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Existence And Uniqueness Of Solutions Of Volterra-Fredholm Nonlinear Integral Equation In Partially Ordered b-Metric Spaces*

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Abstract

In this paper, we prove some coupled coincidence and coupled common fixed point theorems for mappings having a mixed monotone property in partially ordered b-metric spaces. We also investigate the existence and uniqueness theorems for Volterra-Fredholm and Volterra integral equations. The result we have established is illustrated with an example.

1 Introduction

Consider the nonhomogeneous nonlinear Volterra integral equation

$$u(x) = \varphi\left(\int_a^x H(x,t,u(t))dt\right) + g(x) \equiv Tu, \quad u \in X, \tag{1}$$

where $x, t \in [a,b]$, $-\infty < a < b < +\infty$, $g : [a,b] \to \mathbb{R}^n$ is a mapping, H is a continuous function on the domain

$$D := \{(x,t,u) : x \in [a,b], t \in [a,x], u \in X\}$$

where $X := (C[a,b], \mathbb{R}^n)$, with the metric $d(f,g) = \max_{x\in[a,b]} |f(x) - g(x)|$, for all $f, g \in X$ and assume that φ is a bounded linear transformation on X. In this case, we define $\|\varphi\| = \sup\{\|\varphi x\|; \quad x \in X, \quad \|x\| = 1\}$. Thus, φ is bounded if and only if $\|\varphi\| < \infty$, [14].

Consider the system of Volterra-Fredholm integral equations

$$u(x) = g(x) + \int_a^x H(x,t,u(t))dt + \int_a^b G(x,t,u(t))dt, \quad x \in I = [a,b] \quad u \in X, \tag{2}$$

where $g : I \to X$, $H, G : I \times I \times X \to X$ are continuous. Nonlinear integral equations have studied by many authors in the literature and many authors have been studied the problems of existence, uniqueness, continuation and other properties of these type or special forms of the equations (1) and (2), see, for example, [12, 16, 21, 22, 23, 24]. The concept of a b-metric space was introduced by S. Czerwik (see [8, 9]). We recall from [8] the following definition.

Definition 1 ([8]) *Let X be a non-empty set and $s \geq 1$ a given real number. A function $d : X \times X \to \mathbb{R}^+$ is called a b-metric provided that, for all $x, y, z \in X$,*

(bm-1) $d(x,y) = 0$ *iff* $x = y$,

(bm-2) $d(x,y) = d(y,x)$,

(bm-3) $d(x,z) \leq s[d(x,y) + d(y,z)]$.

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The pair (X, d) is called a b-metric space with parameter s.

We remark that a metric space is evidently a b-metric space. However, S. Czerwik (see [8, 9]) has shown that a b-metric on X may not be a metric on X. For more considerations and examples of b-metric spaces, see, for example, [2, 5, 10, 11, 13]. The existence of coupled coincidence and coupled common fixed point theorems in partially ordered metric spaces has been considered recently by several authors, see, for example, [6, 7, 15, 17, 18, 19].

In this work, we establish coupled coincidence and coupled common fixed point results for a mixed g-monotone mapping in partially ordered b-metric spaces. Our results generalize recent results obtained by Luong and Thuan [20] and Berinde [3]. Also we will use an iterative method to prove that equations (1) and (2) have the mentioned cases under some appropriate conditions. Finally, we offer an example to illustrate verify the application of this kind of nonlinear functional-integral equations. First we introduce some new definitions in partially ordered metric spaces.

The concept of a mixed monotone property has been introduced by Bhaskar and Lakshmikantham in [4].

Definition 2 ([4]) *Let $(X, \preceq)$ be a partially ordered set. A mapping $F : X \times X \to X$ is said to have mixed monotone property if $F(x, y)$ is monotone nondecreasing in x and is monotone nonincreasing in y; that is, for any $x, y \in X$,*

$$x_1, x_2 \in X, x_1 \preceq x_2 \quad \textit{implies} \quad F(x_1, y) \preceq F(x_2, y),$$

$$y_1, y_2 \in X, y_1 \preceq y_2 \quad \textit{implies} \quad F(x, y_2) \preceq F(x, y_1).$$

The authors in [17] introduced the concept of a g-mixed monotone mapping.

Definition 3 ([17]) *Let $(X, \preceq)$ be a partially ordered set. Let us consider mappings $F : X \times X \to X$ and $g : X \to X$. The map F is said to have mixed g-monotone property if $F(x, y)$ is monotone g-nondecreasing in x and is monotone g-nonincreasing in y; that is, for any $x, y \in X$,*

$$x_1, x_2 \in X, gx_1 \preceq gx_2 \quad \textit{implies} \quad F(x_1, y) \preceq F(x_2, y),$$

$$y_1, y_2 \in X, gy_1 \preceq gy_2 \quad \textit{implies} \quad F(x, y_2) \preceq F(x, y_1).$$

An element $(x, y) \in X \times X$ is called a coupled fixed point of a mapping $F : X \times X \to X$ if $F(x, y) = x$ and $F(y, x) = y$ [4].

Also, an element $(x, y) \in X \times X$ is called a coupled coincidence point of the mappings $F : X \times X \to X$ and $g : X \to X$ if $F(x, y) = gx$ and $F(y, x) = gy$ [1].

The following definition was given by Luong and Thuan [20], that used in this paper.

Definition 4 *Let Θ denote the class of those functions $\theta : [0, +\infty)^2 \to [0, 1)$ which satisfies the condition: for any sequences $\{t_n\}$, $\{s_n\}$ of positive real numbers,*

$$\theta(t_n, s_n) \to 1 \quad \textit{implies} \quad t_n \to 0 \textit{ and } s_n \to 0.$$

For examples, $\theta_1(t_1, t_2) = k$, for all $(t_1, t_2) \in [0, +\infty)^2$, where $k \in [0, 1)$,

$$\theta_2(t_1, t_2) = \frac{ln(1 + k_1t_1 + k_2t_2)}{k_1t_1 + k_2t_2}, \; \forall \; (t_1, t_2) \in [0, +\infty)^2 \backslash \{(0, 0)\} \;\; \text{and} \;\; \theta_2(0, 0) \in [0, 1),$$

where $k_1, k_2 > 0$, are in Θ.

In [3], Berinde established some generalized coupled fixed point results for the mixed monotone mappings.

Theorem 1 ([3, Theorem 2.1]) *Let $(X, \preceq)$ be a partially ordered set and suppose there exists a metric d on X such that (X, d) is a complete metric space. Let $F : X \times X \to X$ be a mixed monotone mapping for which there exists a constant $k \in [0, 1)$ such that*

$$d(F(x, y), F(u, v)) + d(F(y, x), F(v, u)) \leq k \; [d(x, u) + d(y, v)],$$

for all $x, y, u, v \in X$ with $x \succeq u$ and $y \preceq v$. If there exist $x_0, y_0 \in X$ such that

$$x_0 \preceq F(x_0, y_0) \quad and \quad y_0 \succeq F(y_0, x_0),$$

or

$$x_0 \succeq F(x_0, y_0) \quad and \quad y_0 \preceq F(y_0, x_0),$$

then there exist $\bar{x}, \bar{y} \in X$ such that $\bar{x} = F(\bar{x}, \bar{y})$ and $\bar{y} = F(\bar{y}, \bar{x})$.

