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Exponential Decay For The Lamé System With Fractional Time Delays And Boundary Feedbacks*

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Abstract

Our interest in this paper is to analyse the asymptotic behaviour of a Lamé system with internal fractional delay and boundary damping of Neumann type. Assuming the weights of the delay are small enough, we show that the system is well-posed using the semigroup theory. Furthermore, we introduce a Lyapunov functional that gives the exponential decay.

1 Introduction

This work is devoted to the study of well-posedness and boundary stabilization of the Lamé system in a bounded domain Ω of $\mathbb{R}^n$ with smooth boundary $\partial\Omega$ of class C^2. We assume that $\Gamma = \Gamma_0 \cup \Gamma_1$, where Γ_0 and Γ_1 are closed subsets of Γ with $\Gamma_0 \cap \Gamma_1 = \emptyset$. The system is given by:

$$\begin{cases} u_{tt} - \mu\Delta u - (\mu+\lambda)\nabla(\operatorname{div} u) + a_1\partial_t^{\alpha,\kappa}u(x,t-\tau) = 0 & \text{in } \Omega\times(0,+\infty), \\ u = 0 & \text{in } \Gamma_0\times(0,+\infty), \\ \mu\frac{\partial u}{\partial\nu} + (\mu+\lambda)(\operatorname{div} u)\nu = -a_2 u_t(x,t) & \text{in } \Gamma_1\times(0,+\infty), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) & \text{in } \Omega, \\ u_t(x,t-\tau) = f_0(x,t-\tau) & \text{in } \Omega\times(0,\tau), \end{cases} \tag{P}$$

where μ, λ are Lamé constants, $u = (u_1, u_2, ..., u_n)^T$. Moreover, $a_1 > 0$, $a_2 > 0$ and the constant $\tau > 0$ is the time delay. ν stands for the unit normal vector of $\partial\Omega$ pointing towards the exterior of Ω and $\frac{\partial u}{\partial\nu}$ is the normal derivative. The notation $\partial_t^{\alpha,\kappa}$ stands for the generalized Caputo's fractional derivative (see [5]) of order α with respect to the time variable and is defined by

$$\partial_t^{\alpha,\kappa}w(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}e^{-\kappa(t-s)}\frac{dw}{ds}(s),\, ds \qquad 0<\alpha<1, \qquad \kappa>0.$$

One very active area of mathematical control theory has been the investigation of the delay effect in the stabilization of hyperbolic systems. It is well known that an arbitrarily small delay can have a destabilizing effect to systems that are asymptotically stable in the absence of delay (see [1], [7], [8], [14] and [10]).

In particular, the following boundary stabilization problem for the n-dimensional wave equation with interior delay was studied in [1],

$$\begin{cases} u_{tt}(x,t) - \Delta u(x,t) + a u_t(x,t-\tau) = 0 & x\in\Omega,\ t>0, \\ u = 0 & x\in\Gamma_0,\ t>0, \\ \frac{\partial u}{\partial\nu} = -k u_t(x,t) & x\in\Gamma_1,\ t>0, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) & x\in\Omega, \\ u_t(x,t) = g(x,t), & x\in\Omega,\ t\in(-\tau,0), \end{cases} \tag{PA}$$

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where the authors showed an exponential stability result under the usual Lions geometric condition on the domain Ω, providing that the delay coefficient a is sufficiently small.

In the absence of the delay in system (PA), that is for $\tau = 0$, a large amount of literature is available on this model, addressing problems of the existence, uniqueness and asymptotic behavior in time when some damping effects are considered, such as: frictional damping, viscoelastic damping and thermal dissipation.

Moreover, the result in [1] was extended to the Timoshenko system in [15] (see also [9]), where the authors studied a Timoshenko beam system given by two coupled hyperbolic equations, with delay terms in the first and second equation and two boundary controls, they proved the exponential decay of the total energy.

To our best knowledge the Lamé system with internal fractional time delay terms is not considered previously. Motivated by the above research, we will consider the Lamé with internal fractional time delays and boundary feedbacks (P). The main objectives of the present work are to establish the global well-posedness and exponential stability of system (P).

The outline of the paper is as follows. In Section 2, we take advantage of the complete monotonicity of the power function integral kernel to represent it as a superposition of exponentials and derive what we call the "augmented model", while in Section 3, we deal with the well-posedness result of the problem using the semigroup theory. Lastly, in Section 4, we obtain exponential stability results by constructing an appropriate Lyapunov functional as in [1].

2 Preliminaries

This section is concerned with the reformulation of the model (P) into an augmented system. For that, we need the following claims.

Theorem 1 (see [13]) *Let ω be the function:*

$$\omega(\xi) = |\xi|^{(2\alpha-1)/2}, \quad -\infty < \xi < +\infty,\ 0 < \alpha < 1. \tag{1}$$

Then the relationship between the "input" U and the "output" O of the system

$$\partial_t \phi(\xi, t) + (\xi^2 + \kappa)\phi(\xi, t) - U(t)\omega(\xi) = 0, \quad -\infty < \xi < +\infty, \kappa > 0, t > 0, \tag{2}$$

$$\phi(\xi, 0) = 0, \tag{3}$$

$$O(t) = (\pi)^{-1} \sin(\alpha\pi) \int_{-\infty}^{+\infty} \omega(\xi)\phi(\xi, t)\, d\xi \tag{4}$$

is given by

$$O = I^{1-\alpha,\kappa} U = D^{\alpha,\kappa} U, \tag{5}$$

where

$$[I^{\alpha,\kappa} f](t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} e^{-\kappa(t-\tau)} f(\tau)\, d\tau.$$

Lemma 2 (see [4]) *If $\lambda > 0$, then for ω in (1),*

$$\int_{-\infty}^{+\infty} \frac{\omega^2(\xi)}{\lambda + \xi^2} d\xi = \frac{\pi}{\sin \alpha\pi} \lambda^{\alpha-1}.$$

