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Some Refinements Of Numerical Radius Inequalities Via Convex Functions*

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Abstract

The main goal of this paper is to obtain some refinement of numerical radius inequalities involving convex functions.

1 Introduction and Preliminaries

Let $\mathcal{B}(\mathcal{H})$ denote the C^*-algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$ with inner product $\langle \cdot, \cdot \rangle$. For $T \in \mathcal{B}(\mathcal{H})$, let $\omega(T)$ and $\|T\|$ denote the numerical radius and the operator norm of T, respectively. Recall that $\omega(T) = \sup_{\|x\|=1} |\langle Tx, x\rangle|$. It is well-known that $\omega(\cdot)$ defines a norm on $\mathcal{B}(\mathcal{H})$, which is equivalent to the operator norm $\|\cdot\|$. In fact, for every $T \in \mathcal{B}(\mathcal{H})$,

$$\frac{1}{2}\|T\| \leq \omega(T) \leq \|T\|. \tag{1}$$

An important inequality for $\omega(T)$ is the power inequality stating that

$$\omega(T^n) \leq \omega^n(T)$$

for all $n = 1, 2, \ldots$.

In [6], Kittaneh gave the following estimate of the numerical radius which refines the second inequality in (1): For every $T \in \mathcal{B}(\mathcal{H})$,

$$\omega(T) \leq \frac{1}{2}\|T\| + \frac{1}{2}\left\|T^2\right\|^{\frac{1}{2}}. \tag{2}$$

The following estimate of the numerical radius has been given in [1]:

$$\omega(T) \leq \frac{1}{2}\sqrt{\|T^*T + TT^*\| + 2\omega(T^2)}. \tag{3}$$

The inequality (3) also refines the inequality (2). This can be seen by using the fact that

$$\|T^*T + TT^*\| \leq \|T\|^2 + \left\|T^2\right\|. \tag{4}$$

For other properties of the numerical radius and related inequalities, the reader may consult [10, 11, 12, 13, 15]. In this article, we give several refinements of numerical radius inequalities. Our results mainly extend and improve the inequalities in [6, 14].

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2 Main Results

In the sequel the following lemmas will be needed.

Lemma 1 (**[8]**) *Let A be an operator in $\mathcal{B}(\mathcal{H})$ and $x, y \in \mathcal{H}$ be any vectors.*

1. *If $0 \le \alpha \le 1$, then $|\langle Tx, y\rangle|^2 \le \left\langle |T|^{2\alpha} x, x\right\rangle \left\langle |T^*|^{2(1-\alpha)} y, y\right\rangle$.*
2. *If f and g are non-negative continuous functions on $[0, \infty)$ satisfying $f(t)\, g(t) = t$, $(t \ge 0)$, then $|\langle Tx, y\rangle| \le \|f(|T|)\, x\| \, \|g(|T^*|)\, y\|$.*

Lemma 2 (**[9]**) *Let A be a self-adjoint operator in $\mathcal{B}(\mathcal{H})$ with the spectra contained in the interval J, and let h be convex function on J. Then for any unit vector $x \in \mathcal{H}$,*

$$h(\langle Ax, x\rangle) \le \langle h(A)\, x, x\rangle .$$

In [15, Lemma 2.4], the authors present an improvement of the Young inequality as follows:

Lemma 3 *Let $a, b > 0$ and $\min\{a, b\} \le m < M \le \max\{a, b\}$. Then*

$$\sqrt{ab} \le \frac{2\sqrt{Mm}}{M+m} \frac{a+b}{2}. \tag{5}$$

We recall the following refinement of the Cauchy–Schwarz inequality obtained by Dragomir in [4]. It says that

$$|\langle a, b\rangle| \le |\langle a, e\rangle \langle e, b\rangle| + |\langle a, b\rangle - \langle a, e\rangle \langle e, b\rangle| \le \|a\| \, \|b\| , \tag{6}$$

where a, b, e are vectors in $\mathcal{H}$ and $\|e\| = 1$.

