Applied Mathematics E-Notes, 21(2021), 669-677 © ISSN 1607-2510
Available free at mirror sites of http://www.math.nthu.edu.tw/∼amen/

An Algebraic Note On Print Gallery*

Anood E. Alkatheeri†, Waldo G. Arriagada‡

Received 22 October 2020

Abstract

In 2003 mathematicians H. Lenstra and B. de Smit developed a satisfactory technique to *fill the gap* in Escher's 1956 lithography *Print Gallery.* Their procedure consists in the *untangling* of the distorted image via a complex transformation and subsequent continuation of the image onto the blank space using the Droste effect. The method is interesting from the algebraic point of view as well. We prove, for example, that the transformation proposed by Lenstra and de Smit expands the volume of the underlying lattice. It hence enlarges the Euclidean volume of the distorted elliptic curve (a complex torus) in the exact sequence. In this note we compute a nondegenerate polarization on the underlying lattice. Since the Riemann conditions are automatically satisfied in dimension one, the complex torus has the structure of a complex abelian variety: it admits sufficiently many meromorphic functions.

1 Introduction

The theory of conformal mappings is one of the cornerstones of modern mathematics and it finds immediate applications in many areas of physics and engineering. One of the relative recent achievements of conformal analysis is the completion of the much celebrated Escher's 1956 lithography *Prententententoonstelling* ("Print Gallery") by Dutch mathematicians B. de Smit and H. Lenstra Jr. [2, Figure 1]. Their method consists in the *untangling* of the image via a complex transformation

$$w \mapsto z = w^{\alpha} \tag{1}$$

where α is a complex number carefully determined by the authors. The untangled image contains a blank gap bounded by two spirals [2, Figure 12]. The white space is then filled in by continuation via the Droste effect. The final picture, observed in several documentaries and textbooks, corresponds to the image of the straightened (and *completed*) frame under the inverse $z \mapsto w = z^{1/\alpha}$ of the transformation (1). The number α encapsulates Escher's idea of bringing a straight frame into a distorted one in an *almost conformal* way.

Lenstra *et al.* proved that the map (1) is closely related to a (linear) transformation between universal covers which descends (via the exponential map) as a complex transformation of elliptic curves. The first isomorphism theorem implies that this map is indeed a change of coordinates between two complex tori of dimension one (see diagram (9) below). A complex torus of dimension one is a torus of real dimension 2 which carries the structure of a complex manifold. It can always be obtained as the quotient of a one-dimensional complex vector space by a lattice of rank 2. In this note we prove that the transformation between the tori is an expanding map of the underlying lattice. Then we compute a polarization on the frame. (A polarization is a positive definite Hermitian form with image in $\mathbb{Z}$ when restricted to the lattice). Through a well known result in algebraic geometry [6, p. 35], we prove that each complex torus has the structure of a complex abelian variety. This means that the complex tori are isomorphic to projective algebraic varieties over $\mathbb{C}$ and the latter are in turn endowed with a group structure. The existence of a polarization guarantees that the complex torus admits sufficiently many meromorphic functions. This provides a wider characterization of Escher's lithography from an algebraic viewpoint.

---

*Mathematics Subject Classifications: 00A09, 00A66, 06-02, 06B20.

†School of Engineering, Khalifa University, P.O. Box 127788, Abu Dhabi, United Arab Emirates

‡Department of Mathematics, Khalifa University, P.O. Box 127788, Abu Dhabi, United Arab Emirates

2 Complex Tori

A lattice $\Lambda \subset \mathbb{C}^n$ is a discrete subgroup of $\mathbb{C}^n$ which is isomorphic to $\mathbb{Z}^{2n}$ and which spans $\mathbb{R}^{2n}$:

$$\begin{aligned}\Lambda &= \mathbb{Z}\,\omega_1 + \cdots + \mathbb{Z}\,\omega_{2n}\\ &= \{m_1\omega_1 + \cdots + m_{2n}\,\omega_{2n} \,:\, m_1, \ldots, m_{2n} \in \mathbb{Z}\}\end{aligned}$$

where the set $\{\omega_1, \ldots, \omega_{2n}\}$ is a basis for $\mathbb{R}^{2n}$. Such a collection is called the basis of the lattice and the numbers $\omega_1, \ldots, \omega_{2n}$ are the generators of Λ. A lattice in $\mathbb{C}^n$ is thus an abelian group with a basis of (maximal) rank $2n$ and which acts on $\mathbb{C}^n$ by addition. Different bases can generate the same lattice, but the number

$$\mathrm{d}(\Lambda) = \big|\det[\omega_1, \ldots, \omega_{2n}]\big|, \tag{2}$$

which is sometimes called the covolume of the lattice, is an invariant under changes of bases. A lattice divides the whole of $\mathbb{C}^n$ into identical polyhedra. These correspond to shifted copies of an n-dimensional parallelepiped $\mathfrak{F}$ called the fundamental region of the lattice or fundamental parallelogram. The number $\mathrm{d}(\Lambda)$ corresponds to the hypervolume (i.e. the Euclidean volume in $\mathbb{R}^{2n}$) of this polyhedron.

