-u)^{\beta-1}}{\Gamma(\beta)} \int_0^u \frac{(u-v)^{\gamma-1}}{\Gamma(\gamma)} dvduds \\
&+ (W_2 \|y-x\|_{\infty} + W_3 D^p \|y-x\|_{\infty}) \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \int_0^s \frac{(s-u)^{\beta-1}}{\Gamma(\beta)} \int_0^u \frac{(u-v)^{\gamma-1}}{\Gamma(\gamma)} dvduds \\
&+ (W_2 \|y-x\|_{\infty} + W_3 D^p \|y-x\|_{\infty}) |B_1| t^{\alpha+\beta} \int_0^1 \frac{(t-s)^{2\alpha-1}}{\Gamma(2\alpha)} \int_0^s \frac{(s-u)^{\beta-1}}{\Gamma(\beta)} \int_0^u \frac{(u-v)^{\gamma-1}}{\Gamma(\gamma)} dvduds \\
&+ (W_4 \|y-x\|_{\infty} + W_5 J^q \|y-x\|_{\infty}) \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \int_0^s \frac{(s-u)^{\beta-1}}{\Gamma(\beta)} \int_0^u \frac{(u-v)^{\gamma-1}}{\Gamma(\gamma)} dvduds \\
&+ (W_4 \|y-x\|_{\infty} + W_5 J^q \|y-x\|_{\infty}) |B_1| t^{\alpha+\beta} \left(\int_0^1 \frac{(t-s)^{2\alpha-1}}{\Gamma(2\alpha)} \int_0^s \frac{(s-u)^{\beta-1}}{\Gamma(\beta)} \int_0^u \frac{(u-v)^{\gamma-1}}{\Gamma(\gamma)} dvduds \right).
\end{aligned}$$

Consequently,

$$\begin{aligned}
\sup_{t \in I} |\Phi y(t) - \Phi x(t)| \leq\ & \Bigg\{ (|k| W_1 + W_2 + W_3 + W_4) \left[\frac{1}{\Gamma(\alpha+\beta+\gamma+1)} + |B_1| \frac{1}{\Gamma(2\alpha+\beta+\gamma+1)}\right] \\
&+ W_5 \left[\frac{1}{\Gamma(\alpha+\beta+\gamma+q+1)} + |B_1| \frac{1}{\Gamma(2\alpha+\beta+\gamma+q+1)}\right] \Bigg\} \|y-x\|_Z .
\end{aligned}$$

Hence, $\|\Phi y - \Phi x\|_{\infty} \leq T_1 \|y-x\|_Z$.

B: Let $x, y \in Z$. So we can remark that

$$\begin{aligned}
&\|D^{\alpha} \Phi y - D^{\alpha} \Phi x\|_{\infty} \\
\leq\ & (|k| W_1) \|D^{\alpha} y - D^{\alpha} x\|_{\infty} \left[\sup_{t \in I} \frac{t^{\beta+\gamma}}{\Gamma(\beta+\gamma+1)} + |B_5| \sup_{t \in I} \frac{t^{2\alpha+2\beta+\gamma}}{\Gamma(2\alpha+\beta+\gamma+1)}\right]
\end{aligned}$$

$$+\left(W_2\left\|y-x\right\|_\infty+W_3\left\|D^py-D^px\right\|_\infty\right)\left[\sup_{t\in I}\frac{t^{\beta+\gamma}}{\Gamma\left(\beta+\gamma+1\right)}+\left|B_5\right|\sup_{t\in I}\frac{t^{2\alpha+2\beta+\gamma}}{\Gamma\left(2\alpha+\beta+\gamma+1\right)}\right]$$
$$+\left(W_4\left\|y-x\right\|_\infty+W_5\left\|J^qy-J^qx\right\|_\infty\right)\left[\sup_{t\in I}\frac{t^{\beta+\gamma}}{\Gamma\left(\beta+\gamma+1\right)}+\left|B_5\right|\sup_{t\in I}\frac{t^{2\alpha+2\beta+\gamma}}{\Gamma\left(2\alpha+\beta+\gamma+1\right)}\right].$$

