Applied Mathematics E-Notes, 21(2021), 505-524 © ISSN 1607-2510
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On Nonlinear Third-Order Boundary Value Problems Involving Non-Separated Type Strip-Multi-Point Boundary Conditions*

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Received 29 June 2020

Abstract

We investigate the existence of solutions for nonlinear third order ordinary differential equations and inclusions equipped with non-separated type strip-multi-point boundary conditions on an arbitrary domain. We make use of standard fixed point theorems for single-valued and multi-valued maps to obtain the desired results. Several new results appear as special cases from the present work by fixing the parameters involved in the problems.

1 Introduction

In this paper, we discuss the existence and uniqueness of solutions for a third order ordinary differential equation equipped with non-separated type strip-multi-point boundary conditions given by

$$\begin{cases} u'''(t) = f(t, u(t)), \ a < t < T, \ a, T \in \mathbb{R}, \\ \alpha_1 u(a) + \alpha_2 u(T) = \alpha_3 \int_a^\xi u(s)ds + \sum_{j=1}^m \gamma_j u(\sigma_j), \\ \beta_1 u'(a) + \beta_2 u'(T) = \beta_3 \int_a^\xi u'(s)ds + \sum_{j=1}^m \rho_j u'(\sigma_j), \\ \delta_1 u''(a) + \delta_2 u''(T) = \delta_3 \int_a^\xi u''(s)ds + \sum_{j=1}^m \nu_j u''(\sigma_j), \end{cases} \tag{1}$$

where $f : [a, T] \times \mathbb{R} \to \mathbb{R}$ is a given continuous function, $a < \xi < \sigma_1 < \sigma_2 < \cdots < \sigma_m < T$, and $\alpha_j, \beta_j, \delta_j \in \mathbb{R}$ $(j = 1, 2, 3)$, $\gamma_j, \rho_j, \nu_j \in \mathbb{R}^+$ $(j = 1, 2, \ldots, m)$. Secondly, we extend our discussion to the multi-valued analogue of the problem (1):

$$\begin{cases} u'''(t) \in F(t, u(t)), \ a < t < T, \ a, T \in \mathbb{R}, \\ \alpha_1 u(a) + \alpha_2 u(T) = \alpha_3 \int_a^\xi u(s)ds + \sum_{j=1}^m \gamma_j u(\sigma_j), \\ \beta_1 u'(a) + \beta_2 u'(T) = \beta_3 \int_a^\xi u'(s)ds + \sum_{j=1}^m \rho_j u'(\sigma_j), \\ \delta_1 u''(a) + \delta_2 u''(T) = \delta_3 \int_a^\xi u''(s)ds + \sum_{j=1}^m \nu_j u''(\sigma_j), \end{cases} \tag{2}$$

where $F : [a, T] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map, $\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$.

Nonlinear boundary value problems of differential equations appear extensively in a variety of areas such as underground water flow, blood flow problems, population dynamics, chemical engineering, thermoelasticity, etc. For more details and explanation, see [1, 2]. Widespread applications of boundary value problems motivated many researchers to develop the existence criteria, analytic techniques and numerical methods for solving these problems. During the last few decades, there has been considerable development

*Mathematics Subject Classifications: 34A08, 34B10, 34B15.

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in the study of boundary value problems involving nonlocal and integral boundary conditions. In contrast to the classical boundary data, nonlocal boundary conditions help to take into account certain peculiarities of physical, chemical or other processes happening inside the domain. For the theoretical development of nonlocal boundary value problems, see [3]–[13] and the references cited therein. On the other hand, integral boundary conditions enable to formulate the real world problems involving arbitrary shaped structures, e.g., blood vessels in fluid flow problems. For application details and recent development of the topic, we refer the reader to the works [14]–[23]. On the other hand, inclusions problems play a significant role in the study of dynamical systems and stochastic processes. Examples include control problems [24, 25], granular systems [26], dynamics of wheeled vehicles [27], etc. In the text [28], one can find details of the pressing issues in stochastic processes, queueing networks, optimization and their application in finance, control, climate control, etc.

The objective of the present paper is to enhance the scope of third-order boundary value problems in the context of strip and multi-point boundary data on an arbitrary domain. It is well-known that a significant feature in the study of nonlinear boundary value problems is to examine how the properties of nonlinear function/functions present in a problem influence the nature of its solutions. Keeping this important aspect in mind, we derive a variety of existence results for the problem (1) subject to different kinds of nonlinearities by applying different tools of functional analysis such as Krasnosel'skiĭ fixed point theorem, Leray-Schauder nonlinear alternative for single valued maps and Leray-Schauder degree theory. In order to ensure the uniqueness of solutions for the given problem, we rely on contraction mapping principle. In relation to the problem (2), we have proved two results: the first one deals with convex valued right hand side of the inclusions and is based on nonlinear alternative for Kakutani maps, while the second one is concerned with the nonconvex valued right hand side of the inclusions and relies on Covitz and Nadler fixed point theorem.

We organize the rest of the paper as follows. In Section 2, we prove an auxiliary lemma related to the linear variant of the problem (1). The existence and uniqueness results for the boundary value problem (1), together with illustrative examples, are presented in Section 3. Section 4 deals with the existence of solutions for multi-valued boundary value problem (2) involving convex valued as well as non-convex valued maps. We conclude the paper with Section 5 in which several special cases are presented.

2 Preliminary Result

The following lemma plays a key role in defining the solution for the problems at hand.

Lemma 1 *Let $h \in C([a,T],\mathbb{R})$ and*

$$\left(\delta_1+\delta_2-\delta_3(\xi-a)-\sum_{j=1}^{m}\nu_j\right)\left(\beta_1+\beta_2-\beta_3(\xi-a)-\sum_{j=1}^{m}\rho_j\right)\left(\alpha_1+\alpha_2-\alpha_3(\xi-a)-\sum_{j=1}^{m}\gamma_j\right)\neq 0.$$

Then the following linear problem

$$\begin{cases} u'''(t)=h(t),\ a<t<T,\\ \alpha_1u(a)+\alpha_2u(T)=\alpha_3\int_a^{\xi}u(s)ds+\sum_{j=1}^{m}\gamma_ju(\sigma_j),\\ \beta_1u'(a)+\beta_2u'(T)=\beta_3\int_a^{\xi}u'(s)ds+\sum_{j=1}^{m}\rho_ju'(\sigma_j),\\ \delta_1u''(a)+\delta_2u''(T)=\delta_3\int_a^{\xi}u''(s)ds+\sum_{j=1}^{m}\nu_ju''(\sigma_j),\end{cases} \tag{3}$$

is equivalent to the integral equation

$$\begin{aligned} u(t) &= \int_a^t\frac{(t-s)^2}{2}h(s)ds+\frac{1}{\zeta_4}\Big[-\alpha_2\int_a^T\frac{(T-s)^2}{2}h(s)ds\\ &\quad+\alpha_3\int_a^{\xi}\frac{(\xi-s)^3}{3!}h(s)ds+\sum_{j=1}^{m}\gamma_j\int_a^{\sigma_j}\frac{(\sigma_j-s)^2}{2}h(s)ds\Big]\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{\Delta}P_1(t)\Big[-\beta_2\int_a^T(T-s)h(s)ds+\beta_3\int_a^\xi\frac{(\xi-s)^2}{2}h(s)ds\\
&+\sum_{j=1}^m\rho_j\int_a^{\sigma_j}(\sigma_j-s)h(s)ds\Big]+\frac{1}{\Delta}P_2(t)\Big[-\delta_2\int_a^T h(s)ds\\
&+\delta_3\int_a^\xi(\xi-s)h(s)ds+\sum_{j=1}^m\nu_j\int_a^{\sigma_j}h(s)ds\Big],
\end{aligned}\tag{4}
$$

where

$$
\begin{cases}
\Delta=\zeta_1\zeta_2\zeta_4,\ \ P_1(t)=\zeta_1\zeta_5-\zeta_1\zeta_4(t-a),\\
P_2(t)=\zeta_3\zeta_5-\zeta_2\zeta_6-\zeta_3\zeta_4(t-a)+\zeta_2\zeta_4\frac{(t-a)^2}{2},
\end{cases}\tag{5}
$$

$$
\begin{cases}
\zeta_1=\delta_1+\delta_2-\delta_3(\xi-a)-\sum_{j=1}^m\nu_j\neq 0,\\
\zeta_2=\beta_1+\beta_2-\beta_3(\xi-a)-\sum_{j=1}^m\rho_j\neq 0,\\
\zeta_3=\beta_2(T-a)-\beta_3\frac{(\xi-a)^2}{2}-\sum_{j=1}^m\rho_j(\sigma_j-a),\\
\zeta_4=\alpha_1+\alpha_2-\alpha_3(\xi-a)-\sum_{j=1}^m\gamma_j\neq 0,\\
\zeta_5=\alpha_2(T-a)-\alpha_3\frac{(\xi-a)^2}{2}-\sum_{j=1}^m\gamma_j(\sigma_j-a),\\
\zeta_6=\alpha_2\frac{(T-a)^2}{2}-\alpha_3\frac{(\xi-a)^3}{3!}-\sum_{j=1}^m\gamma_j\frac{(\sigma_j-a)^2}{2}.
\end{cases}\tag{6}
$$

Proof. Integrating $u'''(t)=h(t)$ three times from a to t, we get

$$
u(t)=c_0+c_1(t-a)+c_2\frac{(t-a)^2}{2}+\int_a^t\frac{(t-s)^2}{2}h(s)ds,\tag{7}
$$

where c_0, c_1 and c_2 are arbitrary real constants. Using the third boundary condition of (3) in $u''(t)=c_2+\int_a^t h(s)ds$, gives

$$
c_2=\frac{1}{\zeta_1}\Big[-\delta_2\int_a^T h(s)ds+\delta_3\int_a^\xi(\xi-s)h(s)ds+\sum_{j=1}^m\nu_j\int_a^{\sigma_j}h(s)ds\Big].\tag{8}
$$