We are now in a position to reformulate system (P). Indeed, by using Theorem 1, system (P) may be recasted into the augmented model:

$$\left\{\begin{array}{ll} u_{tt}-\mu\Delta u-(\mu+\lambda)\nabla(\operatorname{div} u)+\zeta\int_{-\infty}^{+\infty}\omega(\xi)\psi(x,\xi,t)\,d\xi=0 & \text{in } \Omega\times(0,+\infty),\\ \partial_t\psi(x,\xi,t)+(\xi^2+\kappa)\psi(x,\xi,t)-z(x,1,t)\omega(\xi)=0 & \text{in } \Omega\times(-\infty,\infty)\times(0,+\infty),\\ \tau z_t(x,\rho,t)+z_\rho(x,\rho,t)=0 & \text{in } \Omega\times(0,1)\times(0,+\infty),\\ u(x,t)=0 & \text{in } \Gamma_0\times(0,+\infty),\\ \mu\frac{\partial u}{\partial\nu}+(\mu+\lambda)(\operatorname{div} u)\nu=-a_2u_t(x,t) & \text{in } \Gamma_1\times(-\infty,\infty)\times(0,+\infty),\\ z(x,0,t)=u_t(x,t), & \text{in } \Omega\times(0,+\infty),\\ u(x,0)=u_0(x),u_t(x,0)=u_1(x) & \text{in } \Omega,\\ \psi(x,\xi,0)=0 & \text{in } \Omega\times(-\infty,\infty)\\ z(x,\rho,0)=f_0(x,-\rho\tau) & \text{in } \Omega\times(0,1), \end{array}\right. \tag{P'}$$

where $\zeta=a_1(\pi)^{-1}\sin(\alpha\pi)$.

We define the energy of the solution by:

$$\begin{aligned} E(t) &= \frac{1}{2}\sum_{j=1}^{n}\left(\|u_{jt}\|^2_{L^2(\Omega)}+\mu\|\nabla u_j\|^2_{L^2(\Omega)}+\overline{\zeta}\int_\Omega\int_{-\infty}^{+\infty}|\psi_j(x,\xi,t)|^2\,d\xi dx\right)\\ &\quad+\frac{\upsilon}{2}\sum_{j=1}^{n}\int_\Omega\int_0^1|z_j(x,\varrho,t)|^2d\varrho dx+\frac{\mu+\lambda}{2}\|\operatorname{div} u\|^2_{L^2(\Omega)}, \end{aligned} \tag{6}$$

where $\overline{\zeta}=\frac{\upsilon}{2\tau I}$ and $I=\int_0^\infty\frac{\omega^2(\xi)}{\xi^2+\kappa}d\xi$ and υ is a strictly positive real number.

In order to establish the exponential energy decay rate, let us consider the usual geometrical control condition: there exists a point $x_0\in\mathbb{R}^n$ such that

$$m\cdot\nu\le 0\ \text{ on }\Gamma_0,\qquad m\cdot\nu>0\ \text{ on }\Gamma_1, \tag{7}$$

where $m=x-x_0$.

The main result of this paper is the following.

Theorem 3 *For any $a_2>0$, there exist positive constants a_0, C_1, C_2 such that*

$$E(t)\le C_1e^{-C_2t}E(0) \tag{8}$$

for any regular solution of problem (P) with $0\le a_1<a_0$. The constants a_0, C_1, C_2 are independent of the initial data but they depend on a_2 and on the geometry of Ω.

3 Well-posedness

In this section, we give the existence and uniqueness result for system (P') using the semigroup theory (see [6]). We define the energy space $\mathcal{H}$ by

$$\mathcal{H}=\left(H^1_{\Gamma_0}(\Omega)\right)^n\times\left(L^2(\Omega)\right)^n\times\left(L^2(\Omega\times(-\infty,+\infty))\right)^n\times\left(L^2(\Omega\times(0,1))\right)^n,$$

where

$$H^1_{\Gamma_0}(\Omega)=\{u\in H^1(\Omega):u_{|\Gamma_0}=0\}.$$

For any $U=(u,v,\psi,z)^T\in\mathcal{H}$, $\tilde{U}=(\tilde{u},\tilde{v},\tilde{\psi},\tilde{z})^T\in\mathcal{H}$, the inner product in $\mathcal{H}$ is defined as

$$\begin{aligned} \left\langle U,\tilde{U}\right\rangle_{\mathcal{H}} &= \sum_{j=1}^{n}\int_\Omega(v_j\tilde{v}_j+\mu\nabla u_j\nabla\tilde{u}_j)\,dx+(\mu+\lambda)\int_\Omega(\operatorname{div} u)(\operatorname{div}\tilde{u})\,dx\\ &\quad+\overline{\zeta}\sum_{j=1}^{n}\int_\Omega\int_{-\infty}^{\infty}\psi_j(x,\xi)\tilde{\psi}_j(x,\xi)d\xi dx+\upsilon\sum_{j=1}^{n}\int_\Omega\int_0^1 z(x,\varrho)\tilde{z}_j(x,\varrho)d\varrho dx. \end{aligned}$$

For $U=(u,v,\psi,z)^T$, where $v=u_t$, then system (P') is equivalent to an abstract Cauchy problem:

$$\begin{cases} U'=\mathcal{A}U, & t>0,\\ U(0)=U_0=(u_0,u_1,0,f_0)^T \end{cases} \tag{9}$$

and

$$\mathcal{A}\begin{pmatrix} u\\ v\\ \psi\\ z\end{pmatrix}=\begin{pmatrix} v\\ \mu\Delta u+(\mu+\lambda)\nabla(\operatorname{div}u)-\zeta\int_{-\infty}^{\infty}\omega(\xi)\psi(x,\xi)\,d\xi\\ -(\xi^2+\kappa)\psi+z(x,1)\omega(\xi)\\ -\tau^{-1}z_\varrho(x,\varrho)\end{pmatrix} \tag{10}$$

with domain

$$\begin{aligned} D(\mathcal{A}) \;=\; & \Big\{(u,v,\psi,z) \text{ in } \mathcal{H}: u\in\left(H^2(\Omega)\cap H^1_{\Gamma_0}(\Omega)\right)^n,\ v\in\left(H^1(\Omega)\right)^n,\\ & -(\xi^2+\kappa)\psi+z(x,1)\omega(\xi)\in\left(L^2(\Omega\times(-\infty,+\infty))\right)^n,\\ & z\in\left(L^2\left(\Omega;H^1(0,1)\right)\right)^n,\ \mu\frac{\partial u}{\partial v}+(\mu+\lambda)(\operatorname{div}u)\nu+a_2v=0 \text{ in } \Gamma_1,\\ & |\xi|\psi\in\left(L^2(\Omega\times(-\infty,+\infty))\right)^n,\ v=z(.,0) \text{ in } \Omega\Big\} \end{aligned} \tag{11}$$

Remark 1 *The condition $|\xi|\psi(\xi)\in(L^2(\Omega\times\mathbb{R}))^n$ is imposed to insure the existence of*

$$-\bar{\zeta}\sum_{j=1}^n\int_\Omega\int_{-\infty}^{\infty}(\xi^2+\kappa)|\psi_j(x,\xi,t)|^2\,d\xi\,dx$$

and $\omega(\xi)\psi(x,\xi)\in(L^1(\Omega\times\mathbb{R}))^n$.