Consequently, we get

$$\begin{aligned}\left\|D^\alpha\Phi y-D^\alpha\Phi x\right\|_\infty \leq & \Bigg(\left(\left|k\right|W_1+W_2+W_3+W_4\right)\left[\begin{array}{c}\sup_{t\in I}\frac{t^{\beta+\gamma}}{\Gamma(\beta+\gamma+1)}+\\ \left|B_5\right|\sup_{t\in I}\frac{t^{2\alpha+2\beta+\gamma}}{\Gamma(2\alpha+\beta+\gamma+1)}\end{array}\right]\\ & +W_5\left[\begin{array}{c}\sup_{t\in I}\frac{t^{\beta+\gamma+q}}{\Gamma(\beta+\gamma+q+1)}+\\ \left|B_5\right|\sup_{t\in I}\frac{t^{2\alpha+2\beta+\gamma+q}}{\Gamma(2\alpha+\beta+\gamma+q+1)}\end{array}\right]\Bigg)\left\|y-x\right\|_\infty .\end{aligned}$$

Therefore,

$$\begin{aligned}\left\|D^\alpha\Phi y-D^\alpha\Phi x\right\|_\infty \leq & \Bigg\{\left(\left|k\right|W_1+W_2+W_3+W_4\right)\left[\frac{1}{\Gamma\left(\beta+\gamma+1\right)}+\frac{\left|B_5\right|}{\Gamma\left(2\alpha+\beta+\gamma+1\right)}\right]\\ & +W_5\left[\frac{1}{\Gamma\left(\beta+\gamma+q+1\right)}+\left|B_5\right|\frac{1}{\Gamma\left(2\alpha+\beta+\gamma+q+1\right)}\right]\Bigg\}\left\|y-x\right\|_Z .\end{aligned}$$

Thus, $\left\|D^\alpha\Phi y-D^\alpha\Phi x\right\|_\infty\leq T_2\left\|y-x\right\|_Z .$

C: On the other side, for $x,y\in Z$, we observe that

$$\begin{aligned}& \left\|D^p\Phi y-D^p\Phi x\right\|_\infty\\ \leq\ & \left(\left|k\right|W_1\right)\left\|D^\alpha y-D^\alpha x\right\|_\infty\left(\sup_{t\in I}\frac{t^{\alpha+\beta+\gamma-p}}{\Gamma\left(\alpha+\beta+\gamma-p+1\right)}+\left|B_1B_8\right|\sup_{t\in I}\frac{t^{3\alpha+2\beta+\gamma-p}}{\Gamma\left(2\alpha+\beta+\gamma+1\right)}\right)\\ & +\left(W_2\left\|y-x\right\|_\infty+W_3\left\|D^py-D^px\right\|_\infty\right)\left(\sup_{t\in I}\frac{t^{\alpha+\beta+\gamma-p}}{\Gamma\left(\alpha+\beta+\gamma-p+1\right)}+\left|B_1B_8\right|\sup_{t\in I}\frac{t^{3\alpha+2\beta+\gamma-p}}{\Gamma\left(2\alpha+\beta+\gamma+1\right)}\right)\\ & +\left(W_4\left\|y-x\right\|_\infty+W_5\left\|J^qy-J^qx\right\|_\infty\right)\left(\sup_{t\in I}\frac{t^{\alpha+\beta+\gamma-p}}{\Gamma\left(\alpha+\beta+\gamma-p+1\right)}+\left|B_1B_8\right|\sup_{t\in I}\frac{t^{3\alpha+2\beta+\gamma-p}}{\Gamma\left(2\alpha+\beta+\gamma+1\right)}\right).\end{aligned}$$

Consequently,

$$\begin{aligned}\left\|D^p\Phi y-D^p\Phi x\right\|_\infty \leq & \Bigg\{\left(\left|k\right|W_1+W_2+W_3+W_4\right)\left[\frac{1}{\Gamma\left(\alpha+\beta+\gamma-p+1\right)}+\frac{\left|B_1B_8\right|}{\Gamma\left(2\alpha+\beta+\gamma+1\right)}\right]\\ & +W_5\left[\frac{1}{\Gamma\left(\alpha+\beta+\gamma+q-p+1\right)}+\frac{\left|B_1B_8\right|}{\Gamma\left(2\alpha+\beta+\gamma+q+1\right)}\right]\Bigg\}\left\|y-x\right\|_Z .\end{aligned}$$

Thus, it yields that $\left\|D^p\Phi y-D^p\Phi x\right\|_\infty\leq T_3\left\|y-x\right\|_Z .$

The above three main steps guarantee that

$$\parallel \Phi x-\Phi y\parallel_Z\leq T\parallel x-y\parallel_Z .$$

The Banach contraction principle allows us to say that the problem (1) has a unique solution. ■

3.2 Existence Criteria

Theorem 2 *Suppose that* $(H2)$ *and* $(H3)$ *are valid. Then the problem (1) has at least one solution on* I.