Now, by using the second boundary condition of (3) in $u'(t)=c_1+c_2(t-a)+\int_a^t(t-s)h(s)ds$, we obtain

$$
\begin{aligned}
c_1 &= \frac{1}{\zeta_2}\Big[-\beta_2\int_a^T(T-s)h(s)ds+\beta_3\int_a^\xi\frac{(\xi-s)^2}{2}h(s)ds+\sum_{j=1}^m\rho_j\int_a^{\sigma_j}(\sigma_j-s)h(s)ds\Big]\\
&\quad+\frac{\zeta_3}{\zeta_1\zeta_2}\Big[-\delta_2\int_a^T h(s)ds+\delta_3\int_a^\xi(\xi-s)h(s)ds+\sum_{j=1}^m\nu_j\int_a^{\sigma_j}h(s)ds\Big].
\end{aligned}
$$

Finally, by using the first boundary condition of (3) in (7), we find that

$$
\begin{aligned}
c_0 &= \frac{1}{\Delta}\Big\{\Big(\zeta_3\zeta_5-\zeta_2\zeta_6\Big)\Big[-\delta_2\int_a^T h(s)ds+\delta_3\int_a^\xi(\xi-s)h(s)ds+\sum_{j=1}^m\nu_j\int_a^{\sigma_j}h(s)ds\Big]\\
&\quad-\zeta_1\zeta_5\Big[-\beta_2\int_a^T(T-s)h(s)ds+\beta_3\int_a^\xi\frac{(\xi-s)^2}{2}h(s)ds+\sum_{j=1}^m\rho_j\int_a^{\sigma_j}(\sigma_j-s)h(s)ds\Big]\\
&\quad+\zeta_1\zeta_2\Big[-\alpha_2\int_a^T\frac{(T-s)^2}{2}h(s)ds+\alpha_3\int_a^\xi\frac{(\xi-s)^3}{3!}h(s)ds\\
&\quad+\sum_{j=1}^m\gamma_j\int_a^{\sigma_j}\frac{(\sigma_j-s)^2}{2}h(s)ds\Big]\Big\},
\end{aligned}
$$

where Δ and ζ_i $(i=1,\ldots,6)$ are given by (5), (6), respectively. Substituting the values of c_0, c_1 and c_2 into (7), we obtain the solution (4). This completes the proof. ∎

3 Main Results

Let $\mathcal{D} = C([a,T],\mathbb{R})$ denote the Banach space of all continuous functions from $[a,T] \to \mathbb{R}$ endowed with the norm defined by $\|u\| = \sup\{|u(t)|, t \in [a,T]\}$. In view of Lemma 1, we transform problem (1) into an equivalent fixed point problem as

$$u = \mathcal{S}u, \tag{9}$$

where $\mathcal{S} : \mathcal{D} \to \mathcal{D}$ is defined by

$$\begin{aligned}(\mathcal{S}u)(t) \;=\; & \int_a^t \frac{(t-s)^2}{2} f(s,u(s))ds + \frac{1}{\zeta_4}\Big[-\alpha_2 \int_a^T \frac{(T-s)^2}{2} f(s,u(s))ds \\ & +\alpha_3 \int_a^\xi \frac{(\xi-s)^3}{3!} f(s,u(s))ds + \sum_{j=1}^m \gamma_j \int_a^{\sigma_j} \frac{(\sigma_j-s)^2}{2} f(s,u(s))ds\Big] \\ & -\frac{1}{\Delta} P_1(t)\Big[-\beta_2 \int_a^T (T-s) f(s,u(s))ds + \beta_3 \int_a^\xi \frac{(\xi-s)^2}{2} f(s,u(s))ds \\ & +\sum_{j=1}^m \rho_j \int_a^{\sigma_j} (\sigma_j - s) f(s,u(s))ds\Big] + \frac{1}{\Delta} P_2(t)\Big[-\delta_2 \int_a^T f(s,u(s))ds \\ & +\delta_3 \int_a^\xi (\xi-s) f(s,u(s))ds + \sum_{j=1}^m \nu_j \int_a^{\sigma_j} f(s,u(s))ds\Big].\end{aligned} \tag{10}$$

Observe that the problem (1) has solutions if the operator equation (9) has fixed points. For computational convenience, we set

$$\begin{aligned}Q \;=\; & \frac{(T-a)^3}{3!} + \frac{1}{|\zeta_4|}\Big[|\alpha_2|\frac{(T-a)^3}{3!} + |\alpha_3|\frac{(\xi-a)^4}{4!} + \sum_{j=1}^m \gamma_j \frac{(\sigma_j-a)^3}{3!}\Big] \\ & +\frac{\bar{P}_1}{|\Delta|}\Big[|\beta_2|\frac{(T-a)^2}{2} + |\beta_3|\frac{(\xi-a)^3}{3!} + \sum_{j=1}^m \rho_j \frac{(\sigma_j-a)^2}{2}\Big] \\ & +\frac{\bar{P}_2}{|\Delta|}\Big[|\delta_2|(T-a) + |\delta_3|\frac{(\xi-a)^2}{2} + \sum_{j=1}^m \nu_j(\sigma_j-a)\Big],\end{aligned} \tag{11}$$

where $\bar{P}_i = \sup_{t\in[a,T]} \big|P_i(t)\big|,\ i=1,2$. Now we are in a position to present our first existence result which relies on Krasnosel'skiĭ's fixed point theorem [29].

Theorem 1 *Let $f : [a,T] \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying the conditions:*

(H_1) $|f(t,u) - f(t,v)| \le \ell |u-v|,\ \forall t \in [a,T],\ \ell > 0,\ u,v \in \mathbb{R}$;

(H_2) *there exist a function $\chi \in C([a,T],\mathbb{R}^+)$ with $\|\chi\| = \sup_{t\in[a,T]} |\chi(t)|$ such that $|f(t,u)| \le \chi(t),\ \ \forall (t,u) \in [a,T] \times \mathbb{R}$.*

In addition, it is assumed that $\ell Q_1 < 1$, where $Q_1 = Q - \frac{(T-a)^3}{3!}$ (Q is defined by (11)). Then the problem (1) has at least one solution on $[a,T]$.

Proof. Consider $B_r = \{u \in \mathcal{D} : \|u\| \le r\}$, where $r \ge Q\|\chi\|$ and Q is given by (11), and introduce the operators $\mathcal{S}_1$ and $\mathcal{S}_2$ on B_r as

$$(\mathcal{S}_1 u)(t) = \int_a^t \frac{(t-s)^2}{2} f(s,u(s))ds,$$

$$(\mathcal{S}_2 u)(t) \;=\; \frac{1}{\zeta_4}\Big[-\alpha_2 \int_a^T \frac{(T-s)^2}{2} f(s,u(s))ds + \alpha_3 \int_a^\xi \frac{(\xi-s)^3}{3!} f(s,u(s))ds$$

$$+\sum_{j=1}^{m}\gamma_j\int_a^{\sigma_j}\frac{(\sigma_j-s)^2}{2}f(s,u(s))ds\Big]-\frac{1}{\Delta}P_1(t)\Big[-\beta_2\int_a^T(T-s)f(s,u(s))ds$$
$$+\beta_3\int_a^{\xi}\frac{(\xi-s)^2}{2}f(s,u(s))ds+\sum_{j=1}^{m}\rho_j\int_a^{\sigma_j}(\sigma_j-s)f(s,u(s))ds\Big]$$
$$+\frac{1}{\Delta}P_2(t)\Big[-\delta_2\int_a^T f(s,u(s))ds+\delta_3\int_a^{\xi}(\xi-s)f(s,u(s))ds$$
$$+\sum_{j=1}^{m}\nu_j\int_a^{\sigma_j}f(s,u(s))ds\Big].$$

Notice that $(\mathcal{S}u)(t)=(\mathcal{S}_1u)(t)+(\mathcal{S}_2u)(t)$ for all $t\in[a,T]$. For $u,v\in B_r$, we have

$$\begin{aligned}
\|\mathcal{S}_1u+\mathcal{S}_2v\| &= \sup_{t\in[a,T]}\Big\{\Big|\int_a^t\frac{(t-s)^2}{2}f(s,u(s))ds+\frac{1}{\zeta_4}\Big[-\alpha_2\int_a^T\frac{(T-s)^2}{2}f(s,v(s))ds\\
&\quad+\alpha_3\int_a^{\xi}\frac{(\xi-s)^3}{3!}f(s,v(s))ds+\sum_{j=1}^{m}\gamma_j\int_a^{\sigma_j}\frac{(\sigma_j-s)^2}{2}f(s,v(s))ds\Big]\\
&\quad-\frac{1}{\Delta}P_1(t)\Big[-\beta_2\int_a^T(T-s)f(s,v(s))ds+\beta_3\int_a^{\xi}\frac{(\xi-s)^2}{2}f(s,v(s))ds\\
&\quad+\sum_{j=1}^{m}\rho_j\int_a^{\sigma_j}(\sigma_j-s)f(s,v(s))ds\Big]+\frac{1}{\Delta}P_2(t)\Big[-\delta_2\int_a^T f(s,v(s))ds\\
&\quad+\delta_3\int_a^{\xi}(\xi-s)f(s,v(s))ds+\sum_{j=1}^{m}\nu_j\int_a^{\sigma_j}f(s,v(s))ds\Big]\Big|\Big\}\\
&\le \|\chi\|\sup_{t\in[a,T]}\Big\{\frac{(t-a)^3}{3!}+\frac{1}{|\zeta_4|}\Big[|\alpha_2|\frac{(T-a)^3}{3!}+|\alpha_3|\frac{(\xi-a)^4}{4!}\\
&\quad+\sum_{j=1}^{m}\gamma_j\frac{(\sigma_j-a)^3}{3!}\Big]+\frac{1}{|\Delta|}|P_1(t)|\Big[|\beta_2|\frac{(T-a)^2}{2}+|\beta_3|\frac{(\xi-a)^3}{3!}\\
&\quad+\sum_{j=1}^{m}\rho_j\frac{(\sigma_j-a)^2}{2}\Big]+\frac{1}{|\Delta|}|P_2(t)|\Big[|\delta_2|(T-a)+|\delta_3|\frac{(\xi-a)^2}{2}\\
&\quad+\sum_{j=1}^{m}\nu_j(\sigma_j-a)\Big]\Big\}\\
&\le \|\chi\|Q\le r,
\end{aligned}$$