From the inequality (6) we deduce that

$$|\langle a, e\rangle \langle e, b\rangle| \le \frac{1}{2} (\|a\| \, \|b\| + |\langle a, b\rangle|) . \tag{7}$$

Taking $e = x$ with $\|x\| = 1$, $a = Ax$ and $b = A^* x$ in the inequality (7) to get

$$|\langle Ax, x\rangle|^2 \le \frac{1}{2} \left(\|Ax\| \, \|A^* x\| + \left|\left\langle A^2 x, x\right\rangle\right|\right) . \tag{8}$$

Taking the supremum over $x \in \mathcal{H}$ with $\|x\| = 1$ in inequality (8) we obtain

$$\omega^2(A) \le \frac{1}{2} \left(\|A\|^2 + \omega\left(A^2\right)\right) . \tag{9}$$

The above inequality can be found in [5]. In addition to this, we have the following related inequality:

Theorem 4 *Let $A \in \mathcal{B}(\mathcal{H})$, and f, g be non-negative continuous functions on $[0, \infty)$ satisfying $f(t)\, g(t) = t$, $(t \ge 0)$, and let and let h be a non-negative increasing convex function on $[0, \infty)$. If*

$$0 < f^2\left(\left|A^2\right|\right) \le m < M \le g^2\left(\left|\left(A^2\right)^*\right|\right),$$

or

$$0 < g^2\left(\left|\left(A^2\right)^*\right|\right) \le m < M \le f^2\left(\left|A^2\right|\right),$$

then

$$h\left(\omega\left(A^2\right)\right) \le \frac{2\sqrt{Mm}}{M+m} \left\| \frac{h\left(f^2\left(\left|A^2\right|\right)\right) + h\left(g^2\left(\left|\left(A^2\right)^*\right|\right)\right)}{2} \right\| . \tag{10}$$

Proof. Let $x \in \mathcal{H}$ be a unit vector. We have

$$\begin{aligned}
& h\left(\left|\left\langle A^2 x, x\right\rangle\right|\right) \\
& \leq h\left(\sqrt{\left\langle f^2\left(\left|A^2\right|\right) x, x\right\rangle\left\langle g^2\left(\left|\left(A^2\right)^*\right|\right) x, x\right\rangle}\right) \\
& \leq h\left(\frac{2\sqrt{Mm}}{M+m}\left(\frac{\left\langle f^2\left(\left|A^2\right|\right) x, x\right\rangle+\left\langle g^2\left(\left|\left(A^2\right)^*\right|\right) x, x\right\rangle}{2}\right)\right) \\
& \leq \frac{2\sqrt{Mm}}{M+m} h\left(\frac{\left\langle f^2\left(\left|A^2\right|\right) x, x\right\rangle+\left\langle g^2\left(\left|\left(A^2\right)^*\right|\right) x, x\right\rangle}{2}\right) \\
& \leq \frac{2\sqrt{Mm}}{M+m}\left(\frac{h\left(\left\langle f^2\left(\left|A^2\right|\right) x, x\right\rangle\right)+h\left(\left\langle g^2\left(\left|\left(A^2\right)^*\right|\right) x, x\right\rangle\right)}{2}\right) \\
& \leq \frac{2\sqrt{Mm}}{M+m}\left(\frac{\left\langle h\left(f^2\left(\left|A^2\right|\right)\right) x, x\right\rangle+\left\langle h\left(g^2\left(\left|\left(A^2\right)^*\right|\right)\right) x, x\right\rangle}{2}\right) \\
& =\frac{2\sqrt{Mm}}{M+m}\left\langle\frac{h\left(f^2\left(\left|A^2\right|\right)\right)+h\left(g^2\left(\left|\left(A^2\right)^*\right|\right)\right)}{2} x, x\right\rangle \\
& \leq \frac{2\sqrt{Mm}}{M+m}\left\|\frac{h\left(f^2\left(\left|A^2\right|\right)\right)+h\left(g^2\left(\left|\left(A^2\right)^*\right|\right)\right)}{2}\right\|,
\end{aligned}$$

