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On The Coupled System of ψ-Caputo Fractional Differential Equations With Four-Point Boundary Conditions*

Mohamed I. Abbas†

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Abstract

In this paper, we study the existence and uniqueness of solutions of boundary value problems for coupled systems of ψ-Caputo fractional differential equations with four-point boundary conditions. The main results are established by the aid of the Leray-Schauder alternative and the Banach's fixed point theorem. Two examples are provided in order to illustrate the theoretical results.

1 Introduction

Fractional differential equations have been studied extensively in the literature because of their applications in various fields of engineering and science. For examples, we refer the interested reader to the monographs [23, 24, 25, 28, 31]. The study of qualitative properties of solutions for fractional differential equations such as existence, uniqueness, stability, continuous dependence, etc., has become a dignified domain of consideration in recent years. For instance, we mention here a few recent works by Abbas [1, 2, 3, 4, 5, 6, 7] and the references cited therein.

The topic of boundary value problems for fractional differential equations has also received worthy interest through recent decades. At the same time, it has become widely seen that coupled boundary value problems have gained an importance role in view of their great utility in handling of applied nature such as: ecological models [22], anomalous diffusion [27], systems of nonlocal thermoelasticity [29], and the heat equations [34], and so on. For some recent theoretical results on the topic, we refer the reader to a series of papers [9, 10, 11, 15, 16, 17, 26, 32, 35] and the references cited therein.

Very recently, Almeida [13] introduced a Caputo-type fractional derivative with respect to another function, named as ψ-Caputo fractional operator. He also studied some properties like the semigroup law, a relationship between the fractional derivative and the fractional integral, Taylor's theorem, and Fermat's theorem. For the purpose of wholeness, we display briefly some recent works concerning coupled systems of fractional differential equations via different types of fractional derivative operators as well as various kinds of boundary conditions.

In [9], Agarwal et al. discussed the existence of solutions for a nonlinear coupled system of Liouville-Caputo type fractional differential equations with coupled flux conditions unified with nonlocal strip and multi-point boundary conditions:

$$\begin{cases} {}^CD^q x(t) = f(t, x(t), y(t), {}^CD^{\sigma_1} y(t)), \quad t \in [0,1],\ 2 < q \le 3,\ 1 < \sigma_1 < 2, \\ {}^CD^p y(t) = g(t, x(t), {}^CD^{\sigma_2} x(t), y(t)), \quad t \in [0,1],\ 2 < p \le 3,\ 1 < \sigma_2 < 2, \\ x(0) = \psi_1(y),\ x'(0) = e_1 y(w_1), \quad x(1) = a_1 \int_0^\xi y(s)ds + b_1 \sum_{i=1}^{m-2} \alpha_i y(\eta_i), \\ y(0) = \psi_2(x),\ y'(0) = e_2 x(w_2), \quad y(1) = a_2 \int_0^\xi x(s)ds + b_1 \sum_{i=1}^{m-2} \beta_i x(\eta_i), \\ 0 < w_1 < w_2 < \xi < \eta_1 < \eta_2 < \cdots < \eta_{m-2} < 1, \end{cases}$$

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†Department of Mathematics and Computer Science, Faculty of Science, Alexandria University, Alexandria 21511, Egypt

where ${}^C D^q$ and ${}^C D^p$ are the Liouville-Caputo fractional derivatives of order q and p, respectively. Their main results are obtained by applying Banach's fixed point theorem and Leray-Schauder alternative.

In [12], Ahmad et al. studied the existence, uniqueness and Hyers-Ulam stability of the following implicit switched coupled system of FDE involving the Katugampola-Caputo fractional derivative:

$$\begin{cases} {}^\sigma_c D^\alpha u(\sigma) - f(\sigma, u(\sigma), {}^\sigma_c D^\alpha u(\sigma)) = \psi(\sigma, u(\sigma), v(\sigma)), \ \sigma \in [0,T], \ 0<\alpha<1, \\ {}^\sigma_c D^\alpha v(\sigma) - g(\sigma, v(\sigma), {}^\sigma_c D^\alpha v(\sigma)) = \phi(\sigma, u(\sigma), v(\sigma)), \\ u(0) + h(u) = u_0, \quad v(0) + h(v) = v_0, \end{cases}$$

where σ is a positive real number and ${}^\sigma_c D^\alpha$ is the Katugampola-Caputo fractional derivative of order α.

By means of the technique of measure of noncompactness combining with the Mönch's fixed point theorem, Derbazi and Baitiche [19] investigated the Coupled systems of ψ-Caputo differential equations with initial conditions:

$$\begin{cases} {}^C D^{\alpha_1;\psi}_{a^+} u(t) = f_1(t, u(t), v(t)), \quad t \in [a,b], \\ {}^C D^{\alpha_2;\psi}_{a^+} v(t) = f_2(t, u(t), v(t)), \\ u(a) = u_a, \quad v(a) = v_a, \end{cases}$$

where ${}^C D^{\alpha_i;\psi}_{a^+}$ is the ψ-Caputo fractional derivative of order $\alpha_i \in (0,1]$, in the Banach space E.

In [30], Rao et al. considered the existence and uniqueness of solutions for a coupled system of fractional differential equation with fractional non-separated coupled boundary conditions:

$$\begin{cases} {}^C D^{\alpha}_{0+} u(t) = f(t, u(t), v(t)), \quad 0<t<1, \\ {}^C D^{\beta}_{0+} v(t) = g(t, u(t), v(t)), \quad 0<t<1, \\ u(0) = \lambda_1 v(1), \quad {}^C D^{\gamma}_{0+} u(1) = \lambda_2 {}^C D^{\gamma}_{0+} v(\xi), \ 0<\gamma<1, \\ v(0) = \mu_1 u(1), \quad {}^C D^{\gamma}_{0+} v(1) = \mu_2 {}^C D^{\gamma}_{0+} u(\xi), \ 0<\gamma<1, \end{cases}$$

where $\alpha, \beta \in (1,2]$, $\xi \in (0,1)$, ${}^C D^{\alpha}_{0+}$ and ${}^C D^{\beta}_{0+}$ are the Caputo fractional derivatives of order α and β, respectively. They investigated their problem using Banach contraction principle and the Leray-Schauder fixed point theorem. For more recent works involving ψ-Caputo fractional operator, we refer the reader to a series of papers [8, 14, 18, 20, 33].