where Q is given by (11). Thus $\mathcal{S}_1u+\mathcal{S}_2v\in B_r$. Using the assumption (H_1) and $Q_1=Q-\dfrac{(T-a)^3}{3!}$, we obtain

$$\begin{aligned}
\|\mathcal{S}_2u-\mathcal{S}_2v\| &\le \sup_{t\in[a,T]}\Big\{\frac{1}{|\zeta_4|}\Big[|\alpha_2|\int_a^T\frac{(T-s)^2}{2}\Big|f(s,u(s))-f(s,v(s))\Big|ds+|\alpha_3|\int_a^{\xi}\frac{(\xi-s)^3}{3!}\\
&\quad\times\Big|f(s,u(s))-f(s,v(s))\Big|ds+\sum_{j=1}^{m}\gamma_j\int_a^{\sigma_j}\frac{(\sigma_j-s)^2}{2}\Big|f(s,u(s))-f(s,v(s))\Big|ds\Big]\\
&\quad+\frac{1}{|\Delta|}|P_1(t)|\Big[|\beta_2|\int_a^T(T-s)\Big|f(s,u(s))-f(s,v(s))\Big|ds+|\beta_3|\int_a^{\xi}\frac{(\xi-s)^2}{2}
\end{aligned}$$

$$
\begin{aligned}
& \times\Big|f(s,u(s))-f(s,v(s))\Big|ds+\sum_{j=1}^{m}\rho_j\int_a^{\sigma_j}(\sigma_j-s)\Big|f(s,u(s))-f(s,v(s))\Big|ds\Big] \\
& +\frac{1}{|\Delta|}|P_2(t)|\Big[|\delta_2|\int_a^T\Big|f(s,u(s))-f(s,v(s))\Big|ds+|\delta_3|\int_a^{\xi}(\xi-s) \\
& \times\Big|f(s,u(s))-f(s,v(s))\Big|ds+\sum_{j=1}^{m}\nu_j\int_a^{\sigma_j}\Big|f(s,u(s))-f(s,v(s))\Big|ds\Big]\Big\} \\
\leq\ & \ell\|u-v\|\Big\{\frac{1}{|\zeta_4|}\Big[|\alpha_2|\frac{(T-a)^3}{3!}+|\alpha_3|\frac{(\xi-a)^4}{4!}+\sum_{j=1}^{m}\gamma_j\frac{(\sigma_j-a)^3}{3!}\Big] \\
& +\frac{|\bar{P}_1|}{|\Delta|}\Big[|\beta_2|\frac{(T-a)^2}{2}+|\beta_3|\frac{(\xi-a)^3}{3!}+\sum_{j=1}^{m}\rho_j\frac{(\sigma_j-a)^2}{2}\Big] \\
& +\frac{|\bar{P}_2|}{|\Delta|}\Big[|\delta_2|(T-a)+|\delta_3|\frac{(\xi-a)^2}{2}+\sum_{j=1}^{m}\nu_j(\sigma_j-a)\Big]\Big\} \\
\leq\ & \ell Q_1\|u-v\|,
\end{aligned}
$$

which implies that $\mathcal{S}_2$ is a contraction in view of the given condition $\ell Q_1 < 1$.

Next we show that $\mathcal{S}_1$ is compact and continuous. Notice that continuity of f implies that the operator $\mathcal{S}_1$ is continuous. Also, $\mathcal{S}_1$ is uniformly bounded on B_r as

$$\|\mathcal{S}_1 u\| \leq \|\chi\|\frac{(T-a)^3}{3!}.$$

Let us fix $\sup_{(t,u)\in[a,T]\times B_r}|f(t,u)| = \bar{f}$, and take $t_1, t_2 \in [a,T]$ with $t_1 < t_2$. Then

$$
\begin{aligned}
|(\mathcal{S}_1u)(t_2)-(\mathcal{S}_1u)(t_1)| &= \Big|\int_a^{t_1}\Big[\frac{(t_2-s)^2}{2}-\frac{(t_1-s)^2}{2}\Big]f(s,u(s))ds \\
&\quad +\int_{t_1}^{t_2}\frac{(t_2-s)^2}{2}f(s,u(s))ds\Big| \\
&\leq \bar{f}\frac{(t_2-t_1)^3}{3}+\frac{1}{3!}\big|(t_2-a)^3-(t_1-a)^3\big| \to 0 \text{ as } t_2\to t_1,
\end{aligned}
$$

independently of $u \in B_r$. This implies that $\mathcal{S}_1$ is relatively compact on B_r. Hence, it follows by the Arzelá-Ascoli theorem that the operator $\mathcal{S}_1$ is compact on B_r. Thus all the assumptions of Krasnosel'skiĭ's fixed point theorem are satisfied. In consequence, we deduce that the problem (1) has at least one solution on $[a,T]$. ■

Now we show the existence of solutions for the problem (1) via Leray-Schauder nonlinear alternative for single valued maps [30].

Theorem 2 *Let $f : [a,T]\times\mathbb{R}\to\mathbb{R}$ be a continuous function. Assume that*

(H_3) *there exist a function $p \in C([a,T],\mathbb{R}^+)$, and a nondecreasing function $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $|f(t,u)| \leq p(t)\Psi(\|u\|),\quad \forall(t,u)\in[a,T]\times\mathbb{R}$;*

(H_4) *there exists a constant $M > 0$ such that*

$$\frac{M}{\|p\|\Psi(M)Q} > 1. \tag{12}$$

Then the boundary value problem (1) has at least one solution on $[a,T]$.

Proof. We complete the proof in several steps. In the first step, we show that the operator $\mathcal{S}:\mathcal{D}\to\mathcal{D}$ defined by (10) maps bounded sets into bounded sets in $\mathcal{D}$. For the positive number $\bar{r}$, let $B_{\bar{r}}=\{u\in\mathcal{D}:\|u\|\le\bar{r}\}$ be a bounded set in $\mathcal{D}$. Then, for any $u\in B_{\bar{r}}$, we have

$$\begin{aligned}
\|(\mathcal{S}u)\| &= \sup_{t\in[a,T]}|(\mathcal{S}u)(t)|=\sup_{t\in[a,T]}\Big\{\Big|\int_a^t\frac{(t-s)^2}{2}f(s,u(s))ds+\frac{1}{\zeta_4}\Big[-\alpha_2\int_a^T\frac{(T-s)^2}{2}f(s,u(s))ds\\
&\quad+\alpha_3\int_a^\xi\frac{(\xi-s)^3}{3!}f(s,u(s))ds+\sum_{j=1}^m\gamma_j\int_a^{\sigma_j}\frac{(\sigma_j-s)^2}{2}f(s,u(s))ds\Big]\\
&\quad-\frac{1}{\Delta}P_1(t)\Big[-\beta_2\int_a^T(T-s)f(s,u(s))ds+\beta_3\int_a^\xi\frac{(\xi-s)^2}{2}f(s,u(s))ds\\
&\quad+\sum_{j=1}^m\rho_j\int_a^{\sigma_j}(\sigma_j-s)f(s,u(s))ds\Big]+\frac{1}{\Delta}P_2(t)\Big[-\delta_2\int_a^T f(s,u(s))ds\\
&\quad+\delta_3\int_a^\xi(\xi-s)f(s,u(s))ds+\sum_{j=1}^m\nu_j\int_a^{\sigma_j}f(s,u(s))ds\Big]\Big|\Big\}\\
&\le \|p\|\Psi(\|u\|)\sup_{t\in[a,T]}\Big\{\frac{(t-a)^3}{3!}+\frac{1}{|\zeta_4|}\Big[|\alpha_2|\frac{(T-a)^3}{3!}+|\alpha_3|\frac{(\xi-a)^4}{4!}\\
&\quad+\sum_{j=1}^m\gamma_j\frac{(\sigma_j-a)^3}{3!}\Big]+\frac{1}{|\Delta|}|P_1(t)|\Big[|\beta_2|\frac{(T-a)^2}{2}+|\beta_3|\frac{(\xi-a)^3}{3!}\\
&\quad+\sum_{j=1}^m\rho_j\frac{(\sigma_j-a)^2}{2}\Big]+\frac{1}{|\Delta|}|P_2(t)|\Big[|\delta_2|(T-a)+|\delta_3|\frac{(\xi-a)^2}{2}+\sum_{j=1}^m\nu_j(\sigma_j-a)\Big]\Big\}\\
&\le \|p\|\Psi(\|u\|)Q\le\|p\|\Psi(\bar{r})Q.
\end{aligned}$$