Given a lattice $\Lambda \subset \mathbb{C}^n$, the quotient $\mathbb{T} = \mathbb{C}^n/\Lambda$ is called a complex torus. This set has the structure of a connected complex manifold which is also compact, since Λ is of maximal rank as a discrete subgroup of $\mathbb{C}^n$ (and thus $\mathbb{T}$ is the image of a bounded subset of $\mathbb{C}^n$). The space $\mathbb{C}^n$ may be considered as the universal covering space of $\mathbb{T}$ via the covering map

$$\pi : \mathbb{C}^n \to \mathbb{T} \tag{3}$$

whose kernel is the lattice Λ. The kernel can be identified with the fundamental group $\pi_1(\mathbb{T}) = \pi_1(\mathbb{T}, 0)$. Since Λ is abelian, the fundamental group $\pi_1(\mathbb{T})$ is canonically isomorphic to the first homology group $H_1(\mathbb{T}, \mathbb{Z})$. A one-dimensional complex torus is called an elliptic curve.

2.1 Meromorphic Functions

A meromorphic function $f : D \to \mathbb{C}$ from an open subset $D \subset \mathbb{C}^n$ is locally the quotient of two holomorphic functions (with the denominator not identically equal to zero) defined on D. A zero of the denominator of a meromorphic function is called a *pole*. For example, $f : \mathbb{C}^2 \to \mathbb{C}$ defined by $f(z, w) = z/w$ is a meromorphic function with a pole at $w = 0$. Let $\mathbb{T} = \mathbb{C}^n/\Lambda$ be a complex torus. A function $f : \mathbb{T} \to \mathbb{C}$ is called meromorphic if the composite $f \circ \pi : \mathbb{C}^n \to \mathbb{C}$ is meromorphic, where π is the projection (3). Those meromorphic functions on $\mathbb{C}^n$ which are invariant under addition of elements of a given lattice Λ (i.e. Λ-periodic) may be considered as functions defined on the torus $\mathbb{T} = \mathbb{C}^n/\Lambda$. Notice that constant functions are holomorphic thus meromorphic. Since $\mathbb{T}$ is compact, any holomorphic function on $\mathbb{T}$ is bounded, thus constant (Liouville's theorem). In general, a complex torus of dimension $n \geq 2$ does not admit *any* (nonconstant) meromorphic function. Those complex tori which indeed admit sufficiently many meromorphic functions are called *abelian varieties*, see §4.1 below. It is known [1] for example that the field of meromorphic functions on $\mathbb{T} = \mathbb{C}/\Lambda$ is generated by the Weierstrass function $\wp : \mathbb{C} \to \mathbb{C}$ along with its derivative $\wp'$, where

$$\wp(z) = \frac{1}{z^2} + \sum_{\substack{w\in\Lambda\\ w\neq 0}} \frac{1}{(z-w)^2} - \frac{1}{w^2}.$$

This function is well-defined and holomorphic in $\mathbb{C}$ minus Λ; it is meromorphic at each $z \in \Lambda$ with a pole of order 2 there and it is even and Λ-periodic.

2.2 Real, Rectangular and Rhombic Lattices

Two nonzero complex numbers z and w are called linearly independent over $\mathbb{R}$ (or simply linearly independent) if whenever $r_1z + r_2w = 0$ for real numbers r_1 and r_2, then necessarily $r_1 = r_2 = 0$. If z and w are linearly independent we write $z \perp w$. The numbers z and w are called linearly dependent over $\mathbb{R}$ (or linearly dependent) if they are not linearly independent. In this case we write $z \parallel w$. Note that $0 \parallel w$ for any $w \in \mathbb{C}$.

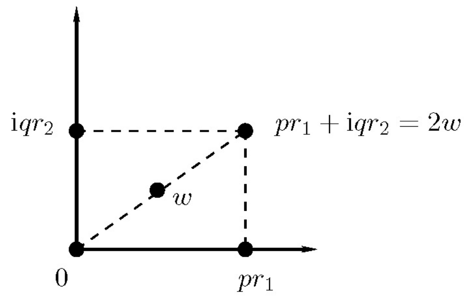

Figure 1: A number $w \in \Lambda \backslash \Lambda_0$.

Lemma 1 *Two nonzero complex numbers z, w are linearly independent ($z \perp w$) if and only if their quotient is non-real.*

Proof. Suppose that there exists a nonzero $r \in \mathbb{R}$ such that $z/w = r$. Then $z - rw = 0$ is a nonzero linear combination and hence z and w are linearly dependent over $\mathbb{R}$.