Recently, Luong and Thuan established some coupled fixed point results for the mixed monotone mappings in [20] and extended above theorem.

Theorem 2 ([20, Theorem 2.1]) *Let $(X, \preceq)$ be a partially ordered set and suppose there exists a metric d on X such that (X, d) is a complete metric space. Let $F : X \times X \to X$ be mapping such that F has the mixed monotone property on X. Suppose that there exists $\theta \in \Theta$ such that*

$$d(F(x, y), F(u, v)) + d(F(y, x), F(v, u)) \leq \theta(d(x, u), d(y, v))(d(x, u) + d(y, v))$$

for all $x, y, u, v \in X$ with $x \succeq u$ and $y \preceq v$. Suppose that either

(a) F is continuous, or

(b) X has the following properties:

(i) if a non-decreasing sequence $\{x_n\}$ converges x, then $x_n \preceq x$ for all $n \geq 0$;

(ii) if a non-increasing sequence $\{y_n\}$ converges y, then $y \preceq y_n$ for all $n \geq 0$.

If there exist $x_0, y_0 \in X$ such that $x_0 \preceq F(x_0, y_0)$ and $y_0 \succeq F(y_0, x_0)$, then F has a coupled fixed point, that is, there exist $x, y \in X$ such that $x = F(x, y)$ and $y = F(y, x)$.

In this paper, we study the existence and uniqueness of solution of the systems (1) and (2). In Sections 2 and 3 we establish coupled coincidence and coupled common fixed point results in partially ordered b-metric spaces. In section 4 we study the existence and uniqueness of solution to a nonlinear integral equation. In section 5 we give an example to illustrate the usefulness of our results.

2 Coupled Coincidence Point Theorems

We start this section by new definition.

Definition 5 *Let (X, d) be a b-metric space and let $g : X \to X$ and $F : X \times X \to X$ be two mappings. The mappings g and F are said to be b-compatible if*

$$\lim_{n\to\infty} d(gF(x_n, y_n), F(gx_n, gy_n)) = 0,$$

and

$$\lim_{n\to\infty} d(gF(y_n, x_n), F(gy_n, gx_n)) = 0,$$

are hold whenever $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $\lim_{n\to\infty} F(x_n, y_n) = \lim_{n\to\infty} gx_n$ and $\lim_{n\to\infty} F(y_n, x_n) = \lim_{n\to\infty} gy_n$.

Proposition 3 *Let (X, d) be a b-metric space. If $\{x_n\}$ and $\{y_n\}$ are b-convergent to x and y respectively, then*

$$\frac{1}{s^2} d(x, y) \leq \liminf_{n\to\infty} d(x_n, y_n) \leq \limsup_{n\to\infty} d(x_n, y_n) \leq s^2 d(x, y).$$

So if $s = 1$, then $\lim_{n\to\infty} d(x_n, y_n)$ exists and equals to $d(x, y)$.

Proof. Suppose $\{x_n\}$ and $\{y_n\}$ are b-convergent to x and y respectively. Then, by (bm-3) we have,

$$\begin{aligned} d(x_n, y_n) &\leq s(d(x_n, x) + d(x, y_n)) \\ &\leq s[d(x_n, x) + s(d(x, y) + d(y, y_n))] \\ &= sd(x_n, x) + s^2(d(x, y) + s^2 d(y, y_n)) \end{aligned}$$

consequently,

$$\limsup_{n\to\infty} d(x_n, y_n) \leq s^2 d(x, y). \tag{3}$$

Also,

$$\begin{aligned} d(x, y) &\leq s(d(x, x_n) + d(x_n, y)) \\ &\leq s[d(x, x_n) + s(d(x_n, y_n) + d(y_n, y))] \\ &= sd(x_n, x) + s^2 d(x_n, y_n) + s^2 d(y_n, y). \end{aligned}$$

Consequently,

$$\frac{1}{s^2} d(x, y) \leq \liminf_{n\to\infty} d(x_n, y_n). \tag{4}$$

Consequently, from (3) and (4), we have

$$\frac{1}{s^2} d(x, y) \leq \liminf_{n\to\infty} d(x_n, y_n) \leq \limsup_{n\to\infty} d(x_n, y_n) \leq s^2 d(x, y).$$

■

Now we prove our main results.

Theorem 4 *Let $(X, \preceq)$ be a partially ordered set and d be a b-metric on X such that (X, d) is a complete b-metric space with constant $s \geq 1$. Let $F : X \times X \to X$ and $g : X \to X$ are two mappings such that F has the mixed g-monotone property on X and there exists $\theta \in \Theta$ such that*

$$d(F(x, y), F(u, v)) + d(F(y, x), F(v, u)) \leq \frac{1}{s^2}\theta(d(gx, gu), d(gy, gv))(d(gx, gu) + d(gy, gv) \tag{5}$$

for all $x, y, u, v \in X$ with $gx \succeq gu$ and $gy \preceq gv$. Also let $F(X \times X) \subseteq g(X)$, g be continuous and g and F are b-compatible. Suppose that either

(a) F is continuous, or

(b) X has the following properties:

(i) if a nondecreasing sequence $\{x_n\}$ converges to x, then $gx_n \preceq gx$, for all $n \geq 0$;

(ii) if a nonincreasing sequence $\{y_n\}$ converges to y, then $gy \preceq gy_n$, for all $n \geq 0$.

If there exist $x_0, y_0 \in X$ such that $gx_0 \preceq F(x_0, y_0)$ and $gy_0 \succeq F(y_0, x_0)$, then g and F have a coupled coincidence point, that is, there exist $x, y \in X$ such that $gx = F(x, y)$ and $gy = F(y, x)$.

Proof. Let $x_0, y_0 \in X$ such that $gx_0 \preceq F(x_0, y_0)$ and $gy_0 \succeq F(y_0, x_0)$. Since $F(X \times X) \subseteq g(X)$, we can choose $x_1, y_1 \in X$ such that $gx_1 = F(x_0, y_0)$ and $gy_1 = F(y_0, x_0)$. Again since $F(X \times X) \subseteq g(X)$, we can choose $x_2, y_2 \in X$ such that $gx_2 = F(x_1, y_1)$ and $gy_2 = F(y_1, x_1)$. Continuing in this way we construct two sequences $\{x_n\}$ and $\{y_n\}$ in X such that for all $n \geq 0$,

$$gx_{n+1} = F(x_n, y_n) \quad \text{and} \quad gy_{n+1} = F(y_n, x_n). \tag{6}$$

Now we prove that for all $n \geq 0$

$$gx_n \preceq gx_{n+1}, \tag{7}$$

and

$$gy_n \succeq gy_{n+1}. \tag{8}$$

We shall use the mathematical induction. Let $n = 0$. Since $gx_0 \preceq F(x_0, y_0)$ and $gy_0 \succeq F(y_0, x_0)$, in view of $gx_1 = F(x_0, y_0)$ and $gy_1 = F(y_0, x_0)$, we have $gx_0 \preceq gx_1$ and $gy_0 \succeq gy_1$, that is, (7) and (8), hold for $n = 0$. We presume that (7) and (8), hold for some $n \geq 0$. As F has the mixed g-monotone property and $gx_n \preceq gx_{n+1}$ and $gy_n \succeq gy_{n+1}$, from (5), we have

$$gx_{n+1} = F(x_n, y_n) \preceq F(x_{n+1}, y_n) \preceq F(x_{n+1}, y_{n+1}) = gx_{n+2} \tag{9}$$

and

$$gy_{n+1} = F(y_n, x_n) \succeq F(y_{n+1}, x_n) \succeq F(y_{n+1}, x_{n+1}) = gy_{n+2}. \tag{10}$$

Using (9) and (10), we have $gx_{n+1} \preceq gx_{n+2}$ and $gy_{n+1} \succeq gy_{n+2}$.