We show that there exists a positive constant c such that $\mathcal{A}-cI$ is dissipative. Let $U=(u,v,\psi,z)^T\in D(\mathcal{A})$. Then

$$\begin{aligned} \langle\mathcal{A}U,U\rangle_{\mathcal{H}} \;=\; & -a_2\sum_{j=1}^n\|v_j\|^2_{L^2(\Gamma_1)}-\zeta\sum_{j=1}^n\int_\Omega v_j\int_{-\infty}^{+\infty}\omega(\xi)\psi_j(x,\xi)\,d\xi\,d\Gamma\\ & +\bar{\zeta}\sum_{j=1}^n\int_\Omega z_j(x,1)\int_{-\infty}^{+\infty}\omega(\xi)\psi_j(x,\xi)\,d\xi\,dx+\frac{v}{2\tau}\sum_{j=1}^n\|v_j\|^2_{L^2(\Omega)}-\frac{v}{2\tau}\sum_{j=1}^n\|z_j(x,1)\|^2_{L^2(\Omega)}\\ & -\bar{\zeta}\sum_{j=1}^n\int_\Omega\int_{-\infty}^{+\infty}(\xi^2+\kappa)|\psi_j(x,\xi,t)|^2\,d\xi\,dx\\ \le\; & -a_2\sum_{j=1}^n\|v_j\|^2_{L^2(\Gamma_1)}+\left(\frac{\zeta^2I}{2\bar{\zeta}}+\frac{v}{2\tau}\right)\sum_{j=1}^n\|v_j\|^2_{L^2(\Omega)}\\ \le\; & \left(\frac{\zeta^2I}{2\bar{\zeta}}+\frac{v}{2\tau}\right)\sum_{j=1}^n\|v_j\|^2_{L^2(\Omega)}. \end{aligned}$$

This shows that $\mathcal{A}-cI$ is dissipative.

In the sequel, we claim that the operator $\mathcal{A}$ has the property $R(\tilde{\lambda}I-\mathcal{A})=\mathcal{H}$ for fixed $\tilde{\lambda}>0$. Indeed, let $G=(G_1,G_2,G_3,G_4)\in\mathcal{H}$, where $G_i=(g_i^1,g_i^2,...,g_i^n)$, we must solve the problem $(\tilde{\lambda}I-\mathcal{A})U=G$ for some $U=(u,v,\psi,z)\in D(\mathcal{A})$. The equation becomes the system

$$\begin{cases} \tilde{\lambda}u-v=G_1(x),\\ \tilde{\lambda}v-\mu\Delta u-(\mu+\lambda)\nabla(\operatorname{div}u)+\zeta\int_{-\infty}^{+\infty}\omega(\xi)\psi(x,\xi)\,d\xi=G_2(x),\\ \tilde{\lambda}\psi+(\xi^2+\kappa)\psi-z(x,1)\omega(\xi)=G_3(x,\xi),\\ \tilde{\lambda}z(x,\varrho)+\tau^{-1}z_\varrho(x,\varrho)=G_4(x,\varrho). \end{cases} \tag{12}$$

Suppose that we have obtained u with the appropriate regularity, then $(12)_1$ and $(12)_3$ yield

$$v=\tilde{\lambda}u-G_1(x)\in\left(H^1_{\Gamma_0}(\Omega)\right)^n \tag{13}$$

and

$$\psi=\frac{G_3(x,\xi)+\omega(\xi)z(x,1)}{\xi^2+\kappa+\tilde{\lambda}}. \tag{14}$$

We note that the last equation in (12) with $z(x,0)=v(x)$ has a unique solution given by

$$z(x,\varrho)=v(x)e^{-\tilde{\lambda}\varrho\tau}+\tau e^{-\tilde{\lambda}\varrho\tau}\int_0^{\varrho}G_4(x,r)e^{\tilde{\lambda}r\tau}dr. \tag{15}$$

Inserting (13) in (15), we get

$$z(x,\varrho)=\tilde{\lambda}u(x)e^{-\tilde{\lambda}\varrho\tau}-G_1(x)e^{-\tilde{\lambda}\varrho\tau}+\tau e^{-\tilde{\lambda}\varrho\tau}\int_0^{\varrho}G_4(x,r)e^{\tilde{\lambda}r\tau}dr,\qquad x\in\Omega,\ \varrho\in(0,1). \tag{16}$$

In particular,

$$z(x,1)=\tilde{\lambda}u(x)e^{-\tilde{\lambda}\tau}+z_0(x),\quad x\in\Omega, \tag{17}$$

where for $x\in\Omega$

$$z_0(x)=-G_1(x)e^{-\tilde{\lambda}\tau}+\tau e^{-\tilde{\lambda}\tau}\int_0^{1}G_4(x,r)e^{\tilde{\lambda}r\tau}dr. \tag{18}$$

In light of the above results, the function u satisfies the following equation:

$$\tilde{\lambda}^2u-\mu\Delta u-(\mu+\lambda)\nabla(\operatorname{div}u)+\zeta\int_{-\infty}^{+\infty}\omega(\xi)\psi(x,\xi)\,d\xi=G_2(x)+\tilde{\lambda}G_1(x). \tag{19}$$

Then for any $w\in(H^1_{\Gamma_0}(\Omega))^n$, it follows from problem (19) that

$$\begin{aligned}&\int_\Omega\left(\tilde{\lambda}^2u_jw_j-\mu\Delta u_jw_j\right)dx-(\mu+\lambda)\int_\Omega\frac{\partial}{\partial x_j}(\operatorname{div}u)w_jdx+\zeta\sum_{j=1}^n\int_\Omega w_j\int_{-\infty}^{+\infty}\omega(\xi)\psi_j(x,\xi)\,d\xi dx\\ =\ &\int_\Omega(F_2^j(x)+\tilde{\lambda}F_1^j(x))w_jdx.\end{aligned} \tag{20}$$