Conversely, assume that the quotient $z/w = \mathrm{i}\nu$ where $\nu \in \mathbb{R}$ is nonzero and $\mathrm{i} = \sqrt{-1}$ is the complex imaginary unit. Suppose that $r_1 z + r_2 w = 0$ for real numbers r_1 and r_2. If $z = a + \mathrm{i}b$ and $w = c + \mathrm{i}d$ where $a, b, c, d \in \mathbb{R}$ are nonzero, then taking real and imaginary parts yields $r_1\, a + r_2\, c = 0$ and $r_1\, b + r_2\, d = 0$ and then $r_1(ad - bc) = r_2(ad - bc) = 0$. Since $a = -d\nu$ and $b = c\nu$ these equations are equivalent to $r_1\nu(c^2 + d^2) = 0$ and $r_2\nu(c^2 + d^2) = 0$ and then $r_1 = r_2 = 0$. The numbers z and w are thus linearly independent. ■

Let ω_1, ω_2 be two nonzero linearly independent complex numbers. A lattice

$$\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 = \{n\,\omega_1 + m\,\omega_2 : n, m \in \mathbb{Z}\} \subset \mathbb{C} \tag{4}$$

is called *real* if $\omega \in \Lambda$ implies $\overline{\omega} \in \Lambda$. That is, Λ is invariant under the complex conjugation. The lattice Λ is called *rectangular* if the basis (ω_1, ω_2) is such that ω_1 is pure real and ω_2 is pure imaginary. The lattice is called *rhombic* if (ω_1, ω_2) are such that $\omega_2 = \overline{\omega_1}$.

Proposition 2 *A lattice $\Lambda \subset \mathbb{C}$ is real if and only if it is either rectangular or rhombic.*

Proof. Rectangular and rhombic lattices are trivially real. Let us prove the converse statement. Let $\Lambda \subset \mathbb{C}$ be a real lattice. Notice that if z is any number in Λ then both $(z + \overline{z}) \in \mathbb{R}$ and $(z - \overline{z}) \in \mathrm{i}\mathbb{R}$ belong to Λ as well. Hence Λ contains nonzero pure real and nonzero pure imaginary points. Then the subsets $R = \Lambda \cap \mathbb{R}$ and $I = \Lambda \cap \mathrm{i}\mathbb{R}$ are nonempty. The set $\Lambda_0 = R + I$ is thus a sublattice of Λ which is generated by a pure real number r_1 and a pure imaginary number $\mathrm{i}r_2$:

$$\Lambda_0 = \mathbb{Z}\, r_1 + \mathbb{Z}\, \mathrm{i}r_2.$$

If $\Lambda_0 = \Lambda$ then the lattice Λ is rectangular. Otherwise, there exists a complex number $w \in \Lambda \backslash \Lambda_0$ and integers p, q such that w belongs to the parallelogram defined by $0, pr_1, \mathrm{i}qr_2$ and $pr_1 + \mathrm{i}qr_2$, see Figure 1. Notice that $2w = (w + \overline{w}) + (w - \overline{w})$ and then $2w \in \Lambda_0$. Moreover, the condition $w \notin \Lambda_0$ means that the number w is neither pure real nor pure imaginary and hence $2w$ cannot coincide with the vertices $0, pr_1$ or $\mathrm{i}qr_2$. Therefore, $2w = pr_1 + \mathrm{i}qr_2$ and w is a point on the main diagonal of the fundamental parallelogram of the lattice Λ_0. Since the numbers $w = (pr_1 + \mathrm{i}qr_2)/2$ and

$$w - \mathrm{i}qr_2 = \frac{1}{2}(w - \mathrm{i}qr_2) = \overline{w}$$

are linearly independent, they generate Λ and in this case this lattice is rhombic. ■

A fundamental region or fundamental parallelogram for the lattice Λ in (4) is given by the subset

$$\mathfrak{F} = \{t_1\,\omega_1 + t_2\,\omega_2 : 0 \le t_1, t_2 \le 1\}.$$

The elements of the quotient space $\mathbb{C}/\Lambda$ are equivalence classes with respect to the equivalence relation

$$z \equiv w \mod \Lambda \;\Leftrightarrow\; z - w \in \Lambda.$$

Each element in $\mathbb{C}/\Lambda$ has a representative point in $\mathfrak{F}$. The equivalence class of a number z is denoted by $[z] = z + \Lambda$. The addition of complex numbers induces the addition of equivalence classes defined by $[z] + [w] = [z + w]$. It is easy to see that this operation does not depend on the representatives of z and w. As mentioned before, this operation equips the quotient space with the structure of an abelian group. The quotient is thus obtained by gluing the opposite edges of the fundamental region $\mathfrak{F}$. The images of the lines $\overline{0\omega_1}$ and $\overline{0\omega_2}$ correspond to cycles A and B on the torus. These cycles generate the first homology group $H_1(\mathbb{T}, \mathbb{Z})$ which coincides with the fundamental group $\pi_1(\mathbb{T})$. In the one-dimensional case, formula (2) has a simple expression in terms of the complex conjugation.