Proof. We shall use Schaefer fixed point theorem.

Step1: We show that Φ is continuous on Z. Obviously, this step is trivial and hence we can omit it.

Step2: In this step, we show that Φ maps any bounded set into another bounded set in Z. Suppose that $r > 0$ and $Br := \{z \in Z; \|z\|_Z \leq r\}$. For $y \in Br$, we observe that

$$\begin{aligned}
\|\Phi y\|_\infty \leq\; & E_L \sup_{t\in I} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \int_0^s \frac{(s-u)^{\beta-1}}{\Gamma(\beta)} \int_0^u \frac{(u-v)^{\gamma-1}}{\Gamma(\gamma)} dvduds \\
& + |B_1| E_L \sup_{t\in I} \left(\int_0^1 \frac{(1-s)^{2\alpha-1}}{\Gamma(2\alpha)} \int_0^s \frac{(s-u)^{\beta-1}}{\Gamma(\beta)} \int_0^u \frac{(u-v)^{\gamma-1}}{\Gamma(\gamma)} dvduds \right) t^{\alpha+\beta} \\
& + |k| E_f \sup_{t\in I} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \int_0^s \frac{(s-u)^{\beta-1}}{\Gamma(\beta)} \int_0^u \frac{(u-v)^{\gamma-1}}{\Gamma(\gamma)} dvduds \\
& + |B_1| |k| E_f \sup_{t\in I} \left(\int_0^1 \frac{(1-s)^{2\alpha-1}}{\Gamma(2\alpha)} \int_0^s \frac{(s-u)^{\beta-1}}{\Gamma(\beta)} \int_0^u \frac{(u-v)^{\gamma-1}}{\Gamma(\gamma)} dvduds \right) t^{\alpha+\beta} \\
& + E_g \sup_{t\in I} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \int_0^s \frac{(s-u)^{\beta-1}}{\Gamma(\beta)} \int_0^u \frac{(u-v)^{\gamma-1}}{\Gamma(\gamma)} dvduds \\
& + |B_1| E_g \sup_{t\in I} \left(\int_0^1 \frac{(1-s)^{2\alpha-1}}{\Gamma(2\alpha)} \int_0^s \frac{(s-u)^{\beta-1}}{\Gamma(\beta)} \int_0^u \frac{(u-v)^{\gamma-1}}{\Gamma(\gamma)} dvduds \right) t^{\alpha+\beta} \\
& + E_h \sup_{t\in I} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \int_0^s \frac{(s-u)^{\beta-1}}{\Gamma(\beta)} \int_0^u \frac{(u-v)^{\gamma-1}}{\Gamma(\gamma)} dvduds \\
& + |B_1| E_h \sup_{t\in I} \left(\int_0^1 \frac{(1-s)^{2\alpha-1}}{\Gamma(2\alpha)} \int_0^s \frac{(s-u)^{\beta-1}}{\Gamma(\beta)} \int_0^u \frac{(u-v)^{\gamma-1}}{\Gamma(\gamma)} dvduds \right) t^{\alpha+\beta}
\end{aligned}$$

and

$$\|\Phi y\|_\infty \leq [E_L + |k| E_f + E_g + E_h] \left[\frac{1}{\Gamma(\alpha+\gamma+\beta+1)} + |B_1| \frac{1}{\Gamma(2\alpha+\gamma+\beta+1)} \right] < +\infty.$$

So, we obtain

$$\|\Phi y\|_\infty \leq \left([E_L + |k| E_f + E_g + E_h] \left[\frac{1}{\Gamma(\alpha+\gamma+\beta+1)} + |B_1| \frac{1}{\Gamma(2\alpha+\gamma+\beta+1)} \right] \right) < +\infty. \tag{6}$$

Also, we have

$$\begin{aligned}
\|D^\alpha \Phi y\|_\infty \leq\; & \Bigg([E_L + |k| E_f + E_g + E_h] \left[\frac{1}{\Gamma(\gamma+\beta+1)} + |B_5| \frac{1}{\Gamma(2\alpha+\gamma+\beta+1)} \right] \\
& + |B_6 A_2 + B_7 A_1 - B_5 A_3| + |A_2| \Bigg) < +\infty.
\end{aligned} \tag{7}$$

On the other hand, it is a simple task to show that

$$\|D^p \Phi y\|_\infty \leq \Bigg([E_L + |k| E_f + E_g + E_h] \left[\frac{1}{\Gamma(\alpha+\beta+\gamma-p+1)} + |B_1 B_8| \frac{1}{\Gamma(2\alpha+\beta+\gamma+1)} \right]$$

$$+\left|B_2A_2+B_3A_1-B_1A_3\right|+\left|B_4B_9A_2\right|\Big)<+\infty. \tag{8}$$

Thanks to (6), (7) and (8), we can observe that Φ is uniformly bounded on Br.