In order to enrich the topic, we study the existence and uniqueness of solutions of the following four-point boundary value problem for a coupled system of fractional differential equations:

$$\begin{cases} {}^C D^{\alpha;\psi}_{0^+} x(t) = f(t, x(t), y(t)), \quad t \in [0,1], \ 1<\alpha<2, \\ {}^C D^{\beta;\psi}_{0^+} y(t) = g(t, x(t), y(t)), \quad t \in [0,1], \ 1<\beta<2, \\ x(0) = y(0) = 0, \\ x(1) = \lambda x(\eta), \ y(1) = \mu y(\xi), \ 0<\eta,\xi<1, \ \lambda,\mu>0 \end{cases} \tag{1}$$

where ${}^C D^{\alpha;\psi}_{0^+}$, ${}^C D^{\beta;\psi}_{0^+}$ denote the ψ-Caputo fractional derivatives of order α, β and $f, g : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ are continuous functions.

For usefulness, the obtained results in the current paper are novel and generalize many recent results relevant to the topic. Further, to the best knowledge of the author, there are no contributions covered boundary value problems for coupled systems of fractional differential equations via ψ-Caputo till now.

This paper is arranged as follows. In Section 2, we give some definitions and lemmas which are used in the rest of the paper. In Section 3, the existence and uniqueness results for the coupled system (1) are derived. In Section 4, we furnish two examples to illustrate the main outcomes.

2 Preliminary and Lemmas

In this section, we recall some definitions, properties and lemmas of the new ψ-Caputo fractional derivative.

Definition 1 ([13, 14]) *For $\alpha > 0$, the left-sided ψ-Riemann-Liouville fractional integral of order α for an integrable function $\sigma : [a, b] \to \mathbb{R}$ with respect to another function $\psi : [a, b] \to \mathbb{R}$ that is an increasing differentiable function such that $\psi'(t) \neq 0$, for all $t \in [a, b]$ is defined by*

$$I_{a^+}^{\alpha;\psi}\sigma(t) = \frac{1}{\Gamma(\alpha)}\int_a^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\sigma(s)ds, \tag{2}$$

where Γ is the Euler Gamma function.

Definition 2 ([13]) *Let $n \in \mathbb{N}$ and let $\psi, \sigma \in C^n([a, b], \mathbb{R})$ be two functions such that ψ is increasing and $\psi'(t) \neq 0$, for all $t \in [a, b]$. The left-sided ψ-Riemann-Liouville fractional derivative of a function σ of order α is defined by*

$$\begin{aligned} D_{a^+}^{\alpha;\psi}\sigma(t) &= \left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n I_{a^+}^{n-\alpha;\psi}\sigma(t) \\ &= \frac{1}{\Gamma(n-\alpha)}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n \int_a^t \psi'(s)(\psi(t)-\psi(s))^{n-\alpha-1}\sigma(s)ds, \end{aligned}$$

where $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of the real number α.

Definition 3 ([13]) *Let $n - 1 < \alpha < n$, $n \in \mathbb{N}$ and let $\psi, \sigma \in C^n([a, b], \mathbb{R})$ be two functions such that ψ is increasing and $\psi'(t) \neq 0$, for all $t \in [a, b]$. The left-sided ψ-Caputo fractional derivative of a function σ of order α is defined by*

$$^{C}D_{a^+}^{\alpha;\psi}\sigma(t) = D_{a^+}^{\alpha;\psi}\left[\sigma(t) - \sum_{k=0}^{n-1}\frac{\sigma_\psi^{[k]}(a)}{k!}(\psi(t)-\psi(a))^k\right],$$

where $\sigma_\psi^{[k]}(t) = \left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^k \sigma(t)$ and $n = [\alpha] + 1$ for $\alpha \notin \mathbb{N}$, $n = \alpha$ for $\alpha \in \mathbb{N}$. Further, if $\sigma \in C^n([a, b], \mathbb{R})$ and $\alpha \notin \mathbb{N}$, then

$$\begin{aligned} ^{C}D_{a^+}^{\alpha;\psi}\sigma(t) &= I_{a^+}^{n-\alpha;\psi}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n \sigma(t) \\ &= \frac{1}{\Gamma(n-\alpha)}\int_a^t \psi'(s)(\psi(t)-\psi(s))^{n-\alpha-1}\sigma_\psi^{[n]}(s)ds. \end{aligned}$$

Thus, if $\alpha = n \in \mathbb{N}$, then $^{C}D_{a^+}^{\alpha;\psi}\sigma(t) = \sigma_\psi^{[n]}(t)$.