Lemma 3 *The covolume of the lattice* (4) *is* $\mathrm{d}(\Lambda) = \left|\,\mathrm{Im}(\overline{\omega_1}\,\omega_2)\,\right|$.

Proof. The covolume $\mathrm{d}(\Lambda)$ corresponds to the area of the parallelogram spanned by $\vec{\omega}_1$ and $\vec{\omega}_2$ as vectors of $\mathbb{R}^2$:

$$\mathrm{d}(\Lambda) = |\det\,[\omega_1,\, \omega_2]| = \|\vec{\omega}_1 \times \vec{\omega}_2\| = |\mathrm{Im}(\overline{\omega_1}\,\omega_2)|\,.$$

■

In general, if z and w are any two nonzero complex numbers then the Euclidean volume of the parallelogram determined by $0, z, w$ and $z + w$ is defined by

$$\mathrm{d}(z, w) = \begin{cases} |\mathrm{Im}(\overline{z}\,w)| & \text{if } z \perp w \\ 0 & \text{if } z \parallel w. \end{cases} \tag{5}$$

This function is called, by analogy, the covolume of the parallelogram defined by z and w. Lemma 1 implies that the covolume function (5) is indeed well defined.

3 Escher's Print Gallery

In 2003 Dutch mathematicians H. Lenstra Jr. and B. de Smit found a concise and satisfactory way to filling in the blank gap at the center of Escher's famous 1956 lithography *Prententententoonstelling* ("Print Gallery") in a conformal way [2, Figure 1]. The authors first analyzed Escher's grid [2, Figure 11]. This net-like pattern (reticulation) is what the authors call Escher's *distorted* grid. Tracing a *square* loop clockwise around zero corresponds to *walking* in an ideally straightened picture in $\mathbb{C}$, in which the square loop corresponds to a path making three left turns, each time traveling four times as far as the previous time, before making the next turn. In the straight plane the loop is not a closed circuit though. Rather, if the origin is properly chosen, the end point is 256 times the starting point [2, p. 448]. This reflects the original invariance of the frame under a homothety with multiplicative period of 256.

Conversely, walking along a square path in the straightened picture would result in a spiral in Escher's grid ending at a point scaled down by a factor of ~ 22.5836845286 and rotated clockwise by ~ 157.6255960832 degrees. Thus, the distorted grid must have been invariant under multiplication by a complex constant γ associated to those numbers. Since the squares have ideally preserved their local angles, Lenstra inferred that the mapping between the distorted and straight grids must be conformal (at least theoretically). The unfolded image thus obtained contained a blank gap bounded by two spirals which corresponds to the white circular gap in the original lithography. The blank area was then properly filled in by continuation via the Droste effect (with period 256), cf. [2, Figure 13]. The final frame was obtained after application of the transformation (1). The completed frame is depicted in [2, Figure 15].

3.1 Summary of the Method

The disentangled version of the lithography is to be drawn on the complex plane $\mathbb{C}$ with 0 at the center and whose complex coordinate is denoted by z. Lenstra proves that the *distorting* transformation is the composite complex map

$$w = h(z) = \exp(\alpha^{-1}\log(z)) = z^{1/\alpha}. \tag{6}$$

The value of α is obtained through an algebraic argument (see below),

$$\alpha = \frac{2\pi \mathrm{i} + \log 256}{2\pi \mathrm{i}}. \tag{7}$$

The untangled picture is invariant under the homothety $z \mapsto 256\, z$ and hence, the distorted image must be invariant under

$$w = h(z) \mapsto h(256\, z) = (256\, z)^{1/\alpha} = 256^{1/\alpha} w.$$

Hence, while the straightened image is periodic with a multiplicative period of 256, the tangled picture is periodic with a complex period of

$$\gamma = 256^{1/\alpha} = \exp(3.1172277221 + \mathrm{i}\, 2.7510856371).$$

Multiplication by $256^{1/\alpha}$ of course corresponds to applying a homothety of ratio $|\gamma| \sim 22.5836845297$ and a rotation of angle $\arg(\gamma) \sim 157.6255960852°$, which match the numbers at the end of the previous paragraph up to the first eight decimal places. Lenstra proved that the map h in (6) is a bijective correspondence between complex tori. This correspondence is indeed the projection of a linear map between universal cover spaces. The exact relation between h and the linear transformation is determined as follows. The exponential map $\exp : \mathbb{C} \to \mathbb{C}^*/\langle\gamma\rangle$ identifies $\mathbb{C}$ with the universal cover of $\mathbb{C}^*/\langle\gamma\rangle$. The lattice

$$\Lambda_\gamma = \mathbb{Z}\, 2\pi \mathrm{i} + \mathbb{Z} \log \gamma \tag{8}$$