Thus by mathematical induction (7) and (8) are hold for all $n \geq 0$. Therefore,

$$gx_0 \preceq gx_1 \preceq gx_2 \preceq \dots \preceq gx_n \preceq gx_{n+1} \preceq \dots, \tag{11}$$

and

$$gy_0 \succeq gy_1 \succeq gy_2 \succeq \dots \succeq gy_n \succeq gy_{n+1} \succeq \dots. \tag{12}$$

If for some n, we have $(gx_{n+1}, gy_{n+1}) = (gx_n, gy_n)$, then $F(x_n, y_n) = gx_n$ and $F(y_n, x_n) = gy_n$, that is, F and g have a coincidence point. So we may assume that $(gx_{n+1}, gy_{n+1}) \neq (gx_n, gy_n)$, for all $n \in \mathbb{N}$, that is, we assume that either $gx_{n+1} = F(x_n, y_n) \neq gx_n$ or $gy_{n+1} = F(y_n, x_n) \neq gy_n$.

Since $gx_n \succeq gx_{n-1}$ and $gy_n \preceq gy_{n-1}$, from (5) and (6), we get

$$\begin{aligned} d(gx_{n+1}, gx_n) + d(gy_{n+1}, gy_n) &= d(F(x_n, y_n), F(x_{n-1}, y_{n-1})) + d(F(y_n, x_n), F(y_{n-1}, x_{n-1})) \\ &\leq \frac{1}{s^2}\theta(d(gx_n, gx_{n-1}), d(gy_n, gy_{n-1}))(d(gx_n, gx_{n-1}) + d(gy_n, gy_{n-1})). \end{aligned}$$

As $\theta(t_1, t_2) < 1$ and $s \geq 1$, for all $(t_1, t_2) \in [0, +\infty) \times [0, +\infty)$, implies

$$d(gx_{n+1}, gx_n) + d(gy_{n+1}, gy_n) < d(gx_n, gx_{n-1}) + d(gy_n, gy_{n-1}).$$

Set $\delta_n = d(gx_{n+1}, gx_n)) + d(gy_{n+1}, gy_n)$. Then the sequence $\{\delta_n\}$ is monotone decreasing. Therefore, there exists some $\delta \geq 0$ such that

$$\lim_{n\to\infty} \delta_n = \lim_{n\to\infty} [d(gx_{n+1}, gx_n)) + d(gy_{n+1}, gy_n)] = \delta. \tag{13}$$

Now, we show $\delta = 0$. Suppose, to the contrary, that $\delta > 0$. From (13), we get

$$\frac{d(gx_{n+1}, gx_n) + d(gy_{n+1}, gy_n)}{d(gx_n, gx_{n-1}) + d(gy_n, gy_{n-1})} \leq \frac{1}{s^2}\theta(d(gx_n, gx_{n-1}), d(gy_n, gy_{n-1})) < 1.$$

By taking the limit from above inequalities, as $n \to \infty$ and using (13), we get

$$\lim_{n\to\infty} \theta(d(gx_n, gx_{n-1}), d(gy_n, gy_{n-1})) = 1. \tag{14}$$

Since $\theta \in \Theta$, relation (14) implies that $\lim_{n\to\infty} d(gx_{n+1}, gx_n)) = 0$ and $\lim_{n\to\infty} d(gy_{n+1}, gy_n) = 0$, or $d(gx_{n+1}, gx_n)) + d(gy_{n+1}, gy_n) \to 0$ as $n \to \infty$, which is a contradiction. Thus $\delta = 0$, that is,

$$\lim_{n\to\infty} \delta_n = \lim_{n\to\infty} [d(gx_{n+1}, gx_n)) + d(gy_{n+1}, gy_n)] = 0. \tag{15}$$

So, we prove that both $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences in the b-metric space (X, d). Suppose on the contrary that at least one of $\{gx_n\}$ and $\{gy_n\}$ are not a Cauchy sequence. So there exists $\epsilon > 0$ such that we can find subsequences $\{n(k)\}$ and $\{m(k)\}$ of $\mathbb{N}$ with $n(k) > m(k) \geq k$ such that

$$d(gx_{n(k)}, gx_{m(k)}) + d(gy_{n(k)}, gy_{m(k)}) \geq s\epsilon. \tag{16}$$

Further, corresponding to $m(k)$, we can choose $n(k)$ in such a way that it is the smallest integer with $n(k) > m(k) \geq k$ and it satisfies (16). From (15) for large enough k, we have $n(k) - m(k) \geq 2$ and

$$d(gx_{n(k)-1}, gx_{m(k)}) + d(gy_{n(k)-1}, gy_{m(k)}) < s\epsilon. \tag{17}$$

By (16), (17), we get

$$\begin{aligned}
s\epsilon &\leq d(gx_{n(k)}, gx_{m(k)}) + d(gy_{n(k)}, gy_{m(k)}) \\
&\leq s(d(gx_{n(k)}, gx_{n(k)-1}) + d(gx_{n(k)-1}, gx_{m(k)})) + s(d(gy_{n(k)}, gy_{n(k)-1}) + d(gy_{n(k)-1}, gy_{m(k)})) \\
&< s(d(gx_{n(k)}, gx_{n(k)-1}) + d(gy_{n(k)}, gy_{n(k)-1})) + s^2\epsilon.
\end{aligned}$$

Using above inequality and (15) there exists $L > 0$ such that for all $k \in \mathbb{N}$, we get

$$0 < \delta\epsilon \leq d(gx_{n(k)}, gx_{m(k)}) + d(gy_{n(k)}, gy_{m(k)}) \leq L. \tag{18}$$

Now, suppose $\{\epsilon_k\}$ is a positive real number such that $\lim_{k\to\infty} \epsilon_k = 0$. Therefore from above inequality

$$\lim_{k\to\infty} \frac{d(gx_{n(k)}, gx_{m(k)}) + d(gy_{n(k)}, gy_{m(k)}) - \epsilon_k}{d(gx_{n(k)}, gx_{m(k)}) + d(gy_{n(k)}, gy_{m(k)})} = 1. \tag{19}$$