Lemma 1 ([14]) *Let $\alpha > 0$. The following statements hold.*

(i) If $\sigma \in C([a, b], \mathbb{R})$, then

$$^{C}D_{a^+}^{\alpha;\psi} I_{a^+}^{\alpha;\psi}\sigma(t) = \sigma(t), \quad t \in [a, b].$$

(ii) If $\sigma \in C^n([a, b], \mathbb{R})$, $n - 1 < \alpha < n$, then

$$I_{a^+}^{\alpha;\psi}\ {}^{C}D_{a^+}^{\alpha;\psi}\sigma(t) = \sigma(t) - \sum_{k=0}^{n-1} c_k(\psi(t)-\psi(a))^k, \quad t \in [a, b] \quad \textit{where } c_k = \frac{\sigma_\psi^{[k]}(a)}{k!}.$$

Lemma 2 ([14, 23]) *Let $t > a$, $\alpha \geq 0$ and $\beta > 0$. Then*

$$I_{a^+}^{\alpha;\psi}(\psi(t)-\psi(a))^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(\psi(t)-\psi(a))^{\beta+\alpha-1},$$

$$^{C}D_{a^+}^{\alpha;\psi}(\psi(t)-\psi(a))^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(\psi(t)-\psi(a))^{\beta-\alpha-1},$$

$$^{C}D_{a^+}^{\alpha;\psi}(\psi(t)-\psi(a))^{k} = 0 \ \ \textit{for all } k \in \{0,1,\cdots,n-1\},\ n \in \mathbb{N}.$$

Lemma 3 (Leray-Schauder alternative [21]) *Let $\mathcal{T} : E \to E$ be a completely continuous operator (i.e., a map that restricted to any bounded set in E is compact). Let $\mathcal{S}(\mathcal{T}) = \{x \in E : x = \nu\mathcal{T}(x),$ for some $0 < \nu < 1\}$. Then either the set $\mathcal{S}(\mathcal{T})$ is unbounded or $\mathcal{T}$ has at least one fixed point.*

3 Main Results

Let $C([0,1],\mathbb{R})$ be the space of all continuous functions defined on $[0,1]$. Let $X = \{x(t) : x(t) \in C([0,1],\mathbb{R})\}$ be a Banach space endowed with the norm $\|x\|_X = \sup_{t\in[0,1]}|x(t)|$ and $Y = \{y(t) : y(t) \in C([0,1],\mathbb{R})\}$ be a Banach space endowed with the norm $\|y\|_Y = \sup_{t\in[0,1]}|y(t)|$. Thus the product space $(X \times Y, \|\cdot\|_{X\times Y})$ is also a Banach space with the norm $\|(x,y)\|_{X\times Y} = \|x\|_X + \|y\|_Y$ for $(x,y) \in X \times Y$.

Lemma 4 *Let $h \in C([0,1],,\mathbb{R})$ be a given function and $1 < \alpha < 2$. Then the unique solution of*

$$\begin{cases} ^{C}D_{0^+}^{\alpha;\psi}x(t) = h(t), \quad t \in [0,1], \\ x(0) = 0,\ x(1) = \lambda x(\eta), \end{cases} \tag{3}$$

is given by the integral equation

$$\begin{aligned} x(t) \;=\; & \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}h(s)ds + \frac{1}{\Delta_1}\left[\frac{1}{\Gamma(\alpha)}\int_0^1 \psi'(s)(\psi(1)-\psi(s))^{\alpha-1}h(s)ds \right.\\ & \left. -\frac{\lambda}{\Gamma(\alpha)}\int_0^\eta \psi'(s)(\psi(\eta)-\psi(s))^{\alpha-1}h(s)ds\right](\psi(t)-\psi(0)), \end{aligned} \tag{4}$$

where

$$\Delta_1 = \lambda(\psi(\eta)-\psi(0)) - (\psi(1)-\psi(0)) \neq 0.$$

Proof. First, let $x \in C([0,1])$ be a solution of (3). Then, by Lemma 1, we get

$$x(t) = \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}h(s)ds + c_0 + c_1(\psi(t)-\psi(0)). \tag{5}$$

The boundary conditions $x(0) = 0,\ x(1) = \lambda x(\eta)$ imply that $c_0 = 0$ and

$$\begin{aligned} & \frac{1}{\Gamma(\alpha)}\int_0^1 \psi'(s)(\psi(1)-\psi(s))^{\alpha-1}h(s)ds + c_1(\psi(1)-\psi(0)) \\ = \;& \frac{\lambda}{\Gamma(\alpha)}\int_0^\eta \psi'(s)(\psi(\eta)-\psi(s))^{\alpha-1}h(s)ds + \lambda c_1(\psi(\eta)-\psi(0)), \end{aligned}$$

which implies that

$$c_1 = \frac{1}{\Delta_1}\left[\frac{1}{\Gamma(\alpha)}\int_0^1 \psi'(s)(\psi(1)-\psi(s))^{\alpha-1}h(s)ds - \frac{\lambda}{\Gamma(\alpha)}\int_0^\eta \psi'(s)(\psi(\eta)-\psi(s))^{\alpha-1}h(s)ds\right].$$

which, on substituting in (5), completes the solution (4).

Conversely, if $x(t)$ satisfies the integral equation (4), then by applying the ψ-Caputo fractional derivative ${}^{C}D_{0^+}^{\alpha;\psi}$ to both sides of equation (4) and using Lemmas 1 and 2, we obtain

$$ {}^{C}D_{0^+}^{\alpha;\psi}x(t) = h(t). $$

Finally, it remains to show that the boundary conditions in (3) are satisfied. Clearly, $x(0) = 0$ and direct computations lead to $x(1) = \lambda x(\eta)$. This completes the proof. ■