corresponds to the kernel of the exponential and hence the short sequence

$$0 \hookrightarrow \Lambda_\gamma \hookrightarrow \mathbb{C} \xrightarrow{\exp} \mathbb{C}^*/\langle\gamma\rangle \longrightarrow 0$$

is exact. By the first isomorphism theorem, the quotient space $\mathbb{C}^*/\langle\gamma\rangle$ is isomorphic to the complex torus $\mathbb{T}_\gamma = \mathbb{C}/\Lambda_\gamma$ with fundamental group $\pi_1(\mathbb{T}_\gamma) = \Lambda_\gamma$. Appropriate coordinates on the torus enables us to take $h(1) = 1$ and then h^{-1} lifts to a unique automorphism of the complex plane $\alpha \mapsto \alpha w$ which fixes 0. The short sequence induces commutative diagrams [2],

$$\begin{tikzcd} 0 \arrow[r] & \Lambda_\gamma \arrow[r] \arrow[d, "\alpha"] & \mathbb{C} \arrow[r, "\exp"] \arrow[d, "\alpha"] & \mathbb{T}_\gamma \arrow[r] \arrow[d, "h^{-1}"] & 0 \\ 0 \arrow[r] & \Lambda_{256} \arrow[r] & \mathbb{C} \arrow[r, "\exp"] & \mathbb{T}_{256} \arrow[r] & 0 \end{tikzcd} \tag{9}$$

where $\Lambda_{256} = \mathbb{Z}\log 256 + \mathbb{Z}2\pi\mathrm{i}$ is the rectangular lattice corresponding to the kernel of the exponential map of the lower exact sequence and $\mathbb{T}_{256} = \mathbb{C}/\Lambda_{256}$. Proposition 2 ensures that this lattice is also real. The map $z \mapsto \alpha z$ is hence a map between fundamental groups. The element $2\pi\mathrm{i} \in \Lambda_\gamma$ corresponds to a single counterclockwise loop around zero in $\mathbb{C}^*$ which is given, up to homotopy, by the square loop depicted in [2, Figure 11]. In the untangled grid this loop becomes a path around zero. In the quotient $\mathbb{C}^*/\langle 256\rangle$ this path becomes a closed loop which represents the element $2\pi\mathrm{i} + \log 256$ in the fundamental group Λ_{256}. That is, multiplication by α in (9) sends $2\pi\mathrm{i}$ into $2\pi\mathrm{i} + \log 256$, whence the value (7) follows. The relationship between fundamental groups is thus $\alpha\, \Lambda_\gamma = \Lambda_{256}$.

Proposition 4 *The complex map h expands the Euclidean volume of the fundamental parallelogram $\mathfrak{F}_\gamma$ in a factor of* $1 + (\log 256/2\pi)^2$.

Proof. Let ω_1, ω_2 be a pair of generators of Λ_γ. In the diagram (9) the point $\omega_i \in \Lambda_\gamma$ is brought into $\alpha\omega_i \in \Lambda_{256}$ by the transformation $z \mapsto \alpha w$. Lemma 3 yields

$$\mathrm{d}(\Lambda_{256}) = \overline{\alpha}\alpha\big|\mathrm{Im}(\overline{\omega_1}\omega_2)\big| = \overline{\alpha}\alpha\mathrm{d}(\Lambda_\gamma) = \big(1 + (\log 256/2\pi)^2\big)\mathrm{d}(\Lambda_\gamma).$$

■

4 Polarizations

An $\mathbb{R}$-bilinear form on $\mathbb{C}^n$ is a function $E : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{R}$ such that

$$\begin{aligned} E(u+v,w) &= E(u,w) + E(v,w), \\ E(u,v+w) &= E(u,v) + E(u,w), \\ E(\lambda\, u,v) &= E(u,\lambda\, v) = \lambda\, E(u,v) \end{aligned} \tag{10}$$

for any $u, v, w \in \mathbb{C}^n$ and $\lambda \in \mathbb{R}$. Such a form is called alternating if $E(u,u) = 0$ for all $u \in \mathbb{C}^n$. Note that if E is alternating then $E(u,v) = -E(v,u)$. That is, any alternating $\mathbb{R}$-bilinear form is automatically skew-symmetric. A Hermitian form is a map $H : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ which is linear with respect to the first coordinate and satisfies $H(z,w) = \overline{H(w,z)}$ (i.e. it is $\mathbb{C}$-antilinear in the second coordinate). Thus, H is $\mathbb{R}$-bilinear and $H(\mathrm{i}u,v) = \mathrm{i}H(u,v) = H(u,\bar{\mathrm{i}}v)$. It is known that a decomposition $H = S + \mathrm{i}E$ exists where $S, E : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{R}$ are $\mathbb{R}$-bilinear maps. The form S is symmetric and satisfies $S(\mathrm{i}u, \mathrm{i}v) = S(u,v)$ and the form E is skew-symmetric and satisfies $E(\mathrm{i}u, \mathrm{i}v) = E(u,v)$. Moreover, $S(u,v) = E(\mathrm{i}u, v)$.