So, we have

$$\begin{aligned}
& d(gx_{n(k)}, gx_{m(k)}) + d(gy_{n(k)}, gy_{m(k)}) \\
\leq\ & s(d(gx_{n(k)}, gx_{n(k)+1}) + d(gx_{n(k)+1}, gx_{m(k)})) + s(d(gy_{n(k)}, gy_{n(k)+1}) + d(gy_{n(k)+1}, gy_{m(k)})) \\
\leq\ & sd(gx_{n(k)+1}, gx_{n(k)}) + s(sd(gx_{n(k)+1}, gx_{m(k)+1}) + sd(gx_{m(k)+1}, gx_{m(k)})) + sd(gy_{n(k)+1}, gy_{n(k)}) \\
& + s(sd(gy_{n(k)+1}, gy_{m(k)+1}) + sd(gy_{m(k)+1}, gy_{m(k)})) \\
\leq\ & s\delta_{n(k)} + s^2\delta_{m(k)} + s^2[d(gx_{n(k)+1}, gx_{m(k)+1}) + d(gy_{n(k)+1}, gy_{m(k)+1})].
\end{aligned} \tag{20}$$

Since $n(k) > m(k)$, $gx_{n(k)} \succeq gx_{m(k)}$ and $gy_{n(k)} \preceq gy_{m(k)}$, with using (5) and (6) we have

$$\begin{aligned}
& d(gx_{n(k)+1}, gx_{m(k)+1}) + d(gy_{n(k)+1}, gy_{m(k)+1}) \\
=\ & d(F(gx_{n(k)}, gy_{n(k)}), F(gx_{m(k)}, gy_{m(k)})) + d(F(gy_{n(k)}, gx_{n(k)}), F(gy_{m(k)}, gx_{m(k)})) \\
\leq\ & \frac{1}{s^2}[\theta(d(gx_{n(k)}, gx_{m(k)}), d(gy_{n(k)}, gy_{m(k)}))(d(gx_{n(k)}, gx_{m(k)}) + d(gy_{n(k)}, gy_{m(k)}))].
\end{aligned} \tag{21}$$

From (20) and (21), we get

$$\begin{aligned}
& d(gx_{n(k)}, gx_{m(k)}) + d(gy_{n(k)}, gy_{m(k)}) \\
\leq\ & s\delta_{n(k)} + s^2\delta_{m(k)} + s^2\frac{1}{s^2}[\theta(d(gx_{n(k)}, gx_{m(k)}), d(gy_{n(k)}, gy_{m(k)}))(d(gx_{n(k)}, gx_{m(k)}) + d(gy_{n(k)}, gy_{m(k)}))].
\end{aligned}$$

Since $\theta(t_1, t_2) < 1$ for all $(t_1, t_2) \in [0, +\infty) \times [0, +\infty)$, from above inequality

$$\frac{d(gx_{n(k)}, gx_{m(k)}) + d(gy_{n(k)}, gy_{m(k)}) - s\delta_{n(k)} - s^2\delta_{m(k)}}{d(gx_{n(k)}, gx_{m(k)}) + d(gy_{n(k)}, gy_{m(k)})} \leq \theta(d(gx_{n(k)}, gx_{m(k)}), d(gy_{n(k)}, gy_{m(k)})) < 1. \tag{22}$$

By taking the limit from (22), as $k \to \infty$ and using (15) and (19) we get

$$\lim_{k\to\infty} \theta(d(gx_{n(k)}, gx_{m(k)}), d(gy_{n(k)}, gy_{m(k)})) = 1.$$

Since $\theta \in \Theta$, we get

$$\lim_{k\to\infty} d(gx_{n(k)}, gx_{m(k)}) = \lim_{k\to\infty} d(gy_{n(k)}, gy_{m(k)}) = 0.$$

Hence

$$\lim_{k\to\infty}[d(gx_{n(k)}, gx_{m(k)}) + d(gy_{n(k)}, gy_{m(k)})] = 0,$$

and this is a contradiction. Thus we proved that $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences in the b-metric space (X,d). Since (X,d) is complete, there are $x, y \in X$ such that $\{gx_n\}$ and $\{gy_n\}$ are respectively b-convergent to x and y, that is,

$$\lim_{n\to\infty} gx_n = \lim_{n\to\infty} F(x_n, y_n) = x \quad \text{and} \quad \lim_{n\to\infty} gy_n = \lim_{n\to\infty} F(y_n, x_n) = y. \tag{23}$$

b-compatibility of g and F implies that

$$\lim_{n\to\infty} d(F(gx_n, gy_n), gF(x_n, y_n)) = 0, \tag{24}$$

and

$$\lim_{n\to\infty} d(F(gy_n, gx_n), gF(y_n, x_n)) = 0. \tag{25}$$

Now suppose that the assumption (a) holds. Using triangle inequality we have

$$d(F(gx_n, gy_n), gx) \leq s[d(F(gx_n, gy_n), gF(x_n, y_n)) + d(gF(x_n, y_n), gx)].$$

From (23), (24) and continuity of g and F, we get $d(F(x,y), gx) = 0$ and $d(F(y,x), gy) = 0$, that is, $gx = F(x,y)$ and $F(y,x) = gy$.

Finally, suppose that (b) holds. Since $\{gx_n\}$ is nondecreasing and $gx_n \to x$ and as $\{gy_n\}$ is nonincreasing and $gy_n \to y$, we have

$$ggx_n \preceq gx \quad \text{and} \quad ggy_n \succeq gy.$$

Since g and F are b-compatible mapping and g is continuous and from (23), (24) and (25), we have

$$\lim_{n\to\infty} g(gx_n) = gx = \lim_{n\to\infty} gF(x_n, y_n) = \lim_{n\to\infty} F(gx_n, gy_n),$$

and

$$\lim_{n\to\infty} g(gy_n) = gy = \lim_{n\to\infty} gF(y_n, x_n) = \lim_{n\to\infty} F(gy_n, gx_n).$$

Since $ggx_n \preceq gx$ and $ggy_n \succeq gy$, from (5) we get

$$\begin{aligned}
& d(F(gx_n, gy_n), F(x,y)) + d(F(gy_n, gx_n), F(y,x)) \\
\leq\ & \frac{1}{s^2}\theta(d(ggx_n, gx), d(ggy_n, gy))[d(ggx_n, gx) + d(ggy_n, gy)] \\
\leq\ & \frac{1}{s^2}[d(ggx_n, gx) + d(ggy_n, gy)].
\end{aligned}$$

Using Proposition 3 and continuity of g, by taking the limit in above inequality as $n \to \infty$ we conclude that

$$\frac{1}{s^2}[d(gx, F(x,y)) + d(gy, F(y,x))] \leq \frac{1}{s^2}s^2[d(gx, gx) + d(gy, gy)] = 0.$$

Hence $d(gx, F(x,y)) = d(gy, F(y,x)) = 0$. Therefore $gx = F(x,y)$ and $gy = F(y,x)$. This completes the proof. ■

The following theorem is a direct result of Theorem 4.