Next we show that $\mathcal{S}$ maps bounded sets into equicontinuous sets of $\mathcal{D}$. Observe that continuity of $\mathcal{S}$ follows from that of f. Let $t_1,t_2\in[a,T]$ with $t_1<t_2$ and $u\in B_{\bar{r}}$, where $B_{\bar{r}}$ is a bounded set of $\mathcal{D}$. Then we have

$$\begin{aligned}
|(\mathcal{S}u)(t_2)-(\mathcal{S}u)(t_1)| &\le \Big|\int_a^{t_1}\Big[\frac{(t_2-s)^2}{2}-\frac{(t_1-s)^2}{2}\Big]f(s,u(s))ds+\int_{t_1}^{t_2}\frac{(t_2-s)^2}{2}f(s,u(s))ds\Big|\\
&\quad+\Big|\frac{\zeta_1\zeta_4}{\Delta}\Big|(t_2-t_1)\Big[|\beta_2|\int_a^T(T-s)|f(s,u(s))|ds+|\beta_3|\int_a^\xi\frac{(\xi-s)^2}{2}|f(s,u(s))|ds\\
&\quad+\sum_{j=1}^m\rho_j\int_a^{\sigma_j}(\sigma_j-s)|f(s,u(s))|ds\Big]+\frac{1}{|\Delta|}\Big(|\zeta_3\zeta_4|(t_2-t_1)+\frac{|\zeta_2\zeta_4|}{2}(t_2^2-t_1^2)\Big)\\
&\quad\times\Big[|\delta_2|\int_a^T|f(s,u(s))|ds+|\delta_3|\int_a^\xi(\xi-s)|f(s,u(s))|ds+\sum_{j=1}^m\nu_j\int_a^{\sigma_j}|f(s,u(s))|ds\Big]\\
&\le \|p\|\Psi(\bar{r})\Big\{\frac{(t_2-t_1)^3}{3}+\frac{1}{3!}\Big|(t_2-a)^3-(t_1-a)^3\Big|+\Big|\frac{\zeta_1\zeta_4}{\Delta}\Big|(t_2-t_1)\Big[|\beta_2|\frac{(T-a)^2}{2}\\
&\quad+|\beta_3|\frac{(\xi-a)^3}{3!}+\sum_{j=1}^m\rho_j\frac{(\sigma_j-a)^2}{2}\Big]+\frac{1}{|\Delta|}\Big(|\zeta_3\zeta_4|(t_2-t_1)+\frac{|\zeta_2\zeta_4|}{2}(t_2^2-t_1^2)\Big)\\
&\quad\times\Big[|\delta_2|(T-a)+|\delta_3|\frac{(\xi-a)^2}{2}+\sum_{j=1}^m\nu_j(\sigma_j-a)\Big]\Big\}.
\end{aligned}$$

Notice that the right hand side of the above inequality tends to zero independently of $u\in B_{\bar{r}}$ as $(t_2-t_1)\to0$. Since $\mathcal{S}$ satisfies the above assumptions, therefore it follows by the Arzelá-Ascoli theorem that the operator $\mathcal{S}:\mathcal{D}\to\mathcal{D}$ is completely continuous.

The result will follow form the Leray-Schauder nonlinear alternative for single valued maps [30] once the boundendness of the set of all solutions to the equation $u = \lambda \mathcal{S} u$ for $\lambda \in [0,1]$ is established. Let u be a solution. Then, using the computation in the first step, one can find that

$$|u(t)| = |\lambda(\mathcal{S}u)(t)| \leq \|p\|\Psi(\|u\|)Q,$$

which, on taking the norm for $t \in [a,T]$, yields

$$\frac{\|u\|}{\|p\|\Psi(\|u\|)Q} \leq 1.$$

In view of (H_4), there exists M such that $\|u\| \neq M$. Let us set

$$U = \{u \in C([a,T],\mathbb{R}) : \|u\| < M\}.$$

Note that the operator $\mathcal{S} : \overline{U} \to \mathcal{D}$ is continuous and completely continuous. From the choice of U, there is no $u \in \partial U$ such that $u = \lambda \mathcal{S}(u)$ for some $\lambda \in (0,1)$. Consequently, by the nonlinear alternative of Leray-Schauder type [30], we deduce that $\mathcal{S}$ has a fixed point $u \in \overline{U}$ which is a solution of the problem (1). This completes the proof. ■

Our next existence result for the problem (1) is based on Leray-Schauder degree theory.

Theorem 3 *Let $f : [a,T] \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Suppose that*

(H_5) *there exist constants $0 \leq \nu < Q^{-1}$, and $M > 0$ such that*

$$|f(t,u)| \leq \nu|u| + M \quad \text{for all} \quad (t,u) \in [a,T] \times \mathbb{R},$$

where Q is defined by (11).

Then the problem (1) has at least one solution on $[a,T]$.

Proof. We define an operator $\mathcal{S} : \mathcal{D} \to \mathcal{D}$ as in (10). In view of the fixed point problem

$$u = \mathcal{S}u, \tag{13}$$

we will show the existence of at least one solution $u \in \mathcal{D}$ satisfying (13). Set a ball $B_R \subset \mathcal{D}$ as

$$B_R = \{u \in \mathcal{D} : \max_{t \in [a,T]} |u(t)| < R\},$$

with a constant radius $R > 0$. Then we show that the operator $\mathcal{S} : \overline{B}_R \to \mathcal{D}$ satisfies a condition

$$u \neq \theta \mathcal{S} u, \quad \forall u \in \partial B_R, \quad \forall \theta \in [0,1]. \tag{14}$$

Let us set

$$H(\theta, u) = \theta \mathcal{S} u, \quad u \in \mathcal{D}, \quad \theta \in [0,1].$$

As shown in Theorem 2, we have that the operator $\mathcal{S}$ is continuous, uniformly bounded and equicontinuous. Then, by the Arzelá-Ascoli theorem, a continuous map h_θ defined by $h_\theta(u) = u - H(\theta,u) = u - \theta \mathcal{S} u$ is completely continuous. If (14) holds, then the following Leray-Schauder degrees are well defined and it follows by the homotopy invariance of topological degree that

$$\begin{aligned} \deg(h_\theta, B_R, 0) &= \deg(I - \theta \mathcal{S}, B_R, 0) = \deg(h_1, B_R, 0) \\ &= \deg(h_0, B_R, 0) = \deg(I, B_R, 0) = 1 \neq 0, \quad 0 \in B_R, \end{aligned}$$

where I denotes the unit operator. By the nonzero property of Leray-Schauder degree, we have $h_1(u) = u - \mathcal{S}u = 0$ for at least one $u \in B_R$. Let us assume that $u = \theta\mathcal{S}u$ for some $\theta \in [0,1]$ and for all $t \in [a,T]$ so that

$$\begin{aligned}
|u(t)| &= |\theta\mathcal{S}u(t)| \\
&\leq (\nu|u| + M) \sup_{t\in[a,T]} \Big\{ \frac{(t-a)^3}{3!} + \frac{1}{|\zeta_4|}\Big[|\alpha_2|\frac{(T-a)^3}{3!} + |\alpha_3|\frac{(\xi-a)^4}{4!} \\
&\quad + \sum_{j=1}^{m}\gamma_j\frac{(\sigma_j-a)^3}{3!}\Big] + \frac{1}{|\Delta|}|P_1(t)|\Big[|\beta_2|\frac{(T-a)^2}{2} + |\beta_3|\frac{(\xi-a)^3}{3!} \\
&\quad + \sum_{j=1}^{m}\rho_j\frac{(\sigma_j-a)^2}{2}\Big] + \frac{1}{|\Delta|}|P_2(t)|\Big[|\delta_2|(T-a) + |\delta_3|\frac{(\xi-a)^2}{2} + \sum_{j=1}^{m}\nu_j(\sigma_j-a)\Big]\Big\} \\
&\leq (\nu|u| + M)Q,
\end{aligned}$$

which, on taking the norm for $t \in [a,T]$ and solving for $\|u\|$, yields

$$\|u\| \leq \frac{MQ}{1-\nu Q}.$$

If $R = \frac{MQ}{1-\nu Q} + 1$, then the inequality (14) holds. This completes the proof. ■

Finally, we discuss the uniqueness of solutions for the problem (1) via Banach's contraction mapping principle.

Theorem 4 *Assume that $f : [a,T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying the condition (H_1). Then the boundary value problem (1) has a unique solution on $[a,T]$ if $\ell < 1/Q$, where Q is given by (11).*

Proof. Consider a set $B_w = \{u \in \mathcal{D} : \|u\| \leq w\}$, where $w \geq \dfrac{QM}{1-\ell Q}$, $\sup_{t\in[a,T]} |f(t,0)| = M$. In the first step, we show that $\mathcal{S}B_w \subset B_w$, where the operator $\mathcal{S}$ is defined by (10). For any $u \in B_w, t \in [a,T]$, we find that

$$\begin{aligned}
|f(s,u(s))| &= |f(s,u(s)) - f(s,0) + f(s,0)| \leq |f(s,u(s)) - f(s,0)| + |f(s,0)| \\
&\leq \ell\|u\| + M \leq \ell w + M.
\end{aligned}$$

Then, for $u \in B_w$, we obtain

$$\begin{aligned}
\|(\mathcal{S}u)\| &= \sup_{t\in[a,T]}\Big\{\Big|\int_a^t \frac{(t-s)^2}{2}f(s,u(s))ds + \frac{1}{\zeta_4}\Big[-\alpha_2\int_a^T\frac{(T-s)^2}{2}f(s,u(s))ds \\
&\quad +\alpha_3\int_a^\xi\frac{(\xi-s)^3}{3!}f(s,u(s))ds + \sum_{j=1}^{m}\gamma_j\int_a^{\sigma_j}\frac{(\sigma_j-s)^2}{2}f(s,u(s))ds\Big] \\
&\quad -\frac{1}{\Delta}P_1(t)\Big[-\beta_2\int_a^T(T-s)f(s,u(s))ds + \beta_3\int_a^\xi\frac{(\xi-s)^2}{2}f(s,u(s))ds \\
&\quad +\sum_{j=1}^{m}\rho_j\int_a^{\sigma_j}(\sigma_j-s)f(s,u(s))ds\Big] + \frac{1}{\Delta}P_2(t)\Big[-\delta_2\int_a^T f(s,u(s))ds \\
&\quad +\delta_3\int_a^\xi(\xi-s)f(s,u(s))ds + \sum_{j=1}^{m}\nu_j\int_a^{\sigma_j}f(s,u(s))ds\Big]\Big|\Big\} \\
&\leq (\ell w + M)\sup_{t\in[a,T]}\Big\{\frac{(t-a)^3}{3!} + \frac{1}{|\zeta_4|}\Big[|\alpha_2|\frac{(T-a)^3}{3!} + |\alpha_3|\frac{(\xi-a)^4}{4!}
\end{aligned}$$

$$+\sum_{j=1}^{m}\gamma_j\frac{(\sigma_j-a)^3}{3!}\Big]+\frac{1}{|\Delta|}|P_1(t)|\Big[|\beta_2|\frac{(T-a)^2}{2}+|\beta_3|\frac{(\xi-a)^3}{3!}$$
$$+\sum_{j=1}^{m}\rho_j\frac{(\sigma_j-a)^2}{2}\Big]+\frac{1}{|\Delta|}|P_2(t)|\Big[|\delta_2|(T-a)+|\delta_3|\frac{(\xi-a)^2}{2}$$
$$+\sum_{j=1}^{m}\nu_j(\sigma_j-a)\Big]\Big\}\le(\ell w+M)Q\le w,$$

where Q is given by (11). This shows that $\mathcal{S}B_w\subset B_w$.