By using integration by parts, the boundary condition $(11)_4$ and (14), we infer that

$$\begin{aligned}&\sum_{j=1}^n\int_\Omega\left(\tilde{\lambda}^2u_jw_j+\mu\nabla u_j\nabla w_j\,dx\right)+(\mu+\lambda)\int_\Omega(\operatorname{div}u)(\operatorname{div}w)dx\\ &+\tilde{\lambda}\theta\sum_{j=1}^n\int_\Omega u_jw_je^{-\tilde{\lambda}\tau}\,dx+\tilde{\lambda}a_2\sum_{j=1}^n\int_{\Gamma_1}u_jw_j\,d\Gamma\\ =\ &\sum_{j=1}^n\int_\Omega\left(g_2^j(x)+\tilde{\lambda}g_1^j(x)\right)w_j\,dx+a_2\sum_{j=1}^n\int_{\Gamma_1}g_1^j(x)w_j\,dx-\zeta\sum_{j=1}^n\int_\Omega w_j\left(\int_{-\infty}^{+\infty}\frac{\omega(\xi)g_3^j(x,\xi)}{\xi^2+\kappa+\tau}d\xi\right)dx\\ &-\theta\sum_{j=1}^n\int_\Omega w_jz_0(x)\,dx,\end{aligned} \tag{21}$$

where $\theta=\zeta\int_{-\infty}^{+\infty}\frac{\omega^2(\xi)}{\xi^2+\kappa+\tau}d\xi$.

Problem (21) is of the form

$$\mathcal{B}(u,w)=\mathcal{L}(w), \tag{22}$$

where $\mathcal{B} : \left(H^1_{\Gamma_0}(\Omega)\right)^n \times \left(H^1_{\Gamma_0}(\Omega)\right)^n \to \mathbb{R}$ is the bilinear form defined by

$$\begin{aligned}\mathcal{B}(u,w) &= \sum_{j=1}^{n}\int_\Omega \left(\tilde{\lambda}^2 u_j w_j + \mu\nabla u_j \nabla w_j \, dx\right) + (\mu+\lambda)\int_\Omega (\operatorname{div} u)(\operatorname{div} w)dx \\ &\quad + \tilde{\lambda}\theta \sum_{j=1}^{n}\int_\Omega u_j w_j e^{-\tilde{\lambda}\tau}\, dx + \tilde{\lambda} a_2 \sum_{j=1}^{n}\int_{\Gamma_1} u_j w_j \, d\Gamma\end{aligned}$$

and $\mathcal{L} : \left(H^1_{\Gamma_0}(\Omega)\right)^n \to \mathbb{R}$ is the linear form given by

$$\begin{aligned}\mathcal{L}(w) &= \sum_{j=1}^{n}\int_\Omega \left(g_2^j(x) + \tilde{\lambda} g_1^j(x)\right) w_j \, dx - \zeta \sum_{j=1}^{n}\int_\Omega w_j \left(\int_{-\infty}^{+\infty} \frac{\omega(\xi) g_3^j(x,\xi)}{\xi^2+\kappa+\tau} d\xi\right) dx \\ &\quad -\theta \sum_{j=1}^{n}\int_\Omega w_j z_0(x)\, dx + a_2 \sum_{j=1}^{n}\int_{\Gamma_1} g_1^j(x) w_j dx.\end{aligned}$$

It is easy to get that $\mathcal{B}$ is continuous and coercive, and $\mathcal{L}$ is continuous. Therefore, by using Lax-Milgram theorem we can obtain problem (22) has a unique solution $u \in \left(H^1_{\Gamma_0}(\Omega)\right)^n$ for all $w \in \left(H^1_{\Gamma_0}(\Omega)\right)^n$. By the regularity theory for the linear elliptic equations, it follows that $u \in \left(H^2(\Omega)\right)^n$. Thus, the operator $(\tilde{\lambda} I - \mathcal{A})$ is surjective for any $\tilde{\lambda} > 0$. By using Hille-Yosida theorem, we have the following existence result:

Theorem 4 (Existence and uniqueness) *(1) If $U_0 \in D(\mathcal{A})$, then system (9) has a unique strong solution*

$$U \in C^0(\mathbb{R}_+, D(\mathcal{A})) \cap C^1(\mathbb{R}_+, \mathcal{H}).$$

(2) If $U_0 \in \mathcal{H}$, then system (9) has a unique weak solution

$$U \in C^0(\mathbb{R}_+, \mathcal{H}).$$

4 Proof of Theorem 3

The proof will be divided into several technical propositions.

Proposition 5 *For any solution of problem (P') the following estimate holds:*

$$\begin{aligned}E'(t) &\le -a_2 \sum_{j=1}^{n}\int_{\Gamma_1} |u_{jt}(x,t)|^2\, dx + \frac{\zeta I + \upsilon\tau^{-1}}{2}\sum_{j=1}^{n}\int_\Omega u_{jt}^2(x,t)\, dx \\ &\quad + \frac{\zeta I - \upsilon\tau^{-1}}{2I}\sum_{j=1}^{n}\int_\Omega\int_{-\infty}^{+\infty} (\xi^2+\kappa)|\psi_j(x,\xi,t)|^2\, dx\, d\xi. \end{aligned} \tag{23}$$

Proof. Multiplying the first equation in (P') by u_{jt}, integrating over Ω and using integration by parts, we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|u_{jt}\|_2^2 - \mu\int_\Omega \Delta u_j u_{jt}\, dx - (\mu+\lambda)\int_\Omega \frac{\partial}{\partial x_j}(\operatorname{div} u) u_{jt} dx \\ &+\zeta \int_\Omega u_{jt}\int_{-\infty}^{+\infty} \omega(\xi)\psi_j(x,\xi,t) d\xi dx = 0.\end{aligned}$$

Then

$$\frac{1}{2}\frac{d}{dt}\sum_{j=1}^{n}\left(\|u_{jt}\|^2_{L^2(\Omega)}+\mu\|\nabla u_j\|^2_{L^2(\Omega)}\right)+\frac{\mu+\lambda}{2}\|\operatorname{div} u\|^2_{L^2(\Omega)}+a_2\sum_{j=1}^{n}\|u_{jt}\|^2_{L^2(\Gamma_1)}$$
$$+\zeta\sum_{j=1}^{n}\int_{\Omega}u_{jt}\int_{-\infty}^{+\infty}\omega(\xi)\psi_j(x,\xi,t)\,d\xi\,dx=0. \tag{24}$$