where the first inequality follows from Lemma 1 (b), the second inequality obtained from Lemma 3, in the third inequality we used the following simple fact for convex functions: $h(\alpha x) \leq \alpha h(x), (0 \leq \alpha \leq 1)$, the convexity of h implies the fourth inequality, and the fifth inequality follows from Lemma 2. Therefore,

$$h\left(\left|\left\langle A^2 x, x\right\rangle\right|\right) \leq \frac{2\sqrt{Mm}}{M+m}\left\|\frac{h\left(f^2\left(\left|A^2\right|\right)\right)+h\left(g^2\left(\left|\left(A^2\right)^*\right|\right)\right)}{2}\right\|.$$

Taking the supremum over $x \in \mathcal{H}$ with $\|x\|=1$, we deduce the desired result (10). ■

The following result may be stated as well.

Corollary 5 *Let $A \in \mathcal{B}(\mathcal{H})$, and f, g be non-negative continuous functions on $[0, \infty)$ satisfying $f(t) g(t) = t, \ (t \geq 0)$, and let $r \geq 1$. If*

$$0<f^2\left(\left|A^2\right|\right) \leq m<M \leq g^2\left(\left|\left(A^2\right)^*\right|\right),$$

or

$$0<g^2\left(\left|\left(A^2\right)^*\right|\right) \leq m<M \leq f^2\left(\left|A^2\right|\right),$$

then

$$\omega^r\left(A^2\right) \leq \frac{2\sqrt{Mm}}{M+m}\left\|\frac{f^{2r}\left(\left|A^2\right|\right)+g^{2r}\left(\left|\left(A^2\right)^*\right|\right)}{2}\right\|.$$

Remark 6 *Letting $r=1$ in Corollary 5. Therefore, it follows from the inequality (9) that*

$$\omega^2(A) \leq \frac{1}{2}\left(\|A\|^2+\frac{2\sqrt{Mm}}{M+m}\left\|\frac{f^2\left(\left|A^2\right|\right)+g^2\left(\left|\left(A^2\right)^*\right|\right)}{2}\right\|\right).$$

It is worth to mention that the above inequality is sharper than [14, Proposition 2.5].

The following result for several operators holds:

Theorem 7 *Let $A, B, X \in \mathcal{B}(\mathcal{H})$, such that A, B are positive operators, and $0 \leq \alpha \leq 1$, and let h be a non-negative increasing sub-multiplicative convex function on $[0,\infty)$. If*

$$0 < B^{2(1-\alpha)} \leq m < M \leq A^{2\alpha},$$

or

$$0 < A^{2\alpha} \leq m < M \leq B^{2(1-\alpha)},$$

then

$$h\left(\omega\left(A^{\alpha}XB^{1-\alpha}\right)\right) \leq \frac{2\sqrt{Mm}}{M+m} h\left(\|X\|\right) \left\| \frac{h\left(B^{2(1-\alpha)}\right) + h\left(A^{2\alpha}\right)}{2} \right\|. \tag{11}$$