Theorem 5 *Let $(X, \preceq)$ be a partially ordered set and d be a b-metric on X such that (X,d) is a complete b-metric space with constant $s \geq 1$. Let $F : X \times X \to X$ be a mapping such that F has the mixed monotone property on X. Suppose there exists $\theta \in \Theta$ such that*

$$d(F(x,y), F(u,v)) + d(F(y,x), F(v,u)) \leq \frac{1}{s^2}[\theta(d(x,u), d(y,v))(d(x,u) + d(y,v))]$$

for all $x, y, u, v \in X$ with $x \succeq u$ and $y \preceq v$. Suppose that either *(a) F is continuous, or*

(b) X has the following properties:

(i) if a nondecreasing sequence $\{x_n\}$ converges to x, then $x_n \preceq x$, for all $n \geq 0$;

(ii) if a nonincreasing sequence $\{y_n\}$ converges to y, then $y \preceq y_n$, for all $n \geq 0$.

If there exist $x_0, y_0 \in X$ such that $x_0 \preceq F(x_0, y_0)$ and $y_0 \succeq F(y_0, x_0)$, then F has a coupled fixed point, that is, there exist $x, y \in X$ such that $x = F(x, y)$ and $y = F(y, x)$.

Proof. Let $g = I_X$ and apply Theorem 4. So this proof is complete. ■

By considering $\theta(t_1, t_2) = k$ for all $t_1, t_2 \in [0, \infty)$, where $k \in [0, 1)$, and g to be identity mapping in Theorem 4, we conclude the following corollary.

Corollary 6 *Let $(X, \preceq)$ be a partially ordered set and d be a b-metric on X such that (X, d) is a complete b-metric space with constant $s \geq 1$. Let $F : X \times X \to X$ be a mixed monotone mapping for which there exists a constant $k \in [0, 1)$ such that*

$$d(F(x, y), F(u, v)) + d(F(y, x), F(v, u)) \leq \frac{k}{s^2}(d(x, u) + d(y, v)) \tag{26}$$

for all $x, y, u, v \in X$ with $x \succeq u$ and $y \preceq v$. Suppose that either

(a) F is continuous, or

(b) X has the following properties:

(i) if a nondecreasing sequence $\{x_n\}$ converges to x, then $x_n \preceq x$, for all $n \geq 0$;

(ii) if a nonincreasing sequence $\{y_n\}$ converges to y, then $y \preceq y_n$, for all $n \geq 0$.

If there exist $x_0, y_0 \in X$ such that $x_0 \preceq F(x_0, y_0)$ and $y_0 \succeq F(y_0, x_0)$, then F has a coupled fixed point, that is, there exist $x, y \in X$ such that $x = F(x, y)$ and $y = F(y, x)$.

Remark 1 *By considering g to be identity mapping and also with definition $s = 1$ in Theorem 4, we conclude Theorem 2.*

3 Common Fixed Point

Now we shall prove the existence and uniqueness theorem of a coupled common fixed point. If $(X, \preceq)$ is a partially ordered set, we endow the product set $X \times X$ with the partial order $\preceq$ defined by

$$(x, y) \rhd (u, v) \Leftrightarrow x \preceq u \ \text{ and } \ y \succeq v,$$

for $(x, y), (u, v) \in X \times X$.

Theorem 7 *In addition to the hypotheses of Theorem 4, suppose that,*

(c) for every $(x, y), (u, v) \in X \times X$, there exists $(w, z) \in X \times X$ such that $(F(w, z), F(z, w))$ is comparable to both $(F(x, y), F(y, x))$ and $(F(u, v), F(v, u))$.

Then F and g have a unique common fixed point, that is, there exists a unique $p \in X$ such that $p = gp = F(p, p)$.

Proof. From Theorem 4, the set of coupled coincidences is nonempty. We shall show that if (x, y) and (u, v) are coupled coincidence points, that is, if $gx = F(x, y)$, $gy = F(y, x)$, $gu = F(u, v)$ and $gv = F(v, u)$, then

$$gx = gu \quad \text{and} \quad gy = gv. \tag{27}$$

By assumption, there exists $(w, z) \in X \times X$ such that $(F(w, z), F(z, w))$ is comparable to both $(F(x, y), F(y, x))$ and $(F(u, v), F(v, u))$. There is four possible cases.

Case 1.

$$\begin{array}{rcl} (F(x,y),F(y,x)) & \rhd & (F(w,z),F(z,w)) \\ (F(u,v),F(v,u)) & \rhd & (F(w,z),F(z,w)). \end{array}$$

Put $w_0 = w$, $z_0 = z$ and choose $w_1, z_1 \in X$ such that $gw_1 = F(w_0, z_0)$ and $gz_1 = F(z_0, w_0)$. Then, similarly as in the proof of Theorem 4, we can inductively define sequences $\{gw_n\}$ and $\{gz_n\}$ in X by

$$gw_{n+1} = F(w_n, z_n) \quad \textit{and} \quad gz_{n+1} = F(z_n, w_n)$$

for all $n \in \mathbb{N}$. By taking

$$x_0 = x_1 = x_2 = \dots = x_n = \dots = x,$$

$$y_0 = y_1 = y_2 = \dots = y_n = \dots = y,$$

$$u_0 = u_1 = u_2 = \dots = u_n = \dots = u$$

and

$$v_0 = v_1 = v_2 = \dots = v_n = \dots = v,$$

for all $n \in \mathbb{N}$, we have

$$gx_n = F(x, y), \quad gy_n = F(y, x) \;\text{ and }\; gu_n = F(u, v), \quad gv_n = F(v, u).$$

Since

$$(F(x, y), F(y, x)) = (gx_1, gy_1) = (gx, gy) \rhd (F(w, z), F(z, w)) = (gw_1, gz_1),$$

we see that $gx \preceq gw_1$ and $gy \succeq gz_1$. Now we shall prove that

$$gx \preceq gw_n \;\text{ and }\; gy \succeq gz_n \;\; \forall\, n \geq 1. \tag{28}$$

Suppose that (28) holds for some $n \geq 1$. Then by the mixed g-monotone property of F, we have

$$gw_{n+1} = F(w_n, z_n) \succeq F(x, z_n) \succeq F(x, y) = gx,$$

and

$$gz_{n+1} = F(z_n, w_n) \preceq F(y, w_n) \preceq F(y, x) = gy.$$

So (28) holds. From (5), we get

$$\begin{aligned} d(gx, gw_{n+1}) + d(gy, gz_{n+1}) &= d(F(x, y), F(w_n, z_n)) + d(F(y, x), F(z_n, w_n)) \\ &\leq \frac{1}{s^2}\Big[\theta(d(gx, gw_n), d(gy, gz_n))(d(gx, gw_n) + d(gy, gz_n))\Big]. \end{aligned} \tag{29}$$

Therefore,

$$d(gx, gw_{n+1}) + d(gy, gz_{n+1}) \leq d(gx, gw_n) + d(gy, gz_n).$$

Set $\delta_n = d(gx, gw_{n+1})) + d(gy, gz_{n+1})$. Since $\{\delta_n\}$ is nonincreasing and bounded belove, there exists some $\delta \geq 0$ such that

$$\lim_{n\to\infty} \delta_n = \lim_{n\to\infty} [d(gx, gw_{n+1})) + d(gy, gz_{n+1})] = \delta. \tag{30}$$