Multiplying the second equation in (P') by $\overline{\zeta}\psi_j$ and integrating over $\Omega\times(-\infty,+\infty)$, we obtain:

$$\frac{\overline{\zeta}}{2}\frac{d}{dt}\sum_{j=1}^{n}\|\psi_j\|^2_{L^2(\Omega\times(-\infty,+\infty))}+\overline{\zeta}\sum_{j=1}^{n}\int_{\Omega}\int_{-\infty}^{+\infty}(\xi^2+\kappa)|\psi_j(x,\xi,t)|^2d\xi dx$$
$$-\overline{\zeta}\sum_{j=1}^{n}\int_{\Omega}z_j(x,1,t)\int_{-\infty}^{+\infty}\omega(\xi)\psi_j(x,\xi,t)\,d\xi\,dx=0. \tag{25}$$

Multiplying the third equation in (P') by υz_j and integrating over $\Omega\times(0,1)$, we get:

$$\frac{\upsilon}{2}\frac{d}{dt}\sum_{j=1}^{n}\|z_j\|^2_{L^2(\Omega\times(0,1))}+\frac{\upsilon\tau^{-1}}{2}\sum_{j=1}^{n}\int_{\Omega}\left(z_j^2(x,1,t)-u_{jt}^2(x,t)\right)=0. \tag{26}$$

From (6), (24) and (26) we get

$$\begin{aligned}E'(t) &= -a_2\sum_{j=1}^{n}\|u_{jt}\|^2_{L^2(\Gamma_1)}-\overline{\zeta}\sum_{j=1}^{n}\int_{\Omega}\int_{-\infty}^{+\infty}(\xi^2+\kappa)|\psi_j(x,\xi,t)|^2\,d\xi\,dx\\ &\quad -\zeta\sum_{j=1}^{n}\int_{\Omega}u_{jt}\int_{-\infty}^{+\infty}\omega(\xi)\psi_j(x,\xi,t)\,d\xi\,dx+\overline{\zeta}\sum_{j=1}^{n}\int_{\Omega}z_j(x,1,t)\int_{-\infty}^{+\infty}\omega(\xi)\psi_j(x,\xi,t)\,d\xi\,dx\\ &\quad +\frac{\upsilon\tau^{-1}}{2}\sum_{j=1}^{n}\int_{\Omega}u_{jt}^2(x,t)\,dx-\frac{\upsilon\tau^{-1}}{2}\sum_{j=1}^{n}\int_{\Omega}z_j^2(x,1,t)\,dx.\end{aligned} \tag{27}$$

Moreover, we have

$$\left|\int_{-\infty}^{+\infty}\omega(\xi)\psi_j(x,\xi,t)\,d\xi\right|\le\left(\int_{-\infty}^{+\infty}\frac{\omega^2(\xi)}{\xi^2+\kappa}d\xi\right)^{\frac{1}{2}}\left(\int_{-\infty}^{+\infty}(\xi^2+\kappa)|\psi_j(x,\xi,t)|^2\,d\xi\right)^{\frac{1}{2}}.$$

Then

$$\begin{aligned}&\left|\int_{\Omega}z_j(x,1,t)\int_{-\infty}^{+\infty}\omega(\xi)\psi_j(x,\xi,t)\,d\xi\,dx\right|\\ \le\ &\left(\int_{-\infty}^{+\infty}\frac{\omega^2(\xi)}{\xi^2+\kappa}\,d\xi\right)^{\frac{1}{2}}\|z_j(x,1,t)\|_{L^2(\Omega)}\left(\int_{\Omega}\int_{-\infty}^{+\infty}(\xi^2+\kappa)|\psi_j(x,\xi,t)|^2\,dx\,d\xi\right)^{\frac{1}{2}}\end{aligned}$$

and

$$\begin{aligned}&\left|\int_{\Omega}u_{jt}(x,t)\int_{-\infty}^{+\infty}\omega(\xi)\psi_j(x,\xi,t)\,d\xi\,dx\right|\\ \le\ &\left(\int_{-\infty}^{+\infty}\frac{\omega^2(\xi)}{\xi^2+\kappa}\,d\xi\right)^{\frac{1}{2}}\|u_j(x,t)\|_{L^2(\Omega)}\left(\int_{\Omega}\int_{-\infty}^{+\infty}(\xi^2+\kappa)|\psi_j(x,\xi,t)|^2\,dx\,d\xi\right)^{\frac{1}{2}}.\end{aligned}$$

Applying the Cauchy-Schwarz inequality we obtain (23). ∎

Proposition 6 *For any regular solution of problem (P') and for every $\varepsilon, \delta > 0$, we have*

$$\begin{aligned}
&\sum_{j=1}^{n} \frac{d}{dt}\left\{\int_\Omega [2m\cdot\nabla u_j + (n-1)u_j]\, u_{jt}\, dx\right\} \\
\le\ & -\sum_{j=1}^{n}\int_\Omega \left(|u_{jt}|^2 + (\mu - \frac{\varepsilon}{2}C(P))|\nabla u_j|^2\right) dx - (\mu+\lambda)\int_\Omega |\operatorname{div} u|^2\, dx \\
& -\zeta\sum_{j=1}^{n}\int_\Omega [2m\cdot\nabla u_j + (n-1)u_j]\left[\int_{-\infty}^{+\infty}\omega(\xi)\psi_j(x,\xi)\, d\xi\right] dx \\
& +\sum_{j=1}^{n}\int_\Gamma \left(\left(\|m\|_\infty + \frac{(n-1)^2}{2\varepsilon}a_2^2 + 2\frac{\|m\|_\infty^2}{\delta\mu}a_2^2\right)|u_{jt}|^2 - \left(\mu\delta - \frac{\delta\mu}{2}\right)|\nabla u_j|^2\right) d\Gamma \\
& -(\mu+\lambda)\delta\int_\Gamma |\operatorname{div} u|^2\, d\Gamma,
\end{aligned}$$

where $C(P)$ is a sort of Poincaré constant, which is a positive constant depending on Ω and independent of the solution u.