Similarly, the general solution of ${}^{C}D_{0^+}^{\beta;\psi}y(t) = h(t),\ y(0) = 0,\ y(1) = \mu y(\xi)$ can be obtained from

$$ \begin{aligned} y(t) &= \frac{1}{\Gamma(\beta)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\beta-1}h(s)ds + \frac{1}{\Delta_2}\left[\frac{1}{\Gamma(\beta)}\int_0^1 \psi'(s)(\psi(1)-\psi(s))^{\beta-1}h(s)ds\right. \\ &\quad \left. -\frac{\mu}{\Gamma(\beta)}\int_0^\xi \psi'(s)(\psi(\xi)-\psi(s))^{\beta-1}h(s)ds\right](\psi(t)-\psi(0)), \end{aligned} \tag{6} $$

where

$$ \Delta_2 = \mu(\psi(\xi)-\psi(0)) - (\psi(1)-\psi(0)) \neq 0. $$

Lemma 5 *Assume that $f, g : [0,1]\times\mathbb{R}^2 \to \mathbb{R}$ are continuous functions. Then $(x,y) \in X \times Y$ is a solution of (1) if and only if $(x,y) \in X \times Y$ is a solution of the coupled system of integral equations*

$$ \begin{aligned} x(t) &= \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}f(s,x(s),y(s))ds \\ &\quad +\frac{1}{\Delta_1}\left[\frac{1}{\Gamma(\alpha)}\int_0^1 \psi'(s)(\psi(1)-\psi(s))^{\alpha-1}f(s,x(s),y(s))ds\right. \\ &\quad \left. -\frac{\lambda}{\Gamma(\alpha)}\int_0^\eta \psi'(s)(\psi(\eta)-\psi(s))^{\alpha-1}f(s,x(s),y(s))ds\right](\psi(t)-\psi(0)) \end{aligned} $$

and

$$ \begin{aligned} y(t) &= \frac{1}{\Gamma(\beta)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\beta-1}g(s,x(s),y(s))ds \\ &\quad +\frac{1}{\Delta_2}\left[\frac{1}{\Gamma(\beta)}\int_0^1 \psi'(s)(\psi(1)-\psi(s))^{\beta-1}g(s,x(s),y(s))ds\right. \\ &\quad \left. -\frac{\mu}{\Gamma(\beta)}\int_0^\xi \psi'(s)(\psi(\xi)-\psi(s))^{\beta-1}g(s,x(s),y(s))ds\right](\psi(t)-\psi(0)). \end{aligned} $$

Let us define the operator $\mathcal{T} : X \times Y \to X \times Y$ as

$$ \mathcal{T}(x,y)(t) = \begin{pmatrix} \mathcal{T}_1(x,y)(t) \\ \mathcal{T}_2(x,y)(t) \end{pmatrix} \tag{7} $$

where

$$ \begin{aligned} \mathcal{T}_1(x,y)(t) &= \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}f(s,x(s),y(s))ds \\ &\quad +\frac{1}{\Delta_1}\left[\frac{1}{\Gamma(\alpha)}\int_0^1 \psi'(s)(\psi(1)-\psi(s))^{\alpha-1}f(s,x(s),y(s))ds\right. \\ &\quad \left. -\frac{\lambda}{\Gamma(\alpha)}\int_0^\eta \psi'(s)(\psi(\eta)-\psi(s))^{\alpha-1}f(s,x(s),y(s))ds\right](\psi(t)-\psi(0)), \end{aligned} \tag{8} $$

and

$$\begin{aligned} \mathcal{T}_2(x,y)(t) &= \frac{1}{\Gamma(\beta)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\beta-1}g(s,x(s),y(s))ds \\ &+\frac{1}{\Delta_2}\left[\frac{1}{\Gamma(\beta)}\int_0^1 \psi'(s)(\psi(1)-\psi(s))^{\beta-1}g(s,x(s),y(s))ds\right. \\ &\left.-\frac{\mu}{\Gamma(\beta)}\int_0^\xi \psi'(s)(\psi(\xi)-\psi(s))^{\beta-1}g(s,x(s),y(s))ds\right](\psi(t)-\psi(0)). \end{aligned} \tag{9}$$

In order to establish our main results, we introduce the following assumptions.

(A1) The functions $f, g : [0,1]\times\mathbb{R}^2 \to \mathbb{R}$ are continuous and there exist real constants $L_1, L_2 > 0$ such that

$$|f(t,x_1,y_1) - f(t,x_2,y_2)| \leq L_1(|x_1-x_2| + |y_1-y_2|)$$

and

$$|g(t,x_1,y_1) - g(t,x_2,y_2)| \leq L_2(|x_1-x_2| + |y_1-y_2|)$$

for $t\in[0,1]$ and $x_i, y_i \in \mathbb{R},\ i=1,2$.

(A2) There exist real constants $k_i, l_i \geq 0,\ i=1,2$ and $k_0>0, l_0>0$ such that

$$|f(t,x,y)| \leq k_0 + k_1|x| + k_2|y|, \quad |g(t,x,y)| \leq l_0 + l_1|x| + l_2|y|,$$

for $t\in[0,1]$ and $x_i, y_i \in \mathbb{R},\ i=1,2$.

For brevity, we use the following notations:

$$\begin{cases} \gamma_1 = \frac{M_1}{\Gamma(\alpha+1)}\left[(\psi(1)-\psi(0))^\alpha + \frac{(|\mu|+1)}{|\Delta_1|}(\psi(1)-\psi(0))^{\alpha+1}\right], \\ \gamma_2 = \frac{M_2}{\Gamma(\beta+1)}\left[(\psi(1)-\psi(0))^\beta + \frac{(|\mu|+1)}{|\Delta_2|}(\psi(1)-\psi(0))^{\beta+1}\right], \\ \gamma_3 = \frac{L_1}{\Gamma(\alpha+1)}\left[(\psi(1)-\psi(0))^\alpha + \frac{(|\lambda|+1)}{|\Delta_1|}(\psi(1)-\psi(0))^{\alpha+1}\right], \\ \gamma_4 = \frac{L_2}{\Gamma(\beta+1)}\left[(\psi(1)-\psi(0))^\beta + \frac{(|\mu|+1)}{|\Delta_2|}(\psi(1)-\psi(0))^{\beta+1}\right], \end{cases} \tag{10}$$