Now we show that the operator $\mathcal{S}$ is a contraction. For $u,v\in\mathcal{D}$, we have

$$\begin{aligned}
\|\mathcal{S}u-\mathcal{S}v\| &= \sup_{t\in[0,T]}\Big|\mathcal{S}u(t)-\mathcal{S}v(t)\Big| \\
&\le \sup_{t\in[a,T]}\Big\{\int_a^t\frac{(t-s)^2}{2}\Big|f(s,u(s))-f(s,v(s))\Big|ds \\
&\quad +\frac{1}{|\zeta_4|}\Big[|\alpha_2|\int_a^T\frac{(T-s)^2}{2}\Big|f(s,u(s))-f(s,v(s))\Big|ds \\
&\quad +|\alpha_3|\int_a^\xi\frac{(\xi-s)^3}{3!}\Big|f(s,u(s))-f(s,v(s))\Big|ds \\
&\quad +\sum_{j=1}^{m}\gamma_j\int_a^{\sigma_j}\frac{(\sigma_j-s)^2}{2}\Big|f(s,u(s))-f(s,v(s))\Big|ds\Big] \\
&\quad +\frac{|\bar{P}_1|}{|\Delta|}\Big[|\beta_2|\int_a^T(T-s)\Big|f(s,u(s))-f(s,v(s))\Big|ds \\
&\quad +|\beta_3|\int_a^\xi\frac{(\xi-s)^2}{2}\Big|f(s,u(s))-f(s,v(s))\Big|ds \\
&\quad +\sum_{j=1}^{m}\rho_j\int_a^{\sigma_j}(\sigma_j-s)\Big|f(s,u(s))-f(s,v(s))\Big|ds\Big] \\
&\quad +\frac{|\bar{P}_2|}{|\Delta|}\Big[|\delta_2|\int_a^T\Big|f(s,u(s))-f(s,v(s))\Big|ds \\
&\quad +|\delta_3|\int_a^\xi(\xi-s)\Big|f(s,u(s))-f(s,v(s))\Big|ds \\
&\quad +\sum_{j=1}^{m}\nu_j\int_a^{\sigma_j}\Big|f(s,u(s))-f(s,v(s))\Big|ds\Big]\Big\} \\
&\le \ell Q\|u-v\|,
\end{aligned}$$

where we have used (11). By the given assumption: $\ell<1/Q$, it follows that the operator $\mathcal{S}$ is a contraction. Thus, by Banach's contraction mapping principle, we deduce that the operator $\mathcal{S}$ has a fixed point, which corresponds to a unique solution of the problem (1) on $[a,T]$. The proof is completed. ■

Example 1 *Consider the following non-separated multi-point boundary value problem with bounded nonlin-*

earity:

$$\begin{cases} u'''(t) = \frac{t}{15\sqrt{t^2+80}} \tan^{-1} u(t) + e^{-t}, \ t \in [1,3], \\ \alpha_1 u(a) + \alpha_2 u(T) = \alpha_3 \int_a^\xi u(s)ds + \sum_{j=1}^4 \gamma_j u(\sigma_j), \\ \beta_1 u'(a) + \beta_2 u'(T) = \beta_3 \int_a^\xi u'(s)ds + \sum_{j=1}^4 \rho_j u'(\sigma_j), \\ \delta_1 u''(a) + \delta_2 u''(T) = \delta_3 \int_a^\xi u''(s)ds + \sum_{j=1}^4 \nu_j u''(\sigma_j), \end{cases} \tag{15}$$

where $a = 1,\ T = 3,\ m = 4,\ \alpha_1 = 1/4,\ \alpha_2 = 1/2,\ \alpha_3 = 1,\ \beta_1 = 1/5,\ \beta_2 = 3/8,\ \beta_3 = 1,\ \delta_1 = 1/3,\ \delta_2 = 2/3,\ \delta_3 = 1,\ \gamma_1 = 1/2,\ \gamma_2 = 7/10,\ \gamma_3 = 9/10,\ \gamma_4 = 11/10,\ \rho_1 = 1/4,\ \rho_2 = 5/12,\ \rho_3 = 7/12,\ \rho_4 = 9/12,\ \nu_1 = 2/5,\ \nu_2 = 13/20,\ \nu_3 = 9/10,\ \nu_4 = 23/20,\ \xi = 3/2,\ \sigma_1 = 7/4, \sigma_2 = 15/8,\ \sigma_3 = 16/8,\ \sigma_4 = 17/8.$ *Clearly,*

$$|f(t,u)| \leq \frac{\pi|t|}{30\sqrt{t^2+80}} + |e^{-t}|, |f(t,u) - f(t,v)| \leq \ell|u-v|,$$

with $\ell = 1/(5\sqrt{89})$. *Using the given values, we find that* $|\zeta_1| = 2.55 \neq 0,\ |\zeta_2| = 1.93 \neq 0,\ |\zeta_4| = 2.95 \neq 0,\ |\zeta_3| = 3.904167,\ |\zeta_5| = 5.2,\ |\zeta_6| = 2.437825$ *and* $|\Delta| = 28.961625$, *(*ζ_i $(i = 1, \ldots, 6)$ *and* Δ *are given by (6), (5) respectively),* $Q = 23.41751009,\ Q_1 = 22.084177$, *(*$Q$ *is defined by (11) and* $Q_1 = Q - \frac{(T-a)^3}{3!}$*). Furthermore, we note that all the conditions of Theorem 1 are satisfied with* $\ell Q_1 \approx 0.468184 < 1$. *Hence the conclusion of Theorem 1 applies to the problem (15). We also observe that all the conditions of Theorem 4 hold true with* $\ell Q \approx 0.496450 < 1$. *Hence we deduce by the conclusion of Theorem 4 that there exists a unique solution for problem (15) on* $[1,3]$.

Example 2 *Consider the third-order ordinary differential equation with periodic nonlinearity:*

$$u'''(t) = \frac{1}{7\sqrt{t+24}}\Big[\sin u + \frac{3}{4}\Big], \quad t \in [1,3],$$

supplemented with the boundary conditions of the problem (15). Evidently, $|f(t,u)| \leq \frac{1}{7\sqrt{t+24}}\Big[|u| + \frac{3}{4}\Big]$, $\Psi(\|u\|) = \|u\| + \frac{3}{4}, p(t) = \frac{1}{7\sqrt{t+24}}$ *and* $\|p\| = \frac{1}{35}$. *We find by (12) that* $M > 1.516353$. *In consequence, it follows by the conclusion of Theorem 2 that the problem (15) has at least one solution on* $[1,3]$.

4 The Multi-Valued Case

In this section, we prove the existence of solutions for the multi-valued boundary value problem (2) when the multivalued map involved in the problem has convex as well nonconvex values. For the related concepts of multi-valued analysis, we refer the reader to the books [31, 32].

Definition 1 *A function* $u \in C^3([a,T], \mathbb{R})$ *is a solution of the problem (2) if*

$$\alpha_1 u(a) + \alpha_2 u(T) = \alpha_3 \int_a^\xi u(s)ds + \sum_{j=1}^m \gamma_j u(\sigma_j),$$

$$\beta_1 u'(a) + \beta_2 u'(T) = \beta_3 \int_a^\xi u'(s)ds + \sum_{j=1}^m \rho_j u'(\sigma_j),$$

$$\delta_1 u''(a) + \delta_2 u''(T) = \delta_3 \int_a^\xi u''(s)ds + \sum_{j=1}^m \nu_j u''(\sigma_j),$$

and there exists function $v \in L^1([a,T], \mathbb{R})$ *such that* $v(t) \in F(t, u(t))$ *a.e.* $t \in [a,T]$ *and*

$$u(t) \quad = \quad \int_a^t \frac{(t-s)^2}{2} v(s)ds + \frac{1}{\zeta_4}\Big[- \alpha_2 \int_a^T \frac{(T-s)^2}{2} v(s)ds$$

$$
\begin{aligned}
&+\alpha_3\int_a^\xi \frac{(\xi-s)^3}{3!}v(s)ds+\sum_{j=1}^m\gamma_j\int_a^{\sigma_j}\frac{(\sigma_j-s)^2}{2}v(s)ds\Big]\\
&-\frac{1}{\Delta}P_1(t)\Big[-\beta_2\int_a^T(T-s)v(s)ds+\beta_3\int_a^\xi\frac{(\xi-s)^2}{2}v(s)ds\\
&+\sum_{j=1}^m\rho_j\int_a^{\sigma_j}(\sigma_j-s)v(s)ds\Big]+\frac{1}{\Delta}P_2(t)\Big[-\delta_2\int_a^T v(s)ds\\
&+\delta_3\int_a^\xi(\xi-s)v(s)ds+\sum_{j=1}^m\nu_j\int_a^{\sigma_j}v(s)ds\Big]. \qquad (16)
\end{aligned}
$$

4.1 The Carathéodory Case

Here we prove an existence result for the problem (2) by applying nonlinear alternative for Kakutani maps [30] when F has convex values and is of Carathéodory type.