Proof. Let $x \in \mathcal{H}$ be a unit vector. By the Cauchy-Schwarz, we have

$$\begin{aligned}
& h\left(\left|\left\langle A^{\alpha}XB^{1-\alpha}x, x\right\rangle\right|\right) \\
& = h\left(\left|\left\langle XB^{1-\alpha}x, A^{\alpha}x\right\rangle\right|\right) \\
& \leq h\left(\|X\| \left\|B^{1-\alpha}x\right\| \|A^{\alpha}x\|\right) \\
& = h\left(\|X\| \sqrt{\langle B^{1-\alpha}x, B^{1-\alpha}x\rangle \langle A^{\alpha}x, A^{\alpha}x\rangle}\right) \\
& = h\left(\|X\| \sqrt{\left\langle B^{2(1-\alpha)}x, x\right\rangle \left\langle A^{2\alpha}x, x\right\rangle}\right) \\
& \leq h\left(\|X\|\right) h\left(\sqrt{\left\langle B^{2(1-\alpha)}x, x\right\rangle \left\langle A^{2\alpha}x, x\right\rangle}\right) \\
& \leq h\left(\|X\|\right) h\left(\frac{2\sqrt{Mm}}{M+m}\left(\frac{\left\langle B^{2(1-\alpha)}x, x\right\rangle + \left\langle A^{2\alpha}x, x\right\rangle}{2}\right)\right) \\
& \leq \frac{2\sqrt{Mm}}{M+m} h\left(\|X\|\right) h\left(\frac{\left\langle B^{2(1-\alpha)}x, x\right\rangle + \left\langle A^{2\alpha}x, x\right\rangle}{2}\right) \\
& \leq \frac{2\sqrt{Mm}}{M+m} h\left(\|X\|\right) \frac{h\left(\left\langle B^{2(1-\alpha)}x, x\right\rangle\right) + h\left(\left\langle A^{2\alpha}x, x\right\rangle\right)}{2} \\
& \leq \frac{2\sqrt{Mm}}{M+m} h\left(\|X\|\right) \frac{\left\langle h\left(B^{2(1-\alpha)}\right)x, x\right\rangle + \left\langle h\left(A^{2\alpha}\right)x, x\right\rangle}{2} \\
& = \frac{2\sqrt{Mm}}{M+m} h\left(\|X\|\right) \left\langle \left(\frac{h\left(B^{2(1-\alpha)}\right) + h\left(A^{2\alpha}\right)}{2}\right) x, x \right\rangle \\
& \leq \frac{2\sqrt{Mm}}{M+m} h\left(\|X\|\right) \left\| \frac{h\left(B^{2(1-\alpha)}\right) + h\left(A^{2\alpha}\right)}{2} \right\|,
\end{aligned}$$

where the sub-multiplicativity of h obtains the second inequality, the third inequality follows from Lemma 3, in the fourth and fifth inequalities, we used properties of convex functions, and the sixth inequality follows from Lemma 2. Therefore,

$$h\left(\left|\left\langle A^{\alpha}XB^{1-\alpha}x, x\right\rangle\right|\right) \leq \frac{2\sqrt{Mm}}{M+m} h\left(\|X\|\right) \left\| \frac{h\left(B^{2(1-\alpha)}\right) + h\left(A^{2\alpha}\right)}{2} \right\|.$$

Taking the supremum over $x \in \mathcal{H}$ with $\|x\| = 1$, we deduce the desired result (11). ∎

Corollary 8 *Let $A, B, X \in \mathcal{B}(\mathcal{H})$, such that A, B are positive operators, and $0 \leq \alpha \leq 1$, and let $r \geq 1$. If*

$$0 < B^{2(1-\alpha)} \leq m < M \leq A^{2\alpha},$$

or

$$0 < A^{2\alpha} \leq m < M \leq B^{2(1-\alpha)},$$

then

$$\omega^r\left(A^\alpha X B^{1-\alpha}\right) \leq \frac{2\sqrt{Mm}}{M+m}\|X\|^r \left\|\frac{A^{2r\alpha}+B^{2r(1-\alpha)}}{2}\right\|$$

As a consequence of the above, we have:

Corollary 9 *Suppose that the assumptions of Corollary 8 are satisfied. Then*

$$\omega^r\left(A^{\frac{1}{2}} X B^{\frac{1}{2}}\right) \leq \frac{2\sqrt{Mm}}{M+m}\|X\|^r \left\|\frac{A^r+B^r}{2}\right\|. \tag{12}$$

On the other hand, it follows from [14, Theorem 3.3] that

$$\omega^r\left(A^{\frac{1}{2}} X B^{\frac{1}{2}}\right) \leq \|X\|^r \left\|\frac{A^r+B^r}{2}\right\|. \tag{13}$$

Therefore, inequality (12) essentially gives a refinement of the inequality of (13) since $\frac{2\sqrt{Mm}}{M+m} \leq 1$.