and

$$\begin{cases} \Omega_0 = \Bigg(\frac{1}{\Gamma(\alpha+1)} \Bigg[(\psi(1)-\psi(0))^{\alpha} + \frac{(|\lambda|+1)}{|\Delta_1|} (\psi(1)-\psi(0))^{\alpha+1} \Bigg] k_1 \\ \qquad + \frac{1}{\Gamma(\beta+1)} \Bigg[(\psi(1)-\psi(0))^{\beta} + \frac{(|\mu|+1)}{|\Delta_2|} (\psi(1)-\psi(0))^{\beta+1} \Bigg] \Bigg) l_0, \\ \\ \Omega_1 = \frac{1}{\Gamma(\alpha+1)} \Bigg[(\psi(1)-\psi(0))^{\alpha} + \frac{(|\lambda|+1)}{|\Delta_1|} (\psi(1)-\psi(0))^{\alpha+1} \Bigg] k_1 \\ \qquad + \frac{1}{\Gamma(\beta+1)} \Bigg[(\psi(1)-\psi(0))^{\beta} + \frac{(|\mu|+1)}{|\Delta_2|} (\psi(1)-\psi(0))^{\beta+1} \Bigg] l_1, \\ \\ \Omega_2 = \frac{1}{\Gamma(\alpha+1)} \Bigg[(\psi(1)-\psi(0))^{\alpha} + \frac{(|\lambda|+1)}{|\Delta_1|} (\psi(1)-\psi(0))^{\alpha+1} \Bigg] k_2 \\ \qquad + \frac{1}{\Gamma(\beta+1)} \Bigg[(\psi(1)-\psi(0))^{\beta} + \frac{(|\mu|+1)}{|\Delta_2|} (\psi(1)-\psi(0))^{\beta+1} \Bigg] l_2, \\ \\ \Omega^* = \max\{\Omega_1, \Omega_2\}. \end{cases} \tag{11}$$

3.1 Uniqueness Result via Banach's Fixed Point Theorem

Theorem 1 *Assume that (A1) holds. Then the coupled system (1) has a unique solution on* $[0,1]$*, provided that*

$$(\gamma_3 + \gamma_4) < 1, \tag{12}$$

where γ_3 *and* γ_4 *are given in (10).*

Proof. Assume that $r > 0$ is a real number such that

$$r \geq \frac{\gamma_1 + \gamma_2}{1 - (\gamma_3 + \gamma_4)}.$$

First, we shall show that $\mathcal{T}\mathcal{B}_r \subset \mathcal{B}_r$, where $\mathcal{T}$ is given by (7) and

$$\mathcal{B}_r = \{(x,y) \in X \times Y : \|(x,y)\|_{X \times Y} \leq r\}.$$

Set $\sup_{t\in[0,1]}|f(t,0,0)| = M_1 < \infty$ and $\sup_{t\in[0,1]}|g(t,0,0)| = M_2 < \infty$. For $(x,y)\in\mathcal{B}_r,\ t\in[0,1]$, we have

$$\begin{aligned}
|\mathcal{T}_1(x,y)(t)| \leq\ & \left|\frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}f(s,x(s),y(s))ds\right| \\
&+\left|\frac{1}{\Delta_1}\left[\frac{1}{\Gamma(\alpha)}\int_0^1 \psi'(s)(\psi(1)-\psi(s))^{\alpha-1}f(s,x(s),y(s))ds\right.\right. \\
&\left.\left.-\frac{\lambda}{\Gamma(\alpha)}\int_0^\eta \psi'(s)(\psi(\eta)-\psi(s))^{\alpha-1}f(s,x(s),y(s))ds\right](\psi(t)-\psi(0))\right| \\
\leq\ & \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}(|f(s,x(s),y(s))-f(t,0,0)|+|f(t,0,0)|)ds \\
&+\frac{1}{|\Delta_1|}\left[\frac{1}{\Gamma(\alpha)}\int_0^1 \psi'(s)(\psi(1)-\psi(s))^{\alpha-1}(|f(s,x(s),y(s))-f(t,0,0)|+|f(t,0,0)|)ds\right. \\
&\left.+\frac{|\lambda|}{\Gamma(\alpha)}\int_0^\eta \psi'(s)(\psi(\eta)-\psi(s))^{\alpha-1}(|f(s,x(s),y(s))-f(t,0,0)|+|f(t,0,0)|)ds\right]|\psi(t)-\psi(0)| \\
\leq\ & \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}(L_1(|x(s)|+|y(s)|)+|f(t,0,0)|)ds \\
&+\frac{1}{|\Delta_1|}\left[\frac{1}{\Gamma(\alpha)}\int_0^1 \psi'(s)(\psi(1)-\psi(s))^{\alpha-1}(L_1(|x(s)|+|y(s)|)+|f(t,0,0)|)ds\right. \\
&\left.+\frac{|\lambda|}{\Gamma(\alpha)}\int_0^\eta \psi'(s)(\psi(\eta)-\psi(s))^{\alpha-1}(L_1(|x(s)|+|y(s)|)+|f(t,0,0)|)ds\right]|\psi(t)-\psi(0)| \\
\leq\ & \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}(L_1(\|x\|_X+\|y\|_Y)+M_1)ds \\
&+\frac{1}{|\Delta_1|}\left[\frac{1}{\Gamma(\alpha)}\int_0^1 \psi'(s)(\psi(1)-\psi(s))^{\alpha-1}(L_1(\|x\|_X+\|y\|_Y)+M_1)ds\right. \\
&\left.+\frac{|\lambda|}{\Gamma(\alpha)}\int_0^\eta \psi'(s)(\psi(\eta)-\psi(s))^{\alpha-1}(L_1(\|x\|_X+\|y\|_Y)+M_1)ds\right](\psi(1)-\psi(0)) \\
\leq\ & \frac{1}{\Gamma(\alpha+1)}\left[(\psi(1)-\psi(0))^\alpha+\frac{(|\lambda|+1)}{|\Delta_1|}(\psi(1)-\psi(0))^{\alpha+1}\right](L_1r+M_1),
\end{aligned}$$

which implies that

$$\|\mathcal{T}_1(x,y)\|_X \leq \frac{1}{\Gamma(\alpha+1)}\left[(\psi(1)-\psi(0))^\alpha+\frac{(|\lambda|+1)}{|\Delta_1|}(\psi(1)-\psi(0))^{\alpha+1}\right](L_1r+M_1). \tag{13}$$