Proof.

$$\begin{aligned}
&\frac{d}{dt}\left\{\int_\Omega [2m\cdot\nabla u_j + (n-1)u_j]u_{jt}\, dx\right\} \\
=\ & \int_\Omega [2m\cdot\nabla u_{jt} + (n-1)u_{jt}]u_{jt}\, dx \\
& + \int_\Omega [2m\cdot\nabla u_j + (n-1)u_j][\mu\Delta u_j + (\mu+\lambda)\frac{\partial}{\partial x_j}(\operatorname{div} u) - \zeta\int_{-\infty}^{+\infty}\omega(\xi)\psi_j(x,\xi)\, d\xi]\, dx
\end{aligned} \tag{28}$$

For $u \in H^2(\Omega)$, we have the following Rellich's identity

$$\begin{cases} \int_\Omega \Delta u_j(m\cdot\nabla u_j)dx = \int_\Gamma (m\cdot\nabla u_j)\frac{\partial u_j}{\partial v}\,\Gamma - \int_\Omega \nabla u_j\cdot\nabla(m\cdot\nabla u_j)dx, \\ \int_\Omega \frac{\partial(\operatorname{div} u)}{\partial x_j}(m\cdot\nabla u_j)dx = \int_\Gamma (m\cdot\nabla u_j)(\operatorname{div} u)\nu_j\,\Gamma - \int_\Omega (\operatorname{div} u)\frac{\partial}{\partial x_j}(m\cdot\nabla u_j)\, dx \end{cases} \tag{29}$$

Hence

$$\begin{aligned}
&\frac{d}{dt}\left\{\int_\Omega [2m\cdot\nabla u_j + (n-1)u_j]u_{jt}\, dx\right\} \\
=\ & \int_\Omega [2m\cdot\nabla u_{jt} + (n-1)u_{jt}]u_{jt}\, dx \\
& -\zeta\int_\Omega [2m\cdot\nabla u_j + (n-1)u_j][\int_{-\infty}^{+\infty}\omega(\xi)\psi_j(x,\xi)d\xi]dx \\
& +2\mu\int_\Gamma (m\cdot\nabla u_j)\frac{\partial u_j}{\partial v}\, d\Gamma - 2\mu\int_\Omega \nabla u_j\cdot\nabla(m\cdot\nabla u_j)dx \\
& +2(\mu+\lambda)\int_\Gamma (m\cdot\nabla u_j)(\operatorname{div} u)\nu_j\, d\Gamma - 2(\mu+\lambda)\int_\Omega (\operatorname{div} u)\frac{\partial}{\partial x_j}(m\cdot\nabla u_j)\, dx \\
& -(n-1)\mu\int_\Omega |\nabla u_j|^2\, dx + (n-1)\mu\int_\Gamma u_j\frac{\partial u_j}{\partial v}d\Gamma \\
& -(n-1)(\mu+\lambda)\int_\Omega \frac{\partial u_j}{\partial x_j}(\operatorname{div} u)\, dx + (n-1)(\mu+\lambda)\int_\Gamma u_j(\operatorname{div} u)\nu_j\, d\Gamma.
\end{aligned} \tag{30}$$

Moreover, using the following identity

$$2\nabla u_j\cdot\nabla(m\cdot\nabla u_j) = 2|\nabla u_j|^2 + m\cdot\nabla(|\nabla u_j|^2)$$

and integration by parts, we get

$$\begin{cases} 2\int_\Omega \nabla u_j\cdot\nabla(m\cdot\nabla u_j)\,dx=(2-n)\int_\Omega|\nabla u_j|^2\,dx+\int_\Gamma m\cdot\nu|\nabla u_j|^2 d\Gamma, \\ 2\sum\limits_{j=1}^{n}\int_\Omega(\operatorname{div}u)\frac{\partial}{\partial x_j}(m\cdot\nabla u_j)dx=(2-n)\int_\Omega|\operatorname{div}u|^2\,dx+\int_\Gamma m\cdot\nu|\operatorname{div}u|^2\,d\Gamma \end{cases} \tag{31}$$

Substituting (31) into (30), we get

$$\begin{aligned} &\sum_{j=1}^{n}\frac{d}{dt}\left\{\int_\Omega[2m\cdot\nabla u_j+(n-1)u_j]u_{jt}\,dx\right\} \\ =\;&\sum_{j=1}^{n}\int_\Omega[2m\cdot\nabla u_{jt}+(n-1)u_{jt}]u_{jt}\,dx \\ &-\zeta\sum_{j=1}^{n}\int_\Omega[2m\cdot\nabla u_j+(n-1)u_j]\left[\int_{-\infty}^{+\infty}\omega(\xi)\psi_j(x,\xi)\,d\xi\right]dx \\ &+2\mu\sum_{j=1}^{n}\int_\Gamma(m\cdot\nabla u_j)\frac{\partial u_j}{\partial v}\,d\Gamma-\mu\sum_{j=1}^{n}\int_\Gamma m\cdot\nu|\nabla u_j|^2\,d\Gamma \\ &-(\mu+\lambda)\int_\Gamma m\cdot\nu|\operatorname{div}u|^2\,d\Gamma+2(\mu+\lambda)\sum_{j=1}^{n}\int_\Gamma(m\cdot\nabla u_j)(\operatorname{div}u)\nu_j\,d\Gamma \\ &-\mu\sum_{j=1}^{n}\int_\Omega|\nabla u_j|^2\,dx-(\mu+\lambda)\int_\Omega|\operatorname{div}u|^2\,dx+(n-1)\mu\sum_{j=1}^{n}\int_\Gamma u_j\frac{\partial u_j}{\partial v}d\Gamma \\ &+(n-1)(\mu+\lambda)\sum_{j=1}^{n}\int_\Gamma u_j(\operatorname{div}u)\nu_j\,d\Gamma. \end{aligned}$$

Noting that $\nabla u_j=\frac{\partial u_j}{\partial v}\nu$ on Γ_0, it follows that

$$\begin{aligned} &\sum_{j=1}^{n}\frac{d}{dt}\left\{\int_\Omega[2m\cdot\nabla u_j+(n-1)u_j]u_{jt}\,dx\right\} \\ =\;&-\sum_{j=1}^{n}\int_\Omega|u_{jt}|^2\,dx+\sum_{j=1}^{n}\int_\Gamma(m\cdot\nu)|u_{jt}|^2\,d\Gamma \\ &-\zeta\sum_{j=1}^{n}\int_\Omega[2m\cdot\nabla u_j+(n-1)u_j]\left[\int_{-\infty}^{+\infty}\omega(\xi)\psi_j(x,\xi)\,d\xi\right]dx \\ &+\mu\sum_{j=1}^{n}\int_\Gamma m\cdot\nu|\nabla u_j|^2\,d\Gamma+(\mu+\lambda)\int_\Gamma m\cdot\nu|\operatorname{div}u|^2\,d\Gamma \\ &+2\sum_{j=1}^{n}\int_\Gamma(m\cdot\nabla u_j)(\mu\frac{\partial u_j}{\partial v}+(\mu+\lambda)(\operatorname{div}u)\nu_j)\,d\Gamma-\mu\sum_{j=1}^{n}\int_\Gamma m\cdot\nu|\nabla u_j|^2\,d\Gamma \\ &-(\mu+\lambda)\int_\Gamma m\cdot\nu|\operatorname{div}u|^2\,d\Gamma-\mu\sum_{j=1}^{n}\int_\Omega|\nabla u_j|^2\,dx-(\mu+\lambda)\int_\Omega|\operatorname{div}u|^2\,dx \\ &+(n-1)\sum_{j=1}^{n}\int_\Gamma u_j(\mu\frac{\partial u_j}{\partial v}+(\mu+\lambda)(\operatorname{div}u)\nu_j)\,d\Gamma. \end{aligned} \tag{32}$$