Theorem 5 *Assume that $F:[a,T]\times\mathbb{R}\to\mathcal{P}_{cp,c}(\mathbb{R})$ is L^1-Carathéodory satisfying the assumption (H_4), where $\mathcal{P}_{cp,c}(\mathbb{R})=\{Y\in\mathcal{P}(\mathbb{R}):Y$ is compact and convex$\}$. In addition we suppose that:*

(B_1) there exist a function $p\in C([a,T],\mathbb{R}^+)$, and a nondecreasing function $\Psi:\mathbb{R}^+\to\mathbb{R}^+$ such that

$$\|F(t,u)\|=\sup\{|v|:v\in F(t,u)\}\le\|p\|\Psi(\|u\|),\ (t,u)\in[a,T]\times\mathbb{R}.$$

Then the boundary value problem (2) has at least one solution on $[a,T]$.

Proof. To transform the problem (2) into a fixed point problem, we define an operator $\mathcal{F}:\mathcal{D}\longrightarrow\mathcal{P}(\mathcal{D})$ by

$$
\mathcal{F}(u)=\left\{\begin{array}{l}
h\in\mathcal{D}:\\
h(t)=\left\{\begin{array}{l}
\int_a^t\frac{(t-s)^2}{2}v(s)ds+\frac{1}{\zeta_4}\Big[-\alpha_2\int_a^T\frac{(T-s)^2}{2}v(s)ds\\
+\alpha_3\int_a^\xi\frac{(\xi-s)^3}{3!}v(s)ds+\sum_{j=1}^m\gamma_j\int_a^{\sigma_j}\frac{(\sigma_j-s)^2}{2}v(s)ds\Big]\\
-\frac{1}{\Delta}P_1(t)\Big[-\beta_2\int_a^T(T-s)v(s)ds+\beta_3\int_a^\xi\frac{(\xi-s)^2}{2}v(s)ds\\
+\sum_{j=1}^m\rho_j\int_a^{\sigma_j}(\sigma_j-s)v(s)ds\Big]+\frac{1}{\Delta}P_2(t)\Big[-\delta_2\int_a^T v(s)ds\\
+\delta_3\int_a^\xi(\xi-s)v(s)ds+\sum_{j=1}^m\nu_j\int_a^{\sigma_j}v(s)ds\Big],
\end{array}\right.
\end{array}\right\}
$$

for $v\in S_{F,u}$. It is obvious that the fixed points of $\mathcal{F}$ are solutions of the boundary value problem (2). We will show that $\mathcal{F}$ satisfies the assumptions of Leray-Schauder nonlinear alternative [30]. For that, we split the proof into several steps.

Step 1. $\mathcal{F}(u)$ is convex for each $u\in\mathcal{D}$.

This step is obvious since $S_{F,u}$ is convex (F has convex values), and therefore we omit the proof.

Step 2. $\mathcal{F}$ maps bounded sets (balls) into bounded sets in $\mathcal{D}$.

For the positive number r, let $B_r=\{u\in\mathcal{D}:\|u\|\le r\}$ be a bounded set in $\mathcal{D}$. Then, for each $h\in\mathcal{F}(u),u\in B_r$, there exists $v\in S_{F,u}$ such that

$$
\begin{aligned}
h(t) &= \int_a^t\frac{(t-s)^2}{2}v(s)ds+\frac{1}{\zeta_4}\Big[-\alpha_2\int_a^T\frac{(T-s)^2}{2}v(s)ds\\
&+\alpha_3\int_a^\xi\frac{(\xi-s)^3}{3!}v(s)ds+\sum_{j=1}^m\gamma_j\int_a^{\sigma_j}\frac{(\sigma_j-s)^2}{2}v(s)ds\Big]\\
&-\frac{1}{\Delta}P_1(t)\Big[-\beta_2\int_a^T(T-s)v(s)ds+\beta_3\int_a^\xi\frac{(\xi-s)^2}{2}v(s)ds
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{j=1}^{m}\rho_j\int_a^{\sigma_j}(\sigma_j-s)v(s)ds\Big]+\frac{1}{\Delta}P_2(t)\Big[-\delta_2\int_a^T v(s)ds\\
&+\delta_3\int_a^{\xi}(\xi-s)v(s)ds+\sum_{j=1}^{m}\nu_j\int_a^{\sigma_j}v(s)ds\Big].
\end{aligned}
$$

Then, as in Theorem 2, one can find that $\|h\|\leq\|p\|\Psi(r)Q$.

Step 3. $\mathcal{F}$ maps bounded sets into equicontinuous sets of $\mathcal{D}$.

Let $t_1,t_2\in[a,T]$ with $t_1<t_2$ and $u\in B_r$. Then, for each $h\in\mathcal{F}(u)$, we obtain

$$
\begin{aligned}
&|h(t_2)-h(t_1)|\\
\leq\;&\|p\|\Psi(r)\Big\{\frac{(t_2-t_1)^3}{3}+\frac{1}{3!}\Big|(t_2-a)^3-(t_1-a)^3\Big|+\Big|\frac{\zeta_1\zeta_4}{\Delta}\Big|(t_2-t_1)\Big[|\beta_2|\frac{(T-a)^2}{2}\\
&+|\beta_3|\frac{(\xi-a)^3}{3!}+\sum_{j=1}^{m}\rho_j\frac{(\sigma_j-a)^2}{2}\Big]+\frac{1}{|\Delta|}\Big(|\zeta_3\zeta_4|(t_2-t_1)+\frac{|\zeta_2\zeta_4|}{2}(t_2^2-t_1^2)\Big)\\
&\times\Big[|\delta_2|(T-a)+|\delta_3|\frac{(\xi-a)^2}{2}+\sum_{j=1}^{m}\nu_j(\sigma_j-a)\Big]\Big\}\to 0\;\text{ as }\;t_2-t_1\to 0,
\end{aligned}
$$

independently of $u\in B_r$. Therefore, the Ascoli-Arzelá theorem applies and that the operator $\mathcal{F}:\mathcal{D}\to\mathcal{P}(\mathcal{D})$ is completely continuous. Since $\mathcal{F}$ is completely continuous, we have to show that it is upper semi-continuous (u.s.c.). In order to do so, we establish that $\mathcal{F}$ has a closed graph ([31, Proposition 1.2]) in the following step.

Step 4. $\mathcal{F}$ has a closed graph.

Let $u_n\to u_*, h_n\in\mathcal{F}(u_n)$ and $h_n\to h_*$. Then we need to show that $h_*\in\mathcal{F}(u_*)$. Associated with $h_n\in\mathcal{F}(u_n)$, there exists $v_n\in S_{F,u_n}$ such that, for each $t\in[a,T]$,

$$
\begin{aligned}
h_n(t)\;=\;&\int_a^t\frac{(t-s)^2}{2}v_n(s)ds+\frac{1}{\zeta_4}\Big[-\alpha_2\int_a^T\frac{(T-s)^2}{2}v_n(s)ds\\
&+\alpha_3\int_a^{\xi}\frac{(\xi-s)^3}{3!}v_n(s)ds+\sum_{j=1}^{m}\gamma_j\int_a^{\sigma_j}\frac{(\sigma_j-s)^2}{2}v_n(s)ds\Big]\\
&-\frac{1}{\Delta}P_1(t)\Big[-\beta_2\int_a^T(T-s)v_n(s)ds+\beta_3\int_a^{\xi}\frac{(\xi-s)^2}{2}v_n(s)ds\\
&+\sum_{j=1}^{m}\rho_j\int_a^{\sigma_j}(\sigma_j-s)v_n(s)ds\Big]+\frac{1}{\Delta}P_2(t)\Big[-\delta_2\int_a^T v_n(s)ds\\
&+\delta_3\int_a^{\xi}(\xi-s)v_n(s)ds+\sum_{j=1}^{m}\nu_j\int_a^{\sigma_j}v_n(s)ds\Big].
\end{aligned}
$$

Thus it suffices to show that there exists $v_*\in S_{F,u_*}$ such that, for each $t\in[a,T]$,

$$
\begin{aligned}
h_*(t)\;=\;&\int_a^t\frac{(t-s)^2}{2}v_*(s)ds+\frac{1}{\zeta_4}\Big[-\alpha_2\int_a^T\frac{(T-s)^2}{2}v_*(s)ds\\
&+\alpha_3\int_a^{\xi}\frac{(\xi-s)^3}{3!}v_*(s)ds+\sum_{j=1}^{m}\gamma_j\int_a^{\sigma_j}\frac{(\sigma_j-s)^2}{2}v_*(s)ds\Big]\\
&-\frac{1}{\Delta}P_1(t)\Big[-\beta_2\int_a^T(T-s)v_*(s)ds+\beta_3\int_a^{\xi}\frac{(\xi-s)^2}{2}v_*(s)ds
\end{aligned}
$$

$$+\sum_{j=1}^{m}\rho_j\int_a^{\sigma_j}(\sigma_j-s)v_*(s)ds\Big]+\frac{1}{\Delta}P_2(t)\Big[-\delta_2\int_a^T v_*(s)ds$$
$$+\delta_3\int_a^{\xi}(\xi-s)v_*(s)ds+\sum_{j=1}^{m}\nu_j\int_a^{\sigma_j}v_*(s)ds\Big].$$

Let us consider the linear operator $\Theta : L^1([a,T],\mathbb{R}) \to \mathcal{D}$ given by

$$\begin{aligned}
v \mapsto \Theta(v)(t) &= \int_a^t \frac{(t-s)^2}{2}v(s)ds+\frac{1}{\zeta_4}\Big[-\alpha_2\int_a^T\frac{(T-s)^2}{2}v(s)ds\\
&+\alpha_3\int_a^{\xi}\frac{(\xi-s)^3}{3!}v(s)ds+\sum_{j=1}^{m}\gamma_j\int_a^{\sigma_j}\frac{(\sigma_j-s)^2}{2}v(s)ds\Big]\\
&-\frac{1}{\Delta}P_1(t)\Big[-\beta_2\int_a^T(T-s)v(s)ds+\beta_3\int_a^{\xi}\frac{(\xi-s)^2}{2}v(s)ds\\
&+\sum_{j=1}^{m}\rho_j\int_a^{\sigma_j}(\sigma_j-s)v(s)ds\Big]+\frac{1}{\Delta}P_2(t)\Big[-\delta_2\int_a^T v(s)ds\\
&+\delta_3\int_a^{\xi}(\xi-s)v(s)ds+\sum_{j=1}^{m}\nu_j\int_a^{\sigma_j}v(s)ds\Big].
\end{aligned}$$