Similarly, we can find that

$$\|\mathcal{T}_2(x,y)\|_Y \leq \frac{1}{\Gamma(\beta+1)}\left[(\psi(1)-\psi(0))^\beta+\frac{(|\mu|+1)}{|\Delta_2|}(\psi(1)-\psi(0))^{\beta+1}\right](L_2r+M_2). \tag{14}$$

Consequently, from (13) and (14), we get

$$\begin{aligned}\|\mathcal{T}(x,y)\|_{X\times Y} &\leq \frac{1}{\Gamma(\alpha+1)}\left[(\psi(1)-\psi(0))^{\alpha}+\frac{(|\lambda|+1)}{|\Delta_1|}(\psi(1)-\psi(0))^{\alpha+1}\right](L_1r+M_1)\\ &\quad +\frac{1}{\Gamma(\beta+1)}\left[(\psi(1)-\psi(0))^{\beta}+\frac{(|\mu|+1)}{|\Delta_2|}(\psi(1)-\psi(0))^{\beta+1}\right](L_2r+M_2)\\ &= (\gamma_1+\gamma_2)+(\gamma_3+\gamma_4)r\\ &\leq r.\end{aligned}$$

Hence, $\mathcal{T}\mathcal{B}_r\subset\mathcal{B}_r$.

Now, for $(x_1,y_1),(x_2,y_2)\in X\times X$ and for any $t\in[0,1]$, we get

$$\begin{aligned} & |\mathcal{T}_1(x_1,y_1)(t)-\mathcal{T}_1(x_2,y_2)(t)|\\ \leq\ & \frac{1}{\Gamma(\alpha)}\int_0^t\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}|f(s,x_1(s),y_1(s))-f(t,x_2(s),y_2(s))|ds\\ & +\frac{1}{|\Delta_1|}\left[\frac{1}{\Gamma(\alpha)}\int_0^1\psi'(s)(\psi(1)-\psi(s))^{\alpha-1}|f(s,x_1(s),y_1(s))-f(t,x_2(s),y_2(s))|ds\right.\\ & \left.+\frac{|\lambda|}{\Gamma(\alpha)}\int_0^\eta\psi'(s)(\psi(\eta)-\psi(s))^{\alpha-1}|f(s,x_1(s),y_1(s))-f(t,x_2(s),y_2(s))|ds\right]|\psi(t)-\psi(0)|\\ \leq\ & \frac{L_1}{\Gamma(\alpha)}\int_0^t\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}(|x_1(s)-x_2(s)|+|y_1(s)-y_2(s)|)ds\\ & +\frac{L_1}{|\Delta_1|}\left[\frac{1}{\Gamma(\alpha)}\int_0^1\psi'(s)(\psi(1)-\psi(s))^{\alpha-1}(|x_1(s)-x_2(s)|+|y_1(s)-y_2(s)|)ds\right.\\ & \left.+\frac{|\lambda|}{\Gamma(\alpha)}\int_0^\eta\psi'(s)(\psi(\eta)-\psi(s))^{\alpha-1}(|x_1(s)-x_2(s)|+|y_1(s)-y_2(s)|)ds\right]|\psi(t)-\psi(0)|\\ \leq\ & \gamma_3(\|x_1-x_2\|+\|y_1-y_2\|),\end{aligned}$$

which implies that

$$\|\mathcal{T}_1(x_1,y_1)-\mathcal{T}_1(x_2,y_2)\|_X\leq\gamma_3(\|x_1-x_2\|+\|y_1-y_2\|). \tag{15}$$

Similarly, we can find that

$$\|\mathcal{T}_2(x_1,y_1)-\mathcal{T}_2(x_2,y_2)\|_X\leq\gamma_4(\|x_1-x_2\|+\|y_1-y_2\|). \tag{16}$$

It follows from (15) and (16) that

$$\|\mathcal{T}(x_1,y_1)-\mathcal{T}(x_2,y_2)\|_{X\times Y}\leq(\gamma_3+\gamma_4)(\|x_1-x_2\|+\|y_1-y_2\|).$$

From the above inequality, we deduce that $\mathcal{T}$ is a contraction in view of the condition (12). Hence, it follows by Banach's fixed point theorem that there exists a unique fixed point for the operator $\mathcal{T}$, which corresponds to the unique solution of the coupled system (1) on $[0,1]$. This completes the proof. ■

3.2 Existence Result via Leray-Schauder Alternative

Theorem 2 *Assume that (A2) holds. If $\Omega^*<1$, then the coupled system (1) has at least one solution on $[0,1]$, where Ω^* is given in (11).*

Proof. First, we show that the operator $\mathcal{T}:X\times Y\to X\times Y$ is completely continuous. By the continuity of functions f and g, the operator $\mathcal{T}$ is continuous. Let $\mathcal{K}\in X\times Y$ be bounded. Then there exist constants