Definition 1 *Let $\Lambda \subset \mathbb{C}^n$ be a lattice. A Riemann form or polarization on the complex torus $\mathbb{T} = \mathbb{C}^n/\Lambda$ is a Hermitian form $H : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ such that the restriction of $E = \mathrm{Im}(H)$ to Λ is integer valued. If $H(u,u) \geq 0$ for all $u \in \mathbb{C}^n$ then H is positive semi-definite. If H is positive semi-definite and $H(u,u) = 0$ only for $u = 0$ then H is called nondegenerate.*

A nondegenerate polarization on the torus exists exactly when the latter admits sufficiently many meromorphic functions [4, p. 6].

4.1 Abelian Varieties Over $\mathbb{C}$

We denote by $\mathbb{C}[X_1, \ldots, X_n]$ the algebra of polynomials in the n variables $X_1, \ldots, X_n$ with coefficients in $\mathbb{C}$. This algebra is a commutative unitary ring (that is, a ring with an identity element and such that the ring product is commutative). For example, if $n = 2$ then

$$\mathrm{i}X_1X_2^3 + 6X_2 + (2\mathrm{i} - \sqrt{2})X_1^7 \in \mathbb{C}[X_1, X_2].$$

An algebraic variety is the zero locus of a system of polynomial equations:

Definition 2 *An algebraic variety or algebraic set is a subset $M(S) \subset \mathbb{C}^n$ which is the zero locus of a subset of polynomials $S \subset \mathbb{C}[X_1, \ldots, X_n]$:*

$$M(S) = \{(x_1, \ldots, x_n) \in \mathbb{C}^n : \text{for all } f \in S,\ f(x_1, \ldots, x_n) = 0\}.$$

A subset $A \subset \mathbb{C}^n$ is called algebraic if there exists $S \subset \mathbb{C}[X_1, \ldots, X_n]$ such that $A = M(S)$.

For example, the circle $\mathbb{S}^1 = \{(x,y) \in \mathbb{C}^2 : x^2 + y^2 - 1 = 0\}$ and the parabola $\{(x,y,z) \in \mathbb{C}^3 : x - y = 0, x^2 + 4y^2 - z = 0\}$ are algebraic varieties.

A morphism between algebraic varieties is a function which is given locally by polynomials. An algebraic variety X is called *complete* if for any variety Y the projection morphism $X \times Y \to Y$ is a closed map, i.e. it maps closed sets onto closed sets. An *abelian variety* X is a complete algebraic variety over $\mathbb{C}$ with a group structure on the underlying set such that the product $X \times X \to X$, $(x,y) \mapsto x \cdot y$ together with the inverse

$X \to X$, $x \mapsto x^{-1}$, are both morphisms of varieties [6]. An abelian variety is thus a complete algebraic variety whose points form a group, in such a way that the maps defining the group structure are given by morphisms.

A complex torus is an abelian variety if and only if it admits the structure of an algebraic variety [5, p. 367]. Abelian varieties are exactly those complex tori which admit a holomorphic embedding into some projective space. A complex torus which is not an abelian variety does not admit a holomorphic projective embedding [4]. Abelian varieties thus correspond to those complex tori which admit sufficiently many meromorphic functions.

Theorem 5 *[6, p. 35] A complex torus $\mathbb{T}$ is an abelian variety if and only if it admits a nondegenerate polarization.*

It is known that in dimension $n = 1$ every complex torus is analytically isomorphic to an abelian variety (isomorphic as complex analytic varieties [3, p. 91]). Hence the conclusion of the theorem is automatically satisfied in this case [6]. This result is not true in dimension greater than or equal to 2. In higher dimensions there are several equivalent criteria to determine whether a complex torus is isomorphic to an abelian variety. These equivalent formulations are called the Riemann conditions. In this note we will solely treat the case of dimension $n = 1$.

4.2 An Explicit Nondegenerate Polarization

The following construction provides an explicit nondegenerate polarization on the lattices Λ_γ and Λ_{256}, see diagram (9).

Theorem 6 *The map $H : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ given by*

$$H(z, w) = \frac{\overline{z}\, w}{2\pi \log 256} \tag{11}$$

is a nondegenerate polarization on the complex torus $\mathbb{T}_{256}$. A nondegenerate Riemann form $\hat{H} : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ on the complex torus $\mathbb{T}_\gamma$ is obtained by the formula

$$\hat{H}(z, w) = \frac{\mathrm{d}(\Lambda_\gamma)}{\mathrm{d}(\Lambda_{256})}\, H(z, w).$$

In particular, the complex tori $\mathbb{T}_{256}$ and $\mathbb{T}_\gamma$ admit sufficiently many meromorphic functions.