Step3: Equicontinuity of the set Br. Let $t_1,t_2\in I$, such that $t_1<t_2$, and let Br be the above bounded set of Z. So for $y\in Br$ and for each $t\in I$, it can be seen that

$$\begin{aligned}\Phi y(t_1)-\Phi y(t_2) \leq\ & (E_L+|k|E_f+E_g+E_h)\Bigg[\int_0^{t_1}\frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}\int_0^{s}\frac{(s-u)^{\beta-1}}{\Gamma(\beta)}\int_0^{u}\frac{(u-v)^{\gamma-1}}{\Gamma(\gamma)}dvduds\\
&-B_1t_1^{\alpha+\beta}\int_0^1\frac{(1-s)^{2\alpha-1}}{\Gamma(2\alpha)}\int_0^{s}\frac{(s-u)^{\beta-1}}{\Gamma(\beta)}\int_0^{u}\frac{(u-v)^{\gamma-1}}{\Gamma(\gamma)}dvduds\\
&-(B_2A_2+B_3A_1-B_1A_3)\,t_1^{\alpha+\beta}+B_4A_2t_1^{\alpha}+A_1\\
&-\int_0^{t_2}\frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}\int_0^{s}\frac{(s-u)^{\beta-1}}{\Gamma(\beta)}\int_0^{u}\frac{(u-v)^{\gamma-1}}{\Gamma(\gamma)}dvduds\\
&+B_1t_2^{\alpha+\beta}\int_0^1\frac{(1-s)^{2\alpha-1}}{\Gamma(2\alpha)}\int_0^{s}\frac{(s-u)^{\beta-1}}{\Gamma(\beta)}\int_0^{u}\frac{(u-v)^{\gamma-1}}{\Gamma(\gamma)}dvduds\\
&+(B_2A_2+B_3A_1-B_1A_3)\,t_2^{\alpha+\beta}-B_4A_2t_2^{\alpha}+A_1\Bigg].\end{aligned}$$

Hence, we have

$$\begin{aligned}&\left|\Phi y(t_1)-\Phi y(t_2)\right|\\
\leq\ & (E_L+|k|E_f+E_g+E_h)\Bigg[\int_0^{t_1}\frac{(t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}\int_0^{s}\frac{(s-u)^{\beta-1}}{\Gamma(\beta)}\int_0^{u}\frac{(u-v)^{\gamma-1}}{\Gamma(\gamma)}dvduds\\
&+\int_{t_1}^{t_2}\frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}\int_0^{s}\frac{(s-u)^{\beta-1}}{\Gamma(\beta)}\int_0^{u}\frac{(u-v)^{\gamma-1}}{\Gamma(\gamma)}dvduds\Bigg]\\
&+\left(\frac{|B_1|}{\Gamma(2\alpha+\beta+\gamma+1)}+|B_2A_2+B_3A_1-B_1A_3|\right)\left|t_1^{\alpha+\beta}-t_2^{\alpha+\beta}\right|+|B_4A_2|\left|t_1^{\alpha+\beta}-t_2^{\alpha+\beta}\right|.\end{aligned} \tag{9}$$

With the same arguments, we can prove that

$$\begin{aligned}\left|D^{\alpha}\Phi y(t_1)-D^{\alpha}\Phi y(t_2)\right| =\ & (E_L+|k|E_f+E_g+E_h)\Bigg[\int_0^{t_1}\frac{(t_1-s)^{\beta-1}-(t_2-s)^{\beta-1}}{\Gamma(\beta)}\int_0^{s}\frac{(s-u)^{\gamma-1}}{\Gamma(\gamma)}duds\\
&+\int_{t_1}^{t_2}\frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)}\int_0^{s}\frac{(s-u)^{\gamma-1}}{\Gamma(\gamma)}duds\Bigg]\\
&+\Bigg[(E_L+|k|E_f+E_g+E_h)|B_5|\int_0^1\frac{(1-s)^{2\alpha-1}}{\Gamma(2\alpha)}\int_0^{s}\frac{(s-u)^{\beta-1}}{\Gamma(\beta)}\int_0^{u}\frac{(u-v)^{\gamma-1}}{\Gamma(\gamma)}dvduds\\
&+|B_6A_2+B_7A_1-B_5A_3|\Bigg]\left|t_1^{\beta}-t_2^{\beta}\right|.\end{aligned} \tag{10}$$