Now, we show that $\delta = 0$. Suppose, to the contrary, that $\delta > 0$. From (29), we get

$$\frac{d(gx, gw_{n+1}) + d(gy, gz_{n+1})}{d(gx, gw_n) + d(gz_n, gy)} \leq \frac{1}{s^2}\theta(d(gx, gw_n), d(gz_n, gy)) \leq \theta(d(gx, gw_n), d(gz_n, gy)) < 1.$$

By taking the limit from above inequalities, as $n \to \infty$ and using (30), we get

$$\lim_{n\to\infty} \theta(d(gx, gw_n), d(gz_n, gy)) = 1. \tag{31}$$

As $\theta \in \Theta$, relation (31) implies

$$\lim_{n\to\infty} d(gx, gw_n) = 0 \ \text{ and } \ \lim_{n\to\infty} d(gz_n, gy) = 0,$$

which is a contradiction. Thus $\delta = 0$, that is, $\lim\limits_{n\to\infty}[d(gx, gw_n) + d(gz_n, gy)] = 0$, which implies that

$$\lim_{n\to\infty} d(gx, gw_n) = \lim_{n\to\infty} d(gz_n, gy) = 0. \tag{32}$$

Similarly,

$$\lim_{n\to\infty} d(gu, gw_n) = \lim_{n\to\infty} d(gz_n, gv) = 0. \tag{33}$$

Therefore, from (32), (33) and the uniqueness of the limit, we get $gx = gu$ and $gy = gv$. So (27) holds.

Case 2.

$$\begin{aligned}(F(x,y), F(y,x)) &\ \triangleright\ (F(w,z), F(z,w))\\ (F(w,z), F(z,w)) &\ \triangleright\ (F(u,v), F(v,u)).\end{aligned}$$

Since

$$(F(x,y), F(y,x)) \triangleright (F(w,z), F(z,w)),$$

By the same method in **Case 1**, we have

$$\lim_{n\to\infty} d(gw_n, gx) = \lim_{n\to\infty} d(gz_n, gy) = 0. \tag{34}$$

Since

$$(F(w,z), F(z,w)) = (gw_1, gz_1) = (gw, gv) \triangleright (F(u,v), F(v,u)) = (gu, gv),$$

then $gw_1 \preceq gu$ and $gz_1 \succeq gv$. Now we shall prove that

$$gw_n \preceq gu \ \text{ and } \ gz_n \succeq gv \ \ \forall\ n \geq 1. \tag{35}$$

Suppose that (35) holds for some $n \geq 1$. Then by the mixed g-monotone property of F, we have

$$gz_{n+1} = F(z_n, w_n) \succeq F(v, w_n) \succeq F(v,u) = gv,$$

and

$$gw_{n+1} = F(w_n, z_n) \preceq F(u, z_n) \preceq F(u,v) = gu.$$

So (35) holds. From (5), we get

$$\begin{aligned} d(gz_{n+1}, gy) + d(gw_{n+1}, gx) &= d(F(z_n, w_n), F(v,u)) + d(F(w_n, z_n), F(u,v))\\ &\leq \frac{1}{s^2}\theta(d(gz_n, gy), d(gw_n, gx))(d(gw_n, gx) + d(gz_n, gy)). \end{aligned} \tag{36}$$

Therefore,

$$d(gz_{n+1}, gy) + d(gw_{n+1}, gx) \leq d(gw_n, gx) + d(gz_n, gy).$$

Set $\delta_n = d(gz_{n+1}, gy) + d(gw_{n+1}, gx))$. Since $\{\delta_n\}$ is nonincreasing and bounded belove, there exists some $\delta \geq 0$ such that

$$\lim_{n\to\infty} \delta_n = \lim_{n\to\infty} [d(gw_{n+1}, gx)) + d(gz_{n+1}, gy)] = \delta. \tag{37}$$

Now, we show $\delta = 0$. Suppose, to the contrary, that $\delta > 0$. From (36), we get

$$\frac{d(gw_{n+1}, gx) + d(gz_{n+1}, gy)}{d(gw_n, gx) + d(gz_n, gy)} \leq \frac{1}{s^2}\theta(d(gz_n, gy), d(gw_n, gx) \leq \theta(d(gz_n, gy), d(gw_n, gx)) < 1.$$

By taking the limit from above inequalities, as $n \to \infty$ and using (37), we get

$$\lim_{n\to\infty} \theta(d(gz_n, gy), d(gw_n, gx)) = 1. \tag{38}$$

As $\theta \in \Theta$, relation (38) implies

$$\lim_{n\to\infty} d(gw_n, gx) = 0 \quad \text{and} \quad \lim_{n\to\infty} d(gz_n, gy) = 0,$$

which is a contradiction. Thus $\delta = 0$, that is, $\lim_{n\to\infty} [d(gw_n, gx) + d(gz_n, gy)] = 0$, which implies that

$$\lim_{n\to\infty} d(gw_n, gx) = \lim_{n\to\infty} d(gz_n, gy) = 0. \tag{39}$$

Similarly,

$$\lim_{n\to\infty} d(gw_n, gu) = \lim_{n\to\infty} d(gz_n, gv) = 0. \tag{40}$$

Therefore, from (39), (40) and the uniqueness of the limit, we get $gx = gu$ and $gy = gv$. So (27) holds.

Case 3.

$$\begin{aligned} (F(u,v), F(v,u)) &\;\triangleright\; (F(w,z), F(z,w)) \\ (F(w,z), F(z,w)) &\;\triangleright\; (F(x,y), F(y,x)). \end{aligned}$$

This case is similar to the **Case 2**.

Case 4.

$$\begin{aligned} (F(w,z), F(z,w)) &\;\triangleright\; (F(x,y), F(y,x)) \\ (F(w,z), F(z,w)) &\;\triangleright\; (F(u,v), F(v,u)). \end{aligned}$$

Also this case is similar to the **Case 1**.

Suppose (x, y) is a coupled coincidence point of F and g. So (y, x) also is a coincidence point. By (27) we get $gx = gy = p$. Hence $F(x, y) = gx = gy = F(y, x)$.

By define, $x_n = x$ and $y_n = y$ for all $n \in \mathbb{N}$, we have

$$\lim_{n\to\infty} F(x_n, y_n) = F(x, y) = gx = \lim_{n\to\infty} gx_n,$$

and

$$\lim_{n\to\infty} F(y_n, x_n) = F(y, x) = gy = \lim_{n\to\infty} gy_n.$$

b-compatibility of g and F implies that

$$\lim_{n\to\infty} d(gF(x_n, y_n), F(gx_n, gy_n)) = \lim_{n\to\infty} d(gF(y_n, x_n), F(gy_n, gx_n)) = 0.$$

Therefore $gF(x, y) = F(gx, gy)$. Hence $gp = F(p, p)$. This shows that (p, p) is a coincidence point of F and g. From (27) we conclude that $gp = gx = p = gy = gp$. Hence $F(p, p) = gp = p$. Therefore p is a common fixed point of F and g. By (27) we conclude that this fixed point is unique and this complete the proof. ■

The following theorem is a direct result of Theorem 7.