Proof. We first prove that any alternating $\mathbb{R}$-bilinear form $F : \mathbb{C} \times \mathbb{C} \to \mathbb{R}$ is of the type

$$F(z, w) = \nu \operatorname{Im}(\overline{z}\, w) \tag{12}$$

where the number $\nu \in \mathbb{R}$ (to be found) uniquely determines the function. Indeed, this function is alternating and hence skew-symmetric and satisfies properties (10). If $z = a + \mathrm{i}b$ and $w = c + \mathrm{i}d$, with $a, b, c, d \in \mathbb{R}$, are any pair of complex numbers then $F(z, w) = ac\, F(1, 1) + ad\, F(1, \mathrm{i}) + bc\, F(\mathrm{i}, 1) + bd\, F(\mathrm{i}, \mathrm{i})$. However, $F(1, 1) = F(\mathrm{i}, \mathrm{i}) = 0$ and $F(1, \mathrm{i}) = -F(\mathrm{i}, 1)$. Therefore $F(z, w) = F(1, \mathrm{i})\,(ad - bc) = F(1, \mathrm{i}) \operatorname{Im}(\overline{z}\, w)$. In particular, the alternating form is uniquely determined by $\nu = F(1, \mathrm{i})$.

Let $\Lambda \subset \mathbb{C}$ be a lattice with generators $\omega_1, \omega_2 \in \mathbb{C}$. Choose a pair of complex numbers $z = t_1\omega_1 + t_2\omega_2$ and $w = s_1\omega_1 + s_2\omega_2$, where $t_i, s_i \in \mathbb{R}$. If the real number $\operatorname{Im}(\omega_2/\omega_1)$ is positive then the map $E : \mathbb{C} \times \mathbb{C} \to \mathbb{R}$ given by

$$(z, w) \mapsto t_1 s_2 - s_1 t_2 \tag{13}$$

defines an $\mathbb{R}$-bilinear alternating and Riemannian form on the lattice Λ. Indeed, it is easy to prove that (13) verifies equivalences (10). For example, if we take $v = r_1\omega_1 + r_2\omega_2$, with $r_1, r_2 \in \mathbb{R}$, then

$$\begin{aligned} E(z + v, w) &= E\big((t_1\omega_1 + t_2\omega_2) + (r_1\omega_1 + r_2\omega_2), s_1\omega_1 + s_2\omega_2\big) \\ &= \big((t_1 + r_1)\omega_1 + (t_2 + r_2)\omega_2, s_1\omega_1 + s_2\omega_2\big) \\ &= (t_1 s_2 - s_1 t_2) + (r_1 s_2 - s_1 r_2) \\ &= E(z, w) + E(v, w). \end{aligned}$$

Likewise, if $\lambda \in \mathbb{R}$ then

$$E(\lambda\, z, v) = E(\lambda t_1\omega_1 + \lambda t_2\omega_2, r_1\omega_1 + r_2\omega_2) = \lambda t_1 s_2 - \lambda s_1 t_2 = \lambda\, E(z, v).$$

The other equivalences are analogous. Notice that $E(z, z) = E(t_1\omega_1 + t_2\omega_2, t_1\omega_1 + t_2\omega_2) = t_1t_2 - t_1t_2 = 0$ so (13) indeed defines an alternating $\mathbb{R}$-bilinear form on $\mathbb{C}$. Therefore, E can be written in the form (12) and is uniquely determined by $E(1, \mathrm{i})$. Let $\tau_i, \varsigma_i \in \mathbb{R}$ be the coefficients of 1 and i with respect (ω_1, ω_2) :

$$1 = \tau_1\omega_1 + \tau_2\omega_2 \quad \text{and} \quad \mathrm{i} = \varsigma_1\omega_1 + \varsigma_2\omega_2. \tag{14}$$

Then $\omega_1 = \dfrac{\varsigma_2 - \mathrm{i}\tau_2}{\varsigma_2\tau_1 - \varsigma_1\tau_2}$ and $\omega_2 = \dfrac{-\varsigma_1 + \mathrm{i}\tau_1}{\varsigma_2\tau_1 - \varsigma_1\tau_2}$. Hence,

$$E(1, \mathrm{i}) = \tau_1\varsigma_2 - \varsigma_1\tau_2 = (\tau_2^2 + \varsigma_2^2)\,\mathrm{Im}\left\{\frac{(-\varsigma_1 + \mathrm{i}\tau_1)(\varsigma_2 + \mathrm{i}\tau_2)}{\tau_2^2 + \varsigma_2^2}\right\} = (\tau_2^2 + \varsigma_2^2)\,\mathrm{Im}\left(\frac{\omega_2}{\omega_1}\right) > 0. \tag{15}$$

Moreover, if $z, w \in \Lambda$ then there exist integers n, m, p and q such that $z = n\omega_1 + m\omega_2$ and $\omega = p\omega_1 + q\omega_2$ and thus $E(z, w) = nq - pm \in \mathbb{Z}$.