Since Γ_1 is compact and m, ν are sufficiently regular, there exists $\delta > 0$ such that $m(x) \cdot \nu(x) \geq \delta > 0$, for all $x \in \Gamma_1$. from (32) we deduce

$$\begin{aligned}
&\sum_{j=1}^{n} \frac{d}{dt}\{\int_\Omega [2m\cdot\nabla u_j + (n-1)u_j]u_{jt}\,dx\} \\
\leq\; & -\sum_{j=1}^{n}\int_\Omega |u_{jt}|^2\,dx - \mu\sum_{j=1}^{n}\int_\Omega |\nabla u_j|^2\,dx \\
& -(\mu+\lambda)\int_\Omega |\operatorname{div} u|^2\,dx - \zeta\sum_{j=1}^{n}\int_\Omega [2m\cdot\nabla u_j + (n-1)u_j]\left[\int_{-\infty}^{+\infty}\omega(\xi)\psi_j(x,\xi)\,d\xi\right]dx \\
& +\|m\|_\infty\sum_{j=1}^{n}\int_\Gamma |u_{jt}|^2\,d\Gamma - \mu\delta\sum_{j=1}^{n}\int_\Gamma |\nabla u_j|^2\,d\Gamma - (\mu+\lambda)\delta\int_\Gamma |\operatorname{div} u|^2\,d\Gamma \\
& +2\sum_{j=1}^{n}\int_\Gamma (m\cdot\nabla u_j)(\mu\frac{\partial u_j}{\partial v} + (\mu+\lambda)(\operatorname{div} u)\nu_j)\,d\Gamma \\
& +(n-1)\sum_{j=1}^{n}\int_\Gamma u_j(\mu\frac{\partial u_j}{\partial v} + (\mu+\lambda)(\operatorname{div} u)\nu_j)\,d\Gamma,
\end{aligned}$$

where we have used also $m(x) \cdot \nu(x) < 0$ on Γ_0. We can estimate

$$\begin{aligned}
& 2\int_\Gamma (m\cdot\nabla u_j)\left(\mu\frac{\partial u_j}{\partial v} + (\mu+\lambda)(\operatorname{div} u)\nu_j\right)d\Gamma \\
\leq\; & \frac{\delta\mu}{2}\int_\Gamma |\nabla u_j|^2\,d\Gamma + 2\frac{\|m\|_\infty^2}{\delta\mu}\int_\Gamma \left|\mu\frac{\partial u_j}{\partial v} + (\mu+\lambda)(\operatorname{div} u)\nu_j\right|^2 d\Gamma \\
\leq\; & \frac{\delta\mu}{2}\int_\Gamma |\nabla u_j|^2\,d\Gamma + 2\frac{\|m\|_\infty^2\,a_2^2}{\delta\mu}\int_\Gamma |u_{jt}|^2\,d\Gamma.
\end{aligned} \tag{33}$$

Moreover,

$$\begin{aligned}
& (n-1)\int_\Gamma u_j\left(\mu\frac{\partial u_j}{\partial v} + (\mu+\lambda)(\operatorname{div} u)\nu_j\right)d\Gamma \\
\leq\; & \frac{\varepsilon}{2}\int_\Gamma |u_j|^2\,d\Gamma + \frac{(n-1)^2}{2\varepsilon}\int_\Gamma \left|\left(\mu\frac{\partial u_j}{\partial v} + (\mu+\lambda)(\operatorname{div} u)\nu_j\right)\right|^2 d\Gamma \\
\leq\; & \frac{\varepsilon}{2}C(P)\int_\Omega |\nabla u_j|^2\,dx + \frac{(n-1)^2}{2\varepsilon}\int_\Gamma \left|\mu\frac{\partial u_j}{\partial v} + (\mu+\lambda)(\operatorname{div} u)\nu_j\right|^2 d\Gamma \\
\leq\; & \frac{\varepsilon}{2}C(P)\int_\Omega |\nabla u_j|^2\,dx + \frac{(n-1)^2\,a_2^2}{2\varepsilon}\int_\Gamma |u_{jt}|^2\,d\Gamma,
\end{aligned} \tag{34}$$

where we have used trace inequality and Poincaré's theorem. ■

Remark 2 *In the above inequality $C(P)$ is the smallest positive constant such that*

$$\int_\Gamma |\vartheta|^2\,d\Gamma \leq C(P)\int_\Omega |\nabla\vartheta|^2\,dx, \quad \forall\vartheta \in H^1_{\Gamma_0}(\Omega).$$

Then by using the Young inequality and the Sobolev-Poincaré inequality, we can easily get the following corollary.

Corollary 7 *For any regular solution of (P') and for every $\varepsilon, \delta > 0$, we have*

$$\begin{aligned} &\sum_{j=1}^{n} \frac{d}{dt} \left\{ \int_\Omega [2m\cdot\nabla u_j + (n-1)u_j]u_{jt}\,dx \right\} \\ \leq\ & -\sum_{j=1}^{n} \int_\Omega |u_{jt}|^2\,dx - \left(\mu - \frac{\varepsilon}{2}C(P) - \zeta\|m\|_\infty^2 I - \frac{\zeta}{2}I(n-1)^2C(\Omega)\right)\sum_{j=1}^{n}\int_\Omega |\nabla u_j|^2\,dx \\ & -(\mu+\lambda)\int_\Omega |\operatorname{div} u|^2\,dx + (\|m\|_\infty + 2\frac{\|m\|_\infty^2}{\delta\mu}a_2^2 + \frac{(n-1)^2\,a_2^2}{2\varepsilon})\sum_{j=1}^{n}\int_\Gamma |u_{jt}|^2\,d\Gamma \\ & +\frac{3}{2}\zeta\sum_{j=1}^{n}\int_\Omega\int_{-\infty}^{+\infty}(\xi^2+\kappa)|\psi_j(x,\xi,t)|^2\,dx\,d\xi \\ & -\frac{\delta\mu}{2}\sum_{j=1}^{n}\int_\Gamma |\nabla u_j|^2\,d\Gamma - (\mu+\lambda)\delta\int_\Gamma |\operatorname{div} u|^2\,d\Gamma. \end{aligned}$$