$N_1>0,\ N_2>0$ such that $|f(t,x(t),y(t))|\le N_1$ and $|g(t,x(t),y(t))|\le N_2$. Then for any $(x,y)\in\mathcal{K}$, we get

$$\begin{aligned}|\mathcal{T}_1(x,y)(t)| &\le \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}|f(s,x(s),y(s))|ds\\ &\quad+\frac{1}{|\Delta_1|}\left[\frac{1}{\Gamma(\alpha)}\int_0^1 \psi'(s)(\psi(1)-\psi(s))^{\alpha-1}|f(s,x(s),y(s))|ds\right.\\ &\quad\left.+\frac{|\lambda|}{\Gamma(\alpha)}\int_0^\eta \psi'(s)(\psi(\eta)-\psi(s))^{\alpha-1}|f(s,x(s),y(s))|ds\right]|\psi(t)-\psi(0)|\\ &\le \frac{N_1}{\Gamma(\alpha+1)}\left[(\psi(1)-\psi(0))^{\alpha}+\frac{(|\lambda|+1)}{|\Delta_1|}(\psi(1)-\psi(0))^{\alpha+1}\right],\end{aligned}$$

which implies that

$$\|\mathcal{T}_1(x,y)\|_X\le\frac{N_1}{\Gamma(\alpha+1)}\left[(\psi(1)-\psi(0))^{\alpha}+\frac{(|\lambda|+1)}{|\Delta_1|}(\psi(1)-\psi(0))^{\alpha+1}\right]. \tag{17}$$

Similarly, we get

$$\|\mathcal{T}_2(x,y)\|_Y\le\frac{N_2}{\Gamma(\beta+1)}\left[(\psi(1)-\psi(0))^{\beta}+\frac{(|\mu|+1)}{|\Delta_2|}(\psi(1)-\psi(0))^{\beta+1}\right]. \tag{18}$$

From (17) and (18), it follows that $\mathcal{T}$ is uniformly bounded.

Next, we shall show that the operator $\mathcal{T}$ is equicontinuous. Let $t_1,t_2\in[0,1]$ with $t_1<t_2$. Then we have

$$\begin{aligned}&|\mathcal{T}_1(x,y)(t_2)-\mathcal{T}_1(x,y)(t_1)|\\ \le&\left|\frac{1}{\Gamma(\alpha)}\int_0^{t_2}\psi'(s)\big[(\psi(t_2)-\psi(s))^{\alpha-1}-(\psi(t_1)-\psi(s))^{\alpha-1}\big]f(s,x(s),y(s))ds\right|\\ &+\left|\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}\psi'(s)(\psi(t_1)-\psi(s))^{\alpha-1}f(s,x(s),y(s))ds\right|\\ &+\left|\frac{1}{\Delta_1}\left[\frac{1}{\Gamma(\alpha)}\int_0^1\psi'(s)(\psi(1)-\psi(s))^{\alpha-1}f(s,x(s),y(s))ds\right.\right.\\ &\left.\left.-\frac{\lambda}{\Gamma(\alpha)}\int_0^\eta\psi'(s)(\psi(\eta)-\psi(s))^{\alpha-1}f(s,x(s),y(s))ds\right](\psi(t_2)-\psi(t_1))\right|\\ \le&\frac{N_1}{\Gamma(\alpha+1)}\big[(\psi(t_2)-\psi(0))^{\alpha}-(\psi(t_1)-\psi(0))^{\alpha}\big]\\ &+\frac{N_1}{|\Delta_1|\Gamma(\alpha+1)}\big[(\psi(1)-\psi(0))^{\alpha}+|\lambda|(\psi(\eta)-\psi(0))^{\alpha}\big](\psi(t_2)-\psi(t_1)),\end{aligned}$$

which imply that $\|\mathcal{T}_1(x,y)-\mathcal{T}_1(x,y)\|\to0$ independent of $(x,y)\in\mathcal{K}$ as $t_2-t_1\to0$. Also, we get

$$\begin{aligned}|\mathcal{T}_2(x,y)(t_2)-\mathcal{T}_2(x,y)(t_1)| &\le \frac{N_2}{\Gamma(\beta+1)}\big[(\psi(t_2)-\psi(0))^{\beta}-(\psi(t_1)-\psi(0))^{\beta}\big]\\ &\quad+\frac{N_2}{|\Delta_2|\Gamma(\beta+1)}\big[(\psi(1)-\psi(0))^{\beta}+|\mu|(\psi(\xi)-\psi(0))^{\beta}\big](\psi(t_2)-\psi(t_1)),\end{aligned}$$

which imply that $\|\mathcal{T}_2(x,y)-\mathcal{T}_2(x,y)\|\to0$ independent of $(x,y)\in\mathcal{K}$ as $t_2-t_1\to0$. Therefore, the operator $\mathcal{T}$ is equicontinuous.

Consequently, by the Arzelà-Ascoli theorem, we deduce that the operator $\mathcal{T}$ is completely continuous. Finally, we shall show that the set

$$\mathcal{S} = \{(x,y) \in X \times Y : (x,y) = \nu\mathcal{T}(x,y),\ 0 < \nu < 1\}$$

is bounded. Let $(x,y) \in \mathcal{S}$. Then $(x,y) = \nu\mathcal{T}(x,y)$. For any $t \in [0,1]$, we have

$$x(t) = \nu\mathcal{T}_1(x,y)(t), \quad y(t) = \nu\mathcal{T}_2(x,y)(t).$$

Then we have

$$\begin{aligned} |x(t)| &= |\nu\mathcal{T}_1(x,y)(t)| \leq |\mathcal{T}_1(x,y)(t)| \\ &\leq \frac{k_0 + k_1|x| + k_2|y|}{\Gamma(\alpha+1)}\left[(\psi(1)-\psi(0))^{\alpha} + \frac{(|\lambda|+1)}{|\Delta_1|}(\psi(1)-\psi(0))^{\alpha+1}\right], \end{aligned}$$

and

$$\begin{aligned} |y(t)| &= |\nu\mathcal{T}_2(x,y)(t)| \leq |\mathcal{T}_2(x,y)(t)| \\ &\leq \frac{l_0 + l_1|x| + l_2|y|}{\Gamma(\beta+1)}\left[(\psi(1)-\psi(0))^{\beta} + \frac{(|\mu|+1)}{|\Delta_2|}(\psi(1)-\psi(0))^{\beta+1}\right]. \end{aligned}$$