Observe that $\|h_n(t)-h_*(t)\| \to 0$ as $n \to \infty$, and thus, it follows by closed graph result [33] that $\Theta \circ S_F$ is a closed graph operator. Further, we have $h_n(t) \in \Theta(S_{F,u_n})$. Since $u_n \to u_*$, we have

$$\begin{aligned}
h_*(t) &= \int_a^t \frac{(t-s)^2}{2}v_*(s)ds+\frac{1}{\zeta_4}\Big[-\alpha_2\int_a^T\frac{(T-s)^2}{2}v_*(s)ds\\
&+\alpha_3\int_a^{\xi}\frac{(\xi-s)^3}{3!}v_*(s)ds+\sum_{j=1}^{m}\gamma_j\int_a^{\sigma_j}\frac{(\sigma_j-s)^2}{2}v_*(s)ds\Big]\\
&-\frac{1}{\Delta}P_1(t)\Big[-\beta_2\int_a^T(T-s)v_*(s)ds+\beta_3\int_a^{\xi}\frac{(\xi-s)^2}{2}v_*(s)ds\\
&+\sum_{j=1}^{m}\rho_j\int_a^{\sigma_j}(\sigma_j-s)v_*(s)ds\Big]+\frac{1}{\Delta}P_2(t)\Big[-\delta_2\int_a^T v_*(s)ds\\
&+\delta_3\int_a^{\xi}(\xi-s)v_*(s)ds+\sum_{j=1}^{m}\nu_j\int_a^{\sigma_j}v_*(s)ds\Big],
\end{aligned}$$

for some $v_* \in S_{F,u_*}$.

Step 5. We show there exists an open set $U \subseteq \mathcal{D}$ with $u \notin \lambda\mathcal{F}(u)$ for any $\lambda \in (0,1)$ and all $u \in \partial U$.

Let $\lambda \in (0,1)$ and $u \in \lambda\mathcal{F}(u)$. Then there exists $v \in L^1([0,1],\mathbb{R})$ with $v \in S_{F,u}$ such that, for $t \in [a,T]$, we have

$$\begin{aligned}
u(t) &= \lambda\int_a^t \frac{(t-s)^2}{2}v(s)ds+\lambda\frac{1}{\zeta_4}\Big[-\alpha_2\int_a^T\frac{(T-s)^2}{2}v(s)ds\\
&+\alpha_3\int_a^{\xi}\frac{(\xi-s)^3}{3!}v(s)ds+\sum_{j=1}^{m}\gamma_j\int_a^{\sigma_j}\frac{(\sigma_j-s)^2}{2}v(s)ds\Big]\\
&-\lambda\frac{1}{\Delta}P_1(t)\Big[-\beta_2\int_a^T(T-s)v(s)ds+\beta_3\int_a^{\xi}\frac{(\xi-s)^2}{2}v(s)ds
\end{aligned}$$

$$+\sum_{j=1}^{m}\rho_j\int_a^{\sigma_j}(\sigma_j-s)v(s)ds\Big]+\lambda\frac{1}{\Delta}P_2(t)\Big[-\delta_2\int_a^T v(s)ds$$
$$+\delta_3\int_a^{\xi}(\xi-s)v(s)ds+\sum_{j=1}^{m}\nu_j\int_a^{\sigma_j}v(s)ds\Big].$$

Then, for $t\in[a,T]$, using the computation in the first step leads to

$$\|u\|\leq\|p\|\Psi(\|u\|)Q,$$

which can alternatively be expressed as

$$\frac{\|u\|}{\|p\|\Psi(\|u\|)Q}\leq 1.$$

By the condition (H_4), we can find a positive number M such that $\|u\|\neq M$. Let us set

$$U=\{u\in\mathcal{D}:\|u\|<M\}.$$

Note that the operator $\mathcal{F}:\overline{U}\to\mathcal{P}(\mathcal{D})$ is a compact multi-valued map, u.s.c. with convex closed values. From the choice of U, there is no $u\in\partial U$ such that $u\in\lambda\mathcal{F}(u)$ for some $\lambda\in(0,1)$. Consequently, by the nonlinear alternative of Leray-Schauder type [30], we deduce that $\mathcal{F}$ has a fixed point $u\in\overline{U}$ which is a solution of the problem (2). This completes the proof. ■

4.2 The Lipschitz Case

In this subsection, we prove the existence of solutions for the problem (2) for nonconvex valued right hand side by applying a fixed point theorem for multivalued maps due to Covitz and Nadler [34]. Let (X,d) be a metric space induced from the normed space $(X;\|\cdot\|)$. Consider $H_d:\mathcal{P}(X)\times\mathcal{P}(X)\to\mathbb{R}\cup\{\infty\}$ given by

$$H_d(A,B)=\max\{\sup_{a\in A}d(a,B),\sup_{b\in B}d(A,b)\},$$

where $d(A,b)=\inf_{a\in A}d(a;b)$ and $d(a,B)=\inf_{b\in B}d(a;b)$. Then $(\mathcal{P}_{cl,b}(X),H_d)$ is a metric space (see [28]).

Definition 2 *A multivalued operator $N:X\to\mathcal{P}_{cl}(X)$ is called*

(a) *γ-Lipschitz if and only if there exists $\gamma>0$ such that*

$$H_d(N(x),N(y))\leq\gamma d(x,y)\ \textit{for each}\ x,y\in X;$$

(b) *a contraction if and only if it is γ-Lipschitz with $\gamma<1$.*

Lemma 2 ([34]) *Let (X,d) be a complete metric space. If $N:X\to\mathcal{P}_{cl}(X)$ is a contraction, then $FixN\neq\emptyset$.*

Theorem 6 *Assume that:*

(A_1) *$F:[a,T]\times\mathbb{R}\to\mathcal{P}_{cp}(\mathbb{R})$ is such that $F(\cdot,u):[a,T]\to\mathcal{P}_{cp}(\mathbb{R})$ is measurable for each $u\in\mathbb{R}$, where $\mathcal{P}_{cp}(\mathbb{R})=\{Y\in\mathcal{P}(\mathbb{R}):Y\ \text{is compact}\}$;*

(A_2) *for almost all $t\in[a,T]$ and $u_1,w_1\in\mathbb{R}$, $H_d(F(t,u_1),F(t,w_1))\leq m(t)|u_1-w_1|$ with $m\in C(J,\mathbb{R}^+)$ and $d(0,F(t,0))\leq m(t)$, for almost all $t\in[a,T]$.*

Then the boundary value problem (2) has at least one solution on $[a,T]$ if $\|m\|Q<1$ where Q is defined by (11).

Proof. Consider the operator $\mathcal{F}$ defined at the beginning of the proof of Theorem 5. Observe that the set $S_{F,u}$ is nonempty for each $u \in \mathcal{D}$ by the assumption (A_1), so F has a measurable selection (see Theorem III.6 [35]). Now we show that the operator $\mathcal{F}$ satisfies the assumptions of Lemma 2. We show that $\mathcal{F}(u) \in \mathcal{P}_{cl}(\mathcal{D})$ for each $u \in \mathcal{D}$. Let $\{u_n\}_{n\geq 0} \in \mathcal{F}(u)$ be such that $u_n \to u$ $(n \to \infty)$ in $\mathcal{D}$. Then $u \in \mathcal{D}$ and there exists $v_n \in S_{F,u_n}$ such that, for each $t \in [a,T]$,

$$\begin{aligned} u_n(t) &= \int_a^t \frac{(t-s)^2}{2} v_n(s)ds + \frac{1}{\zeta_4}\Big[-\alpha_2 \int_a^T \frac{(T-s)^2}{2} v_n(s)ds \\ &+\alpha_3 \int_a^\xi \frac{(\xi-s)^3}{3!} v_n(s)ds + \sum_{j=1}^m \gamma_j \int_a^{\sigma_j} \frac{(\sigma_j-s)^2}{2} v_n(s)ds\Big] \\ &-\frac{1}{\Delta} P_1(t)\Big[-\beta_2 \int_a^T (T-s)v_n(s)ds + \beta_3 \int_a^\xi \frac{(\xi-s)^2}{2} v_n(s)ds \\ &+\sum_{j=1}^m \rho_j \int_a^{\sigma_j} (\sigma_j - s)v_n(s)ds\Big] + \frac{1}{\Delta} P_2(t)\Big[-\delta_2 \int_a^T v_n(s)ds \\ &+\delta_3 \int_a^\xi (\xi-s)v_n(s)ds + \sum_{j=1}^m \nu_j \int_a^{\sigma_j} v_n(s)ds\Big]. \end{aligned}$$

As F has compact values, we pass onto a subsequence (if necessary) to obtain that v_n converges to v in $L^1([a,T],\mathbb{R})$. Thus, $v \in S_{F,u}$ and for each $t \in [a,T]$, we have

$$\begin{aligned} u_n(t) \to u(t) &= \int_a^t \frac{(t-s)^2}{2} v(s)ds + \frac{1}{\zeta_4}\Big[-\alpha_2 \int_a^T \frac{(T-s)^2}{2} v(s)ds \\ &+\alpha_3 \int_a^\xi \frac{(\xi-s)^3}{3!} v(s)ds + \sum_{j=1}^m \gamma_j \int_a^{\sigma_j} \frac{(\sigma_j-s)^2}{2} v(s)ds\Big] \\ &-\frac{1}{\Delta} P_1(t)\Big[-\beta_2 \int_a^T (T-s)v(s)ds + \beta_3 \int_a^\xi \frac{(\xi-s)^2}{2} v(s)ds \\ &+\sum_{j=1}^m \rho_j \int_a^{\sigma_j} (\sigma_j - s)v(s)ds\Big] + \frac{1}{\Delta} P_2(t)\Big[-\delta_2 \int_a^T v(s)ds \\ &+\delta_3 \int_a^\xi (\xi-s)v(s)ds + \sum_{j=1}^m \nu_j \int_a^{\sigma_j} v(s)ds\Big]. \end{aligned}$$

Hence, $u \in F(u)$.