Theorem 8 *In addition to the hypotheses of Theorem 5, suppose that, for every $(x, y), (z, t) \in X \times X$, there exists a $(u, v) \in X \times X$ such that (u, v) is comparable to (x, y) and (z, t). Then F has a unique fixed point, that is, there exist $p \in X$ such that $p = F(p, p)$.*

Proof. By getting $g = I_X$ and using Theorem 7, we have the desired conclusion. ■

4 Generalized Voltera-Fredholm Integral Equations

In this section, we study the existence and uniqueness of solution to a nonlinear integral equation, as an application to the fixed point theorem proved in Section 2.

Consider the integral equation:

$$u(x) = \Phi\Big(\int_a^{\cdot} H(.,t,u(t))dt, \int_a^b G(.,t,u(t))dt\Big)(x) + g(x), \tag{41}$$

where $x, t \in [a,b]$, $-\infty < a < b < +\infty$, $g : [a,b] \to \mathbb{R}^n$ is a mapping and H and G are tow real continuous function on the domain $D := \{(x,t,u) : x \in [a,b], t \in [a,x], u \in X\}$. For every $u \in X$ we define tow functions U_1 and U_2 as follows:

$$U_1(x) := \int_a^x H(x,t,u(t))dt, U_2(x) := \int_a^b G(x,t,u(t))dt$$

We will analyze Eq. (41) under the following assumption:

(i) $X = C([a,b], \mathbb{R}^n)$ is a partially ordered set with the following:

$$u, v \in C([a,b], \mathbb{R}^n) \;\; u \preceq v \Leftrightarrow u(t) \leq v(t).$$

(ii) $\Phi : X \times X \to X$ has the mixed monotone property and there exists $L > 0$ such that

$$|\Phi(U_1,U_2)(x) - \Phi(V_1,V_2)(x)| \leq L|U_1(x) - V_1(x)| + L|U_2(x) - V_2(x)|,$$

for all $u, v \in X$ and $x \in [a,b]$.(U_1, U_2, V_1 and V_2 are defined above)

(iii) $H, G : D \to \mathbb{R}^n$ and $g : [a,b] \to \mathbb{R}^n$ are continuous.

(iv) There exists integrable functions $p_1, p_2 : [a,b] \times [a,b] \to \mathbb{R}^+$ such that for all $u, v \in X$, if $v \preceq u$ then

$$0 \leq H(x,t,u) - H(x,t,v) \leq p_1(x,t)(u - v),$$

and

$$-p_2(x,t)(u - v) \leq G(x,t,u) - G(x,t,v) \leq 0.$$

(v)

$$2^{3p-2}\left[\sup_{x\in[a,b]}\left(\int_a^b p_1(x,t)dt\right)^p + \sup_{x\in[a,b]}\left(\int_a^b p_2(x,t)dt\right)^p\right]L^p < 1.$$

Theorem 9 *Under assumptions (i)–(iv), Eq. (41) has a unique solution in $C([a,b], \mathbb{R}^n)$.*

Proof. Let $X = C([a,b], \mathbb{R}^n)$. X is a partially ordered set if we define the following order relation in X by

$$u, v \in C([a,b], \mathbb{R}^n) \;\; u \preceq v \Leftrightarrow u(t) \leq v(t),$$

for all $t \in [a,b]$. The space (X,d) is a complete b-metric space with $s = 2^{p-1}$ and

$$d(u,v) = d_1(u,v)^p,$$

where $d_1(u,v) = \sup_{t\in[a,b]} |u(t) - v(t)|$ is a metric for all $u, v \in X$.

Suppose $\{u_n\}$ is a monotone nondecreasing in X that converges to $u \in X$. Then for every $t \in I$, the sequence of real numbers

$$u_1(t) \leq u_2(t) \leq ... \leq u_n(t) \leq ...,$$

converges to $u(t)$. Therefore, for all $t \in [a,b]$ and $n \in \mathbb{N}$ we have $u_n(t) \leq u(t)$. Hence $u_n \preceq u$, for all n.

Similarly, we can verify that limit $v(t)$ of a monotone nonincreasing sequence $v_n(t)$ in X is a lower bound for all the elements in the sequence. That is, $v \preceq v_n$ for all n. So, condition (b) of Corollary 6 holds.

We consider the operator $F : X \times X \to X$ define by

$$F(u,v)(x) \quad = \quad \Phi\left(\int_a^{.} H(.,t,u(t))dt, \int_a^b G(.,t,v(t))dtdt\right)(x) + g(x),$$

for all $x \in [0,1]$.

At first, we prove that F has the mixed monotone property. For every $u_1, u_2, v \in X$ with $u_1 \preceq u_2$, that is, $u_1(t) \leq u_2(t)$, for all $t \in [a,b]$, from (iv) we have

$$\int_a^{.} H(.,t,u_1(t))dt \preceq \int_a^{.} H(.,t,u_2(t))dt.$$

Since Φ has the mixed monotone property, from above inequality we conclude that

$$\begin{aligned} F(u_1,v) &= \Phi\left(\int_a^{.} H(.,t,u_1(t))dt, \int_a^b G(.,t,v(t))dt\right) \preceq \Phi\left(\int_a^{.} H(.,t,u_2(t))dt, \int_a^b G(.,t,v(t))dt\right) \\ &= F(u_2,v). \end{aligned}$$

Similarly, if $u, v_1, v_2 \in X$ with $v_1 \preceq v_2$ we get $F(u,v_2) \preceq F(u,v_1)$. Thus, F has the mixed monotone property.

Now, for every $u, v, w, z \in X$ with $w \preceq u$ and $v \preceq z$, that is, $u(t) \geq w(t)$ and $v(t) \leq z(t)$ for all $t \in [a,b]$, we have

$$\begin{aligned} & |F(u,v)(x) - F(w,z)(x)| \\ = & \Big|\Phi\left(\int_a^{.} H(.,t,u(t))dt, \int_a^b G(.,t,v(t))dt\right)(x) \\ & -\Phi\left(\int_a^{.} H(.,t,w(t))dt, \int_a^b G(.,t,z(t))dt\right)(x)\Big| \\ \leq & L\Big|\int_a^x [H(x,t,u(t)) - H(x,t,w(t))]dt\Big| + L\Big|\int_a^b [G(x,t,v(t)) - G(x,t,z(t))]dt\Big| \\ \leq & L\int_a^x |H(x,t,u(t)) - H(x,t,w(t))|dt + L\int_a^b |G(x,t,v(t)) - G(x,t,z(t))|dt \\ \leq & L\int_a^x p_1(x,t)|u(t) - w(t)|dt + L\int_a^b p_2(x,t)|v(t) - z(t)|dt \\ \leq & Ld_1(u,w)\sup_{x\in[a,b]}\left(\int_a^x p_1(x,t)dt\right) + Ld_1(v,z)\sup_{x\in[a,b]}\left(\int_a^b p_2(x,t)dt\right). \end{aligned}$$

Thus,

$$\begin{aligned} d_1(F(u,v),F(w,z)) &= \sup_{x\in[a,b]}\{|F(u,v)(x) - F(w,z)(x)|\} \\ &\leq Ld_1(u,w)\sup_{x\in[a,b]}\left(\int_a^b p_1(x,t)dt\right) + Ld_1(v,z)\sup_{x\in[a,b]}\left(\int_a^b p_2(x,t)dt\right). \end{aligned}$$