Hence, we get

$$\|x\|_X \leq \frac{1}{\Gamma(\alpha+1)}\left[(\psi(1)-\psi(0))^{\alpha} + \frac{(|\lambda|+1)}{|\Delta_1|}(\psi(1)-\psi(0))^{\alpha+1}\right](k_0 + k_1\|x\|_X + k_2\|y\|_Y),$$

$$\|y\|_Y \leq \frac{1}{\Gamma(\beta+1)}\left[(\psi(1)-\psi(0))^{\beta} + \frac{(|\mu|+1)}{|\Delta_2|}(\psi(1)-\psi(0))^{\beta+1}\right](l_0 + l_1\|x\|_X + l_2\|y\|_Y),$$

which imply that

$$\|x\|_X + \|y\|_Y \leq \Omega_0 + \max\{\Omega_1,\Omega_2\}\|x+y\|_{X\times Y} = \Omega_0 + \Omega^*\|x+y\|_{X\times Y}),$$

where $\Omega_0, \Omega_1, \Omega_2$ and Ω^* are given in (11). Consequently, we get

$$\|(x,y)\|_{X\times Y} \leq \frac{\Omega_0}{1-\Omega^*}, \tag{19}$$

which proves that the set $\mathcal{S}$ is bounded. Therefore, by Lemma 3, the operator $\mathcal{T}$ has at least one fixed point. Hence, the coupled system (1) has at least one solution on $[0,1]$. The proof is completed. ∎

4 Examples

Example 1 *Consider the following coupled system of ψ-Caputo fractional differential equations:*

$$\begin{cases} {}^{C}D_{0^+}^{\frac{3}{2};\psi}x(t) = \frac{e^{-3t}}{75+t}\left(\sin x(t) + |y(t)|\right) + \frac{e^{-t}}{1+t^2}, \quad t \in [0,1], \\ {}^{C}D_{0^+}^{\frac{4}{3};\psi}y(t) = \frac{1}{2t^2+100}\left(\frac{|x(t)|}{1+|x(t)|} + \sin y(t)\right) + \sin t + 1, \\ x(0) = y(0) = 0, \\ x(1) = x(\frac{1}{2}),\ y(1) = y(\frac{1}{3}). \end{cases} \tag{20}$$

Here, $\alpha = \frac{3}{2}$, $\beta = \frac{4}{3}$, $\eta = \frac{1}{2}$, $\xi = \frac{1}{3}$, $\lambda = \mu = 1$,

$$f(t,x,y) = \frac{e^{-3t}}{75+t}\left(\sin x + |y|\right) + \frac{e^{-t}}{1+t^2}$$

and

$$g(t,x,y)=\frac{1}{2t^2+100}\left(\frac{|x|}{1+|x|}+\sin y\right)+\sin t+1.$$

Obviously, on can find that

$$|f(t,x_1,y_1)-f(t,x_2,y_2)|\leq\frac{1}{75}(|x_1-x_2|+|y_1-y_2|)$$

and

$$|g(t,u_1,v_1)-g(t,u_2,v_2)|\leq\frac{1}{100}(|x_1-x_2|+|y_1-y_2|),$$

from which, we get $L_1=\frac{1}{75}$ *and* $L_2=\frac{1}{100}$. *Let us take* $\psi(t)=3t^2$. *Clearly,* ψ *is an increasing function on* $[0,1]$ *and* $\psi'(t)=6t$ *is a continuous function on* $[0,1]$. *Using the given data, the condition (12) becomes*

$$\gamma_3+\gamma_4=0.1910978713+0.3633970871=0.5544949584<1.$$

Thus, all the assumptions of Theorem 1 are satisfied. Hence it follows that the coupled system (20) has a unique solution on $[0,1]$.

Example 2 *Consider the following coupled system of* ψ*-Caputo fractional differential equations:*

$$\begin{cases} {}^{C}D_{0^+}^{\frac{3}{2};\psi}x(t)=\frac{1}{\sqrt{625+t}}\cos t+\frac{e^{-t}}{200}\sin x(t)+\frac{1}{300}\frac{y(t)|x(t)|}{1+|x(t)|}, \quad t\in[0,1],\\ {}^{C}D_{0^+}^{\frac{4}{3};\psi}y(t)=\frac{e^{-2t}}{2\sqrt{1600+t}}+\frac{1}{270}\sin x(t)+\frac{1}{3(60+t)}\sin(y(t)),\\ x(0)=y(0)=0,\\ x(1)=x(\frac{1}{2}),\ y(1)=y(\frac{1}{3}). \end{cases} \tag{21}$$

Obviously, we get

$$|f(t,x,y)|\leq\frac{1}{25}+\frac{1}{200}\|x\|_X+\frac{1}{300}\|y\|_Y,$$

$$|g(t,x,y)|\leq\frac{1}{80}+\frac{1}{270}\|x\|_X+\frac{1}{180}\|y\|_Y.$$

Thus $k_0=\frac{1}{25},\ k_1=\frac{1}{200},\ k_2=\frac{1}{300},\ l_0=\frac{1}{80},\ l_1=\frac{1}{270},\ l_2=\frac{1}{180}$. *Using (11), we find that*

$$\Omega^*=\max\{\Omega_1,\Omega_2\}=\max\{0.2062532154,0.5020208267\}=0.5020208267<1.$$

Therefore, the assumptions of Theorem 2 are satisfied. Hence, the coupled system (21) has at least one solution on $[0,1]$.

5 Conclusion

In the current work, investigation of four-point boundary value problems for coupled system of differential equations via ψ-Caputo fractional derivatives has been studied by means of Leray-Schauder alternative and Banach's fixed point theorem. Further, two examples are inserted to illustrate the applicability of the theoretical results. For future study, we shall consider coupled system of fractional differential equations with p-Laplacian operaor via generalized proportional fractional derivatives with respect to another function.

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