Now, let us introduce the functional

$$\mathcal{S}(t) = \sum_{j=1}^{n}\int_\Omega\int_0^1 e^{-\tau\rho}|z_j(x,\rho,t)|^2 d\,\rho\,dx.$$

We can easily estimate

$$\begin{aligned} \mathcal{S}'(t) &= 2\int_\Omega\int_0^1 e^{-\tau\rho} z_t(x,\rho,t)z(x,\rho,t)d\rho\,dx \\ &= -\frac{2}{\tau}\int_\Omega\int_0^1 e^{-\tau\rho} z_\rho(x,\rho,t)z(x,\rho,t)d\rho\,dx \\ &= -\frac{1}{\tau}\int_\Omega\int_0^1 e^{-\tau\rho}\frac{d}{d\rho}|z(x,\rho,t)|^2 d\rho\,dx \\ &= -\frac{1}{\tau}\int_\Omega e^{-\tau}|z(x,1,t)|^2\,dx + \frac{1}{\tau}\int_\Omega |u_t|^2\,dx \\ &= -\int_\Omega\int_0^1 e^{-\tau\rho}|z(x,\rho,t)|^2 d\,\rho\,dx \\ &\leq \frac{1}{\tau}\int_\Omega |u_t|^2\,dx - \frac{1}{\tau}e^{-\tau}\int_\Omega |z(x,1,t)|^2\,dx - e^{-\tau}\int_\Omega\int_0^1 |z(x,\rho,t)|^2 d\,\rho\,dx. \end{aligned}$$

Let us introduce the Lyapunov functional

$$\mathcal{E}(t) = E(t) + \gamma_1\sum_{j=1}^{n}\int_\Omega [2m\cdot\nabla u_j + (n-1)u_j]u_{jt}\,dx + \gamma_2\mathcal{S}(t),$$

where γ_1, γ_2 are suitable positive small constants that will be precised later on. Note that $\mathcal{E}(t)$ is equivalent to the energy $E(t)$ if γ_1 is small enough. In particular, there exists a positive constant C_1 and suitable positive constants α_1, α_2 such that

$$\alpha_1 E(t) \leq \mathcal{E}(t) \leq \alpha_2 E(t), \quad \forall 0 < \gamma_1, \quad \gamma_1 \leq C_1. \tag{35}$$

Proposition 8 *For every $a_2 > 0$ there exist a_0, c_1, c_2 such that for any solution of problem (P) with $0 \leq a_1 < a_0$ we have*

$$\mathcal{E}(t) \leq c_1 e^{-c_2 t}, \quad t > 0. \tag{36}$$

The constants a_0, c_1, c_2 are independent of the initial data but they depend on a_2 and on the geometry of Ω.

Proof. Differentiating the Lyapunov functional $\mathcal{E}$ and using the propositions above we deduce

$$\begin{aligned}\mathcal{E}'(t) \quad \leq \quad & \left(\frac{\zeta I + \upsilon\tau^{-1}}{2} - \gamma_1 + \frac{\gamma_2}{\tau}\right)\sum_{j=1}^{n}\int_\Omega |u_{jt}|^2\,dx \\ & -\gamma_2 e^{-\tau}\sum_{j=1}^{n}\int_\Omega\int_0^1 |z_j(x,\rho,t)|^2 d\,\rho\,dx - \frac{\gamma_2}{\tau}e^{-\tau}\sum_{j=1}^{n}\int_\Omega |z_j(x,1,t)|^2\,dx \\ & + \left(\frac{\zeta I + \upsilon\tau^{-1}}{2} + \frac{3}{2}\zeta\gamma_1\right)\sum_{j=1}^{n}\int_\Omega\int_{-\infty}^{+\infty}(\xi^2+\kappa)|\psi_j(x,\xi,t)|^2\,dx\,d\xi \\ & -\gamma_1\left(\mu - \frac{\varepsilon}{2}C(P) - \zeta\|m\|_\infty^2 I - \frac{\zeta}{2}I(n-1)^2C(\Omega)\right)\sum_{j=1}^{n}\int_\Omega |\nabla u_j|^2\,dx \\ & + \left(\gamma_1\|m\|_\infty + \gamma_1 a_2^2\left(2\frac{\|m\|_\infty^2}{\delta\mu} + \frac{(n-1)^2}{2\varepsilon}\right) - a_2\right)\sum_{j=1}^{n}\int_\Gamma |u_{jt}|^2\,d\Gamma. \end{aligned} \tag{37}$$

For a fixed $a_2 > 0$ we want to choose $\varepsilon, \gamma_1, \gamma_2 < C_1$ and a_1 sufficiently small in order to obtain

$$\mathcal{E}'(t) \leq -cE(t). \tag{38}$$

Applying the second inequality of (35) estimate (36) easily follows. To show that (37) implies (38) we simply need that

$$\frac{\zeta I + \upsilon\tau^{-1}}{2} - \gamma_1 + \frac{\gamma_2}{\tau} < 0,$$

$$\frac{\zeta I - \upsilon\tau^{-1}}{2} + \frac{3}{2}\zeta\gamma_1 < 0,$$

$$\mu - \frac{\varepsilon}{2}C(P) - \zeta\|m\|_\infty^2 I - \frac{\zeta}{2}I(n-1)^2C(\Omega) > 0,$$

$$\gamma_1\|m\|_\infty + \gamma_1 a_2^2\left(2\frac{\|m\|_\infty^2}{\delta\mu} + \frac{(n-1)^2}{2\varepsilon}\right) - a_2 < 0.$$

For any $a_2 > 0$ this last condition is satisfied for γ_1 sufficiently small. It then remains to the first and third conditions. For the first one, we need to assume that $\gamma_1 > \gamma_2/\tau$, while for the third equation we need to fix ε small enough such that

$$\mu - \frac{\varepsilon}{2}C(P) > 0.$$

Then we now fix γ_1, γ_2 and ε and fulfilling the above requirements and look at the first equation to the third equation as conditions on a_1 and υ. ■

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