On the other hand, we have

$$
\begin{aligned}
&\left|D^p\Phi y(t_1)-D^p\Phi y(t_2)\right| \\
\leq\ & \left(E_L+|k|\,E_f+E_g+E_h\right)\left[\int_0^{t_1}\frac{\left|(t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1}\right|}{\Gamma(\alpha)}\int_0^s\frac{(s-u)^{\beta-1}}{\Gamma(\beta)}\int_0^u\frac{(u-v)^{\gamma-p-1}}{\Gamma(\gamma-p)}dvduds\right. \\
&\left.+\int_{t_1}^{t_2}\frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}\int_0^s\frac{(s-u)^{\beta-1}}{\Gamma(\beta)}\int_0^u\frac{(u-v)^{\gamma-p-1}}{\Gamma(\gamma-p)}dvduds\right] \\
&+\left[\left(E_L+|k|\,E_f+E_g+E_h\right)|B_1B_8|\left[\int_0^1\frac{(1-s)^{2\alpha-1}}{\Gamma(2\alpha)}\int_0^s\frac{(s-u)^{\beta-1}}{\Gamma(\beta)}\int_0^u\frac{(u-v)^{\gamma-1}}{\Gamma(\gamma)}dvduds\right]\right. \\
&\left.+|B_2A_2+B_3A_1-B_1A_3|\,|B_8|\right]\left|t_1^{\alpha+\beta-p}-t_2^{\alpha+\beta-p}\right| \\
&+|B_4B_9A_2|\left|t_1^{\alpha-p}-t_2^{\alpha-p}\right|. \qquad (11)
\end{aligned}
$$

It is clear to affirm that the right-hand sides of (9), (10) and (11) tend to zero, when $t_1\to t_2$. Hence, Φ is a completely continuous operator.

Step4: We have to certify that the set $W:=\{z\in Z: z=\eta\Phi(z), 0<\eta<1\}$ is bounded in Z. Let $y\in W$, for some $0<\eta<1$. We have $y=\eta\Phi(y)$, and then we can write:

$$
\begin{aligned}
\|y\|_\infty \leq\ & \eta\Bigg(\left[E_L+|k|\,E_f+E_g+E_h\right]\left[\frac{1}{\Gamma(\gamma+\beta+1)}+|B_5|\frac{1}{\Gamma(2\alpha+\gamma+\beta+1)}\right] \\
&+|B_6A_2+B_7A_1-B_5A_3|+|A_2|\Bigg).
\end{aligned}
$$

In the same manner, we can prove that

$$
\begin{aligned}
\|D^\alpha\Phi y\|_\infty \leq\ & \eta\Bigg(\left[E_L+|k|\,E_f+E_g+E_h\right]\left[\frac{1}{\Gamma(\gamma+\beta+1)}+|B_5|\frac{1}{\Gamma(2\alpha+\gamma+\beta+1)}\right] \\
&+|B_6A_2+B_7A_1-B_5A_3|+|A_2|\Bigg).
\end{aligned}
$$

Also, we have

$$
\begin{aligned}
\|D^p\Phi y\|_\infty \leq\ & \eta\Bigg(\left[E_L+|k|\,E_f+E_g+E_h\right]\left[\frac{1}{\Gamma(\alpha+\beta+\gamma-p+1)}+|B_1B_8|\frac{1}{\Gamma(2\alpha+\beta+\gamma+1)}\right] \\
&+|B_2A_2+B_3A_1-B_1A_3|+|B_4B_9A_2|\Bigg).
\end{aligned}
$$

Thanks to (6), (7) and (8), we deduce that $\|y\|_Z<\infty$. This implies that W is bounded.

It follows from the above four steps that Φ has at least one fixed point which is a solution of our problem. ■

4 Examples

This section deals with two examples to illustrate our results by.