Therefore,

$$\begin{aligned}
&d(F(u,v),F(w,z)) = \sup_{x\in[a,b]}\{|F(u,v)(x)-F(w,z)(x)|\}^p \\
\leq\ & \left(Ld_1(u,v)\sup_{x\in[a,b]}\left(\int_a^b p_1(x,t)dt\right)+Ld_1(v,z)\sup_{x\in[a,b]}\left(\int_a^b p_2(x,t)dt\right)\right)^p \\
\leq\ & 2^{p-1}\left(L^p d_1(u,w)^p\sup_{x\in[a,b]}\left(\int_a^b p_1(x,t)dt\right)^p+L^p d_1(v,z)^p\sup_{x\in[a,b]}\left(\int_a^b p_2(x,t)dt\right)^p\right).
\end{aligned}$$

Similarly,

$$\begin{aligned}
&d(F(v,u),F(z,w)) \\
\leq\ & 2^{p-1}\left(L^p d_1(v,z)^p\sup_{x\in[a,b]}\left(\int_a^b p_1(x,t)dt\right)^p+L^p d_1(u,w)^p\sup_{x\in[a,b]}\left(\int_a^b p_2(x,t)dt\right)^p\right).
\end{aligned}$$

Hence,

$$\begin{aligned}
&d(F(u,v),(F(w,z))+d(F(v,u),(F(z,w)) \\
\leq\ & 2^{p-1}\left(\sup_{x\in[a,b]}\left(\int_a^b p_1(x,t)dt\right)^p+\sup_{x\in[a,b]}\left(\int_a^b p_2(x,t)dt\right)^p\right)2L^p\left(d_1(v,z)^p+d_1(u,w)^p\right) \\
=\ & \frac{2^{3p-2}\left(\sup_{x\in[a,b]}\left(\int_a^b p_1(x,t)dt\right)^p+\sup_{x\in[a,b]}\left(\int_a^b p_2(x,t)dt\right)^p\right)L^p}{(2^{p-1})^2}\times\left(d(u,w)+d(v,z)\right).
\end{aligned}$$

But from (iv), we have

$$k=2^{3p-2}\left[\sup_{x\in[a,b]}\left(\int_a^b p_1(x,t)dt\right)^p+\sup_{x\in[a,b]}\left(\int_a^b p_2(x,t)dt\right)^p\right]L^p<1.$$

Consequently,

$$d(F(u,v),(F(w,z))+d(F(v,u),(F(z,w))\leq\frac{k}{s^2}(d(u,w)+d(v,z))$$

which is just inequality (26) in Corollary 6. So, Corollary 6 gives us that F has a coupled fixed point $(x,y)\in X\times X$.

Finally, $X\times X=C([a,b],\mathbb{R}^n)\times\mathbb{C}([\eth,],\mathbb{R}^n)$ is a partially ordered set if we define the following order relation in $X\times X$

$$(x,y),(u,v)\in X\times X,(x,y)\rhd(u,v)\Leftrightarrow x(t)\leq u(t)\text{ and }y(t)\geq v(t),\forall\ t\in[a,b].$$

For any $x,y\in X$, $\max\{x(t),y(t)\}$ and $\min\{x(t),y(t)\}$, for each $t\in[a,b]$, are in X . Therefore, for every $(x,y),(u,v)\in X\times X$, there exists a $(\max\{x,u\},\min\{y,v\})\in X$ that is comparable to (x,y) and (u,v). Hence by Theorem 8 we conclude that $x=F(x,x)$ and $x\in C([a,b],\mathbb{R}^n)$ is the unique solution of Eq. (41). This complete the proof. ■

Remark 2 *By considering $\Phi(x,y)=\varphi(x)$ for all $x,y\in X$, in Eq. (41) we conclude Eq. (1).*

Remark 3 *By considering $\Phi(x,y)=x+y$ for all $x,y\in X$ and $a=0$ in Eq. (41) we conclude Eq. (2).*

5 Example

In this section we give an example to illustrate the usefulness of our results and the following example shows that Theorem 4 is a real extension for Eq. (1).

Example 1 *Consider the following nonlinear Volterra-Fredholm integral equation*

$$u(x) = 1 - e^{-\int_1^x \frac{1}{24}\tan(\frac{\pi t^2 x}{32})\frac{u(t)+|u(t)|}{2}dt} - 3\int_1^2 \frac{1}{24}e^{-xtu(t)} + e^{-x^2} - 1. \tag{42}$$

In Eq. (41), *let* $X = C([1,2],\mathbb{R})$ *and we define*

$$H(x,t,u) := \frac{1}{24}\tan(\frac{\pi t^2 x}{32})\frac{u+|u|}{2}, G(x,t,u) := \frac{1}{24}e^{-xtu}, g(x) := e^{-x^2} - 1,$$

for all $t, x \in [1,2]$ *and* $u \in C([1,2],\mathbb{R})$. *Also suppose that* $\Phi(f,h) = 1 - e^{-f} - 3h$ *for all* $f, g \in X$. *Obviously,* Φ *has the mixed monotone property and by using the mean valued theorem*

$$|\Phi(U_1,U_2)(x) - \Phi(V_1,V_2)(x)| \leq 3|U_1(x) - V_1(x)| + 3|U_2(x) - V_2(x)|,$$

for all $x \in [1,2]$ *and* $u, v \in X$.

We consider $p = 2$ *in Theorem* 4. *Then clearly* $C([1,2],\mathbb{R})$ *is a complete b-metric space with* $s = 2^{p-1} = 2$.

Now for all $u, v \in X$, *if* $v \preceq u$ *we have*

$$0 \leq H(x,t,u) - H(x,t,v) \leq p_1(x,t)(u - v),$$

where $p_1(x,t) = \frac{1}{24}\tan(\frac{\pi t^2 x}{32})$ *which is integrable function of* $[1,2]\times[1,2]$ *into* $\mathbb{R}^+$ *and*

$$\sup_{x\in[1,2]}\left(\int_1^2 p_1(x,t)dt\right)^2 \leq \frac{1}{24^2}.$$

Similarly, for all $u, v \in X$, *if* $v \preceq u$ *we have*

$$-p_2(x,t)(u - v) \leq G(x,t,u) - G(x,t,v) \leq 0$$

where $p_2(x,t) = \frac{1}{24}e^{-xt}$ *which is integrable function of* $[1,2]\times[1,2]$ *into* $\mathbb{R}^+$ *and*

$$\sup_{x\in[1,2]}\left(\int_1^2 p_2(x,t)dt\right)^2 \leq \frac{1}{24^2}e^{-2}.$$

Moreover,

$$2^{3p-2}\left[\sup_{x\in[1,2]}\left(\int_1^2 p_1(x,t)dt\right)^p + \sup_{x\in[1,2]}\left(\int_1^2 p_2(x,t)dt\right)^p\right]L^p \leq 2^4[\frac{1}{24^2} + \frac{1}{24^2}e^{-2}]3^2 < 1.$$

Hence the required conditions of Theorem 9 *are satisfied and Eq.* (42) *has a unique solution in complete metric space* $C([1,2],\mathbb{R})$.

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