Next we show that there exists $\delta < 1$ $\Big(\delta := \|m\|Q\Big)$ such that

$$H_d(\mathcal{F}(u), \mathcal{F}(\bar{u})) \leq \delta \|u - \bar{u}\| \quad \text{for each} \quad u, \bar{u} \in C^3([a,T],\mathbb{R}).$$

Let $u, \bar{u} \in C^3([a,T],\mathbb{R})$ and $h_1 \in \mathcal{F}(u)$. Then there exists $v_1(t) \in F(t,u(t))$ such that, for each $t \in [a,T]$,

$$\begin{aligned} h_1(t) &= \int_a^t \frac{(t-s)^2}{2} v_1(s)ds + \frac{1}{\zeta_4}\Big[-\alpha_2 \int_a^T \frac{(T-s)^2}{2} v_1(s)ds \\ &+\alpha_3 \int_a^\xi \frac{(\xi-s)^3}{3!} v_1(s)ds + \sum_{j=1}^m \gamma_j \int_a^{\sigma_j} \frac{(\sigma_j-s)^2}{2} v_1(s)ds\Big] \\ &-\frac{1}{\Delta} P_1(t)\Big[-\beta_2 \int_a^T (T-s)v_1(s)ds + \beta_3 \int_a^\xi \frac{(\xi-s)^2}{2} v_1(s)ds \end{aligned}$$

$$+\sum_{j=1}^{m}\rho_j\int_a^{\sigma_j}(\sigma_j-s)v_1(s)ds\Big]+\frac{1}{\Delta}P_2(t)\Big[-\delta_2\int_a^T v_1(s)ds$$
$$+\delta_3\int_a^{\xi}(\xi-s)v_1(s)ds+\sum_{j=1}^{m}\nu_j\int_a^{\sigma_j}v_1(s)ds\Big].$$

By (A_2), we have $H_d(F(t,u(t)),F(t,\bar{u}(t))\le m(t)|u(t)-\bar{u}(t)|$. So, there exists $z\in F(t,u(t))$ such that $|v_1(t)-z|\le m(t)|u(t)-\bar{u}(t)|$ for almost all $t\in[a,T]$. Define the multifunction $U:[a,T]\to\mathcal{P}(\mathbb{R})$ by

$$U(t)=\{z\in\mathbb{R}:\ |v_1(t)-z|\le m(t)|u(t)-\bar{u}(t)|,\ \text{for almost all } t\in[a,T]\}.$$

It is easy to check that the multifunction $U(\cdot)\cap F(\cdot,u(\cdot))$ is measurable. Hence, we can choose $v_2\in S_{F,u}$ such that

$$|v_1(t)-v_2(t)|\le m(t)|u(t)-\bar{u}(t)|,$$

for almost all $t\in[a,T]$. For each $t\in[a,T]$, let us define

$$\begin{aligned}
h_2(t) &= \int_a^t\frac{(t-s)^2}{2}v_2(s)ds+\frac{1}{\zeta_4}\Big[-\alpha_2\int_a^T\frac{(T-s)^2}{2}v_2(s)ds\\
&\quad+\alpha_3\int_a^{\xi}\frac{(\xi-s)^3}{3!}v_2(s)ds+\sum_{j=1}^{m}\gamma_j\int_a^{\sigma_j}\frac{(\sigma_j-s)^2}{2}v_2(s)ds\Big]\\
&\quad-\frac{1}{\Delta}P_1(t)\Big[-\beta_2\int_a^T(T-s)v_2(s)ds+\beta_3\int_a^{\xi}\frac{(\xi-s)^2}{2}v_2(s)ds\\
&\quad+\sum_{j=1}^{m}\rho_j\int_a^{\sigma_j}(\sigma_j-s)v_2(s)ds\Big]+\frac{1}{\Delta}P_2(t)\Big[-\delta_2\int_a^T v_2(s)ds\\
&\quad+\delta_3\int_a^{\xi}(\xi-s)v_2(s)ds+\sum_{j=1}^{m}\nu_j\int_a^{\sigma_j}v_2(s)ds\Big].
\end{aligned}$$

Thus we have

$$\begin{aligned}
|h_1(t)-h_2(t)| &\le \sup_{t\in[a,T]}\Big\{\int_a^t\frac{(t-s)^2}{2}|v_1(s)-v_2(s)|ds+\frac{1}{|\zeta_4|}\Big[|\alpha_2|\int_a^T\frac{(T-s)^2}{2}|v_1(s)-v_2(s)|ds\\
&\quad+|\alpha_3|\int_a^{\xi}\frac{(\xi-s)^3}{3!}|v_1(s)-v_2(s)|ds+\sum_{j=1}^{m}\gamma_j\int_a^{\sigma_j}\frac{(\sigma_j-s)^2}{2}|v_1(s)-v_2(s)|ds\Big]\\
&\quad+\frac{1}{|\Delta|}|P_1(t)|\Big[|\beta_2|\int_a^T(T-s)|v_1(s)-v_2(s)|ds+|\beta_3|\int_a^{\xi}\frac{(\xi-s)^2}{2}|v_1(s)-v_2(s)|ds\\
&\quad+\sum_{j=1}^{m}\rho_j\int_a^{\sigma_j}(\sigma_j-s)|v_1(s)-v_2(s)|ds\Big]+\frac{1}{|\Delta|}|P_2(t)|\Big[|\delta_2|\int_a^T|v_1(s)-v_2(s)|ds\\
&\quad+|\delta_3|\int_a^{\xi}(\xi-s)|v_1(s)-v_2(s)|ds+\sum_{j=1}^{m}\nu_j\int_a^{\sigma_j}|v_1(s)-v_2(s)|ds\Big]\Big\}\\
&\le \|m\|\|u-\bar{u}\|\Big\{\frac{(T-a)^3}{3!}+\frac{1}{|\zeta_4|}\Big[|\alpha_2|\frac{(T-a)^3}{3!}+|\alpha_3|\frac{(\xi-a)^4}{4!}\\
&\quad+\sum_{j=1}^{m}\gamma_j\frac{(\sigma_j-a)^3}{3!}\Big]+\frac{1}{|\Delta|}|P_1(t)|\Big[|\beta_2|\frac{(T-a)^2}{2}+|\beta_3|\frac{(\xi-a)^3}{3!}\\
&\quad+\sum_{j=1}^{m}\rho_j\frac{(\sigma_j-a)^2}{2}\Big]+\frac{1}{|\Delta|}|P_2(t)|\Big[|\delta_2|(T-a)+|\delta_3|\frac{(\xi-a)^2}{2}+\sum_{j=1}^{m}\nu_j(\sigma_j-a)\Big]\Big\}
\end{aligned}$$

$$\leq \quad \|m\|Q\|u-\bar{u}\|,$$

which implies that

$$\|h_1 - h_2\| \leq \|m\|Q\|u-\bar{u}\|.$$

Analogously, interchanging the roles of u and $\bar{u}$, we obtain

$$H_d(\mathcal{F}(u), \mathcal{F}(\bar{u})) \leq \|m\|Q\|u-\bar{u}\|.$$

So $\mathcal{F}$ is a contraction. Therefore, it follows by Lemma 2 that $\mathcal{F}$ has a fixed point u which is a solution of (2). This completes the proof. ■

5 Conclusions

We have presented the existence theory for a new class of single-valued and multi-valued third-order boundary value problems on an arbitrary domain involving non-separated integro-multi-point boundary conditions. It is worthwhile to mention that the boundary conditions considered in the present work are of fairly general nature and specialize to various cases by fixing the involved parameters appropriately. For instance, the boundary conditions in problems (1) and (2) reduce to non-separated type multi-point boundary conditions for $\alpha_3 = \beta_3 = \delta_3 = 0$, while the non-separated type strip boundary conditions follow by fixing $\gamma_j = \rho_j = \nu_j = 0, \forall j$. Moreover, the given boundary conditions become anti-periodic boundary conditions if we take $\alpha_3 = \beta_3 = \delta_3 = 0$, $\gamma_j = \rho_j = \nu_j = 0, \forall j$ and $\alpha_i = \beta_i = \delta_i = 1, i = 1, 2$. Similarly, initial and terminal integro-multi-point boundary conditions can be obtained by fixing $\alpha_1 = \beta_1 = \delta_1 = 1, \alpha_2 = \beta_2 = \delta_2 = 0$ and $\alpha_1 = \beta_1 = \delta_1 = 0$, $\alpha_2 = \beta_2 = \delta_2 = 1$, respectively. In case we set $\alpha_i = \beta_i = \delta_i = 0, i = 1, 2$, our results correspond to integro-multi-point boundary conditions of the form:

$$\begin{aligned} \alpha_3 \int_a^\xi u(s)ds + \sum_{j=1}^m \gamma_j u(\sigma_j) &= 0, \quad \beta_3 \int_a^\xi u'(s)ds + \sum_{j=1}^m \rho_j u'(\sigma_j) = 0, \\ \delta_3 \int_a^\xi u''(s)ds + \sum_{j=1}^m \nu_j u''(\sigma_j) &= 0. \end{aligned} \tag{17}$$

Observe that the boundary data in (17) only contain the interior points of the interval $[a, T]$ and these boundary condition can be interpreted as the sum of scalar multiple of the strip condition and sum of discrete values of u, u', u'' at interior points σ_j $(j = 1, 2, \ldots, m)$ amount to zero. In the nutshell, our results are not only new in the given configuration but also correspond to several new ones as special cases with appropriate choice of the parameters involved in the boundary conditions.

Acknowledgment. We thank the referee for his/her useful remarks on our paper.

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