Example 1 *We consider the following problem*

$$\begin{cases} D^{0.71}D^{0.69}D^{0.61}z(t) + 0.08f\left(t, D^{0.61}z(t)\right) + g\left(t, z(t), D^{0.5}z(t)\right) + h\left(t, z(t), J^{0.4}z(t)\right) = L(t) \\ z(0) = \pi + \sqrt{2}, \quad D^{0.61}z(0) = -1, \quad J^{0.61}z(1) = \frac{2}{23}, \quad t \in [0,1]. \end{cases} \tag{12}$$

It is clear that

$$\alpha = 0.61, \quad \beta = 0.69, \quad \gamma = 0.71, \quad k = 0.08, \quad p = 0.5 \quad q = 0.4$$

and

$$\begin{aligned} f(t, a_1) &= \frac{\cos(3+t^2)}{\sqrt{1+t^3}} + \frac{4}{9}a_1, \\ g(t, a_1, a_2) &= \frac{\sin(2+t)}{\sqrt{6+t^3}} + \frac{1}{8}a_1 + \frac{6}{998}a_2, \\ h(t, a_1, a_2) &= \frac{1}{80+t^4} + \frac{\sqrt{2\pi-3}}{5}a_1 + \frac{6}{71}a_2, \\ L(t) &= \frac{2t+3}{7}. \end{aligned}$$

Taking $t \in [0,1]$ *and* $a_1, a_2, b_1, b_2 \in \mathbb{R}$, *we have*

$$|f(t, a_1) - f(t, b_1)| \leq \frac{4}{9}|a_1 - b_1|,$$

$$|g(t, a_1, a_2) - g(t, b_1, b_2)| \leq \frac{1}{8}|a_1 - b_1| + \frac{6}{998}|a_2 - b_2|$$

and

$$|h(t, a_1, a_2) - h(t, b_1, b_2| \leq \frac{\sqrt{2\pi-3}}{5}|a_1 - b_1| + \frac{6}{71}|a_2 - b_2|.$$

Hence,

$$W_1 = \frac{4}{9}, \quad W_2 = \frac{1}{8}, \quad W_3 = \frac{6}{998}, \quad W_4 = \frac{\sqrt{2\pi-3}}{5}, \quad W_5 = \frac{6}{71}.$$

Therefore, we obtain

$$T_1 = 0.531\,60, \quad T_2 = 0.786\,97, \quad T_3 = 0.670\,76.$$

Consequently, we can write

$$T = \max(T_1, T_2, T_3) = 0.786\,97 < 1.$$

By Theorem 1, we confirm that (12) has a unique solution on $[0,1]$.

Example 2 *Let us now consider the following second problem:*

$$\begin{cases} D^{0.70}D^{0.58}D^{0.73}z(t) + 0.11f\left(t, D^{0.73}z(t)\right) + g\left(t, z(t), D^{0.6}z(t)\right) + h\left(t, z(t), J^{0.55}z(t)\right) = L(t) \\ z(0) = 3\pi - \sqrt{2}, \quad D^{0.73}z(0) = -1, \quad J^{0.73}z(1) = \frac{5}{18}, \quad t \in [0,1], \end{cases} \tag{13}$$

under the conditions

$$\begin{cases} f(t, a_1) = \frac{e^{-3t}}{1+2ta_1^2} \\ g(t, a_1, a_2) = \frac{\cos(a_1 a_2)}{5+t^2(a_1+a_2)^2} \\ h(t, a_1, a_2) = \frac{\sin(a_1+a_2)}{\left(\sqrt{2+\pi}+e^{3t}\right)^2} \\ L(t) = \frac{t+3}{8}, \\ \alpha = 0.73, \quad \beta = 0.58, \quad \gamma = 0.70 \quad k = 0.11, \quad p = 0.6 \quad q = 0.55. \end{cases}$$

Clearly, we have

$$|f(t,a_1)| \leq 1 = E_f, \quad |g(t,a_1,a_2)| \leq \frac{1}{5} = E_g,$$

$$|h(t,a_1,a_2)| \leq \frac{1}{2+\pi} = E_h \quad \textit{and} \quad \|L(t)\|_\infty = \frac{1}{2} = E_L.$$

Then, by Theorem 2, we state that (13) has a solution.

5 Conclusion

In the current research, we propose a new fractional problem of Duffing type. With its parameters, this problem dos not satisfy the (CO-SG) properties, and it allows us, in particular, to obtain both the standard form of Duffing equation and the fractional equation in [9]. Under some sufficient conditions, we establish an existence and uniqueness criteria, then under some new sufficient conditions, we establish an existence criteria for the studied problem. Two illustrative examples are also discussed.

Acknowledgment. This paper is supported by DGRSDT Algerie.

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