We give an example to clarify the situation.

Example 10 *Let $A = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $X = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$ and $r = 2$. Then, we can choose $m = 0.14$ and $M = 6.9$. A simple calculation shows that*

$$\omega^2\left(A^{\frac{1}{2}} X B^{\frac{1}{2}}\right) \approx 3.24,$$

$$\|X\|^2 \left\|\frac{A^2+B^2}{2}\right\| \approx 32.62,$$

and

$$\left(\frac{M+m}{2\sqrt{Mm}}\right)^{-1} \|X\|^2 \left\|\frac{A^2+B^2}{2}\right\| \approx 9.1.$$

The following result is of interest in itself.

Theorem 11 *Let $A \in \mathcal{B}(\mathcal{H})$, and let and let h be a non-negative increasing convex function on $[0, \infty)$.*

$$h\left(\omega^2(A)\right) \leq \frac{1}{4}\left(h\left(\|A^*A + AA^*\|\right) + h\left(\|A^*A - AA^*\|\right)\right) + \frac{1}{2} h\left(\omega\left(A^2\right)\right).$$

In particular, for any $r \geq 1$,

$$w^{2r}(A) \leq \frac{1}{4}\left(\|A^*A + AA^*\|^r + \|A^*A - AA^*\|^r\right) + \frac{1}{2} w^r\left(A^2\right).$$

Proof. The celebrated Boas–Bellman inequality asserts that

$$\sum_{i=1}^n |\langle a, b_i\rangle|^2 \leq \|a\|^2 \left(\max_{1\leq i\leq n} \|b_i\|^2 + \left(\sum_{1\leq i\neq j\leq n} |\langle b_i, b_j\rangle|^2\right)^{\frac{1}{2}}\right)$$

for any $a \in \mathcal{H}$ (see [2, 3]).

Evidently, the case $n = 2$ in the above reduces to

$$|\langle z, x\rangle|^2 + |\langle z, y\rangle|^2 \le \|z\|^2 \left(\max\left(\|x\|^2, \|y\|^2\right) + |\langle x, y\rangle|\right).$$

On choosing $x = Ax$, $y = A^*x$, and $z = x$ with $\|x\| = 1$ we infer that

$$\begin{aligned}
&|\langle x, Ax\rangle|^2 + |\langle x, A^*x\rangle|^2 \\
&\le \max\left(\|Ax\|^2, \|A^*x\|^2\right) + |\langle Ax, A^*x\rangle| \\
&= \frac{1}{2}\left(|\langle A^*A + AA^*x, x\rangle| + |\langle A^*A - AA^*x, x\rangle|\right) + \left|\left\langle A^2x, x\right\rangle\right|,
\end{aligned} \tag{14}$$

thanks to $\max(a, b) = \frac{|a+b|+|a-b|}{2}$.

Applying the arithmetic-geometric mean inequality for the left hand side of the above inequality, we get

$$\begin{aligned}
&|\langle A^*x, x\rangle| \, |\langle Ax, x\rangle| \\
&\le \frac{1}{4}\left(|\langle A^*A + AA^*x, x\rangle| + |\langle A^*A - AA^*x, x\rangle|\right) + \frac{1}{2}\left|\left\langle A^2x, x\right\rangle\right|.
\end{aligned}$$