Next, define the complex form

$$H(u, v) = E(\mathrm{i}u, v) + \mathrm{i}E(u, v). \tag{16}$$

Notice that this form satisfies $H(\mathrm{i}u, v) = E(\mathrm{i}^2 u, v) + \mathrm{i}E(\mathrm{i}u, v) = -E(u, v) + \mathrm{i}E(\mathrm{i}u, v) = \mathrm{i}H(u, v)$. Similarly, $\mathrm{i}H(u, v) = \mathrm{i}E(\mathrm{i}u, v) - E(u, v) = E(\mathrm{i}u, \bar{\mathrm{i}}v) + \mathrm{i}^2 E(u, v) = H(u, \bar{\mathrm{i}}v)$ and moreover,

$$H(v, u) = E(\mathrm{i}v, u) + \mathrm{i}E(v, u) = -E(u, \mathrm{i}v) - \mathrm{i}E(u, v) = \overline{H(u, v)}.$$

Hence, the map (16) is a Hermitian form. Estimate (15) ensures that H also defines a nondegenerate polarization on Λ_{256}. Relative to this lattice, the coefficients in (14) are given by $\tau_1 = 1/\log 256$, $\tau_2 = 0$, $\varsigma_1 = 0$ and $\varsigma_2 = 1/2\pi$ and thus $E(1, \mathrm{i}) = \tau_1\varsigma_2 - \varsigma_1\tau_2 = 1/2\pi \log 256$. This yields the form (11). Finally, note that Proposition 4 implies that

$$\hat{H}(z, w) = \frac{\mathrm{d}(\Lambda_\gamma)}{\mathrm{d}(\Lambda_{256})}\, H(z, w) = \frac{1}{\alpha\overline{\alpha}} H(z, w) = H(\alpha^{-1}z, \alpha^{-1}w)$$

and hence the map $\hat{H}$ is integer-valued on the lattice (8). The nondegeneracy is clear and then this map defines a nondegenerate polarization on the complex torus $\mathbb{T}_\gamma$. Theorem 5 ensures that the tori $\mathbb{T}_{256}$ and $\mathbb{T}_\gamma$ admit nonzero meromorphic functions. The theorem is proved. ∎

Lemma 3 carries the following corollary.

Corollary 7 *Let z and w be any two complex numbers. Then the covolume function* (5) *corresponds to*

$$\mathrm{d}(z, w) = \begin{cases} 2\pi \log(256)\,\big|\,\mathrm{Im}\big(H(z, w)\big)\big| & \text{if} \quad z \perp w \in \mathbb{C} \\ 0 & \text{if} \quad z \parallel w \end{cases}$$

where H is the polarization (11). *In particular,* $\mathrm{d}(\Lambda_{256}) = 2\pi \log 256$.

Consider the form $E = \mathrm{Im}(H)$, where H is the polarization (11). The *alternatization* of E is the map $\mathbb{C} \times \mathbb{C} \to \mathbb{R}$ defined by $(z, w) \mapsto E(z, w) - E(w, z)$. It is easy to prove that this map is $\mathbb{R}$-bilinear. The latter plays a crucial role in identifying the second cohomology group of Λ_γ with the group of alternating bilinear forms on the lattice. We look forward, in future publications, to addressing the problem of computing the second cohomology group of different admissible lattices Λ_δ where δ is a complex number such that $|\delta| \neq 1$.

Acknowledgment. The first author is grateful to Dr. Aaesha Al Naqbi and the Office of Student Services of Khalifa University for support granted during the preparation of this manuscript. This material has been presented at the *Second American University of Sharjah Regional Students' Conference on Mathematics March 4, 2017.*

The remarks and suggestions raised by the referee(s) allowed us to improve this manuscript to a great extent. Both authors are grateful to the unknown referee(s).

References

[1] L. Ahlfors, Complex Analysis, 3^{rd} ed., McGraw-Hill, 1979.

[2] B. de Smit, H. W. Lenstra, D. J. Dunham and R. Sarhangi, Artful mathematics: the heritage of M. C. Escher, Notices Amer. Math. Soc., 50(2003), 446–457.

[3] M. Hindry and J. H. Silverman, Diophantine Geometry, an Introduction, Graduate Texts in Mathematics 201, Springer, 2000.

[4] H. Lange and C. Birkenhake, Complex Abelian Varieties, Springer Verlag Grundlehren der mathematischen Wissenschaften 302, 1992.

[5] S. Lefschetz, On certain numerical invariants of algebraic varieties with application to Abelian varieties, Trans. Amer. Math. Soc., 22(1921), 327–482.

[6] D. Mumford, Abelian Varieties with Appendices by C.P. Ramanujam and Y. Manin, 2^{nd} ed., TIFR and Oxford Univ. Press, reprint, 1985.