Whence,

$$\begin{aligned}
&h\left(|\langle A^*x, x\rangle| \, |\langle Ax, x\rangle|\right) \\
&\le h\left(\frac{1}{4}\left(|\langle A^*A + AA^*x, x\rangle| + |\langle A^*A - AA^*x, x\rangle|\right) + \frac{1}{2}\left|\left\langle A^2x, x\right\rangle\right|\right) \\
&= h\left(\frac{\frac{1}{2}\left(|\langle A^*A + AA^*x, x\rangle| + |\langle A^*A - AA^*x, x\rangle|\right) + \left|\left\langle A^2x, x\right\rangle\right|}{2}\right) \\
&\le \frac{1}{2}\left(h\left(\frac{|\langle A^*A + AA^*x, x\rangle| + |\langle A^*A - AA^*x, x\rangle|}{2}\right) + h\left(\left|\left\langle A^2x, x\right\rangle\right|\right)\right) \\
&\le \frac{1}{4}\left(h\left(|\langle A^*A + AA^*x, x\rangle|\right) + h\left(|\langle A^*A - AA^*x, x\rangle|\right)\right) + \frac{1}{2}h\left(\left|\left\langle A^2x, x\right\rangle\right|\right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&h\left(|\langle A^*x, x\rangle| \, |\langle Ax, x\rangle|\right) \\
&\le \frac{1}{4}\left(h\left(|\langle A^*A + AA^*x, x\rangle|\right) + h\left(|\langle A^*A - AA^*x, x\rangle|\right)\right) + \frac{1}{2}h\left(\left|\left\langle A^2x, x\right\rangle\right|\right).
\end{aligned}$$

Now, by taking supremum over $x \in \mathcal{H}$ with $\|x\| = 1$ we reach the desired inequality. ∎

Remark 12 *If we choose A as a normal operator and use the fact that for normal operators we have $\omega(A) = \|A\|$ and $\omega\left(A^2\right) = \left\|A^2\right\| = \|A\|^2$ then we get on both sides of (15) the same quantity $\|A\|^2$. This shows the sharpness of the inequality (15).*

Corollary 13 *Let $A \in \mathcal{B}(\mathcal{H})$ be an invertible operator. Then*

$$\omega(A) \le \sqrt{\frac{1}{2}\|A\|^2 + \frac{3}{4}\|A^2\| - \frac{1}{4}\|A^{-1}\|^{-2}}.$$

Proof. By [7, (34)],

$$\|A^*A - AA^*\| \le \|A\|^2 - \left\|A^{-1}\right\|^{-2}. \tag{15}$$

On the other hand, from Theorem 11, we have

$$\omega^2(A) \le \frac{1}{4}\left(\|A^*A + AA^*\| + \|A^*A - AA^*\|\right) + \frac{1}{2}\omega\left(A^2\right).$$

Hence

$$\begin{aligned}\omega^2(A) &\le \frac{1}{4}\left(\|A^*A+AA^*\|+\|A^*A-AA^*\|\right)+\frac{1}{2}\omega\left(A^2\right)\\ &\le \frac{1}{4}\left(\|A^*A+AA^*\|+\|A\|^2-\left\|A^{-1}\right\|^{-2}\right)+\frac{1}{2}\omega\left(A^2\right) \quad \text{(by (15))}\\ &\le \frac{1}{4}\left(2\|A\|^2+\left\|A^2\right\|-\left\|A^{-1}\right\|^{-2}\right)+\frac{1}{2}\omega\left(A^2\right) \quad \text{(by (4))}\\ &\le \frac{1}{2}\|A\|^2+\frac{3}{4}\left\|A^2\right\|-\frac{1}{4}\left\|A^{-1}\right\|^{-2} \quad \text{(by (1))}\end{aligned}$$

as required. ∎

The following upper bound for the nonnegative difference $\omega^2(A)-\omega\left(A^2\right)$ can be obtained:

Corollary 14 *Let $A\in\mathcal{B}(\mathcal{H})$. Then*

$$w^2(A)-w\left(A^2\right)\le\frac{1}{4}\left(\left\||A|^2+|A^*|^2\right\|+\left\||A|^2-|A^*|^2\right\|\right).$$

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