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Progressively Type-II Right Censored Order Statistics From Odds Generalized Exponential-Pareto Distribution And Related Inference*

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Abstract

In this paper some recurrence relations satisfied by single and product moments of progressively Type-II right censored order statistics from odds generalized exponential-Pareto distribution are obtained. Then we use these results to compute the moments for all sample sizes and all censoring schemes. This allows us to obtain best linear unbiased estimators of location and scale parameters based on progressively Type-II right censored samples. The best linear unbiased predictors of censored failure times are discussed briefly. Two numerical examples are presented to illustrate the estimation and prediction methods discussed here.

1 Introduction

The scheme of progressive Type-II censoring is of importance in reliability and life-testing experiments. It allows the experimenter to remove units from a life test at various stages during the experiment which may lead to a saving of costs and of time (see Cohen [12] and Sen [30]). In such a random experiment, a group of n independent and identical experimental units is put on a life test at time zero with continuous, identically distributed failure times $X_1, X_2, ..., X_n$. After the j^{th} failure, a prespecified number $R_j \geq 0$ of the $n-j-\sum_{i=0}^{j-1} R_i$ remaining (or surviving) units are randomly withdrawn from the experiment, $1 \leq j \leq m, m \leq n,\ R_0 = 0$. Removed units thus become right censored at the time of failure of other units. This progressive censoring leads to m ordered observed failure times denoted by $X_{1:m:n}^{(R_1,R_2,...,R_m)},\ X_{2:m:n}^{(R_1,R_2,...,R_m)}, ..., X_{m:m:n}^{(R_1,R_2,...,R_m)}$, and these are called progressively Type-II right censored order statistics of size m from a sample of size n with progressive censoring scheme $(R_1, R_2, ..., R_m)$. Thus, in this type of sampling, m failures are observed, $\sum_{j=1}^{m} R_j$ units are progressively censored and $n = m + \sum_{j=1}^{m} R_j$ denotes the number of units in the life test. The withdrawal of units may be seen as a model describing drop-outs of units due to failures which have causes other than the specific one under study. In this sense, progressive censoring schemes are applied in clinical trials as well. Here, the drop-outs of patients may be caused by migration, lack of interest or by personal or ethical decisions, and they are regarded as random withdrawals. For a detailed discussion of progressive censoring and the relevant developments in this area, one may refer to Sen [30] and Balakrishnan and Aggarwala [4].

The situation with no censoring corresponds to the special case with $m = n$ and $R_1 = R_2 = ... = R_m = 0$, whereas the situation with ordinary Type-II right censoring at a given order statistic corresponds to the special case with $m < n$, $R_1 = R_2 = ... = R_{m-1} = 0$ and $R_m = n - m$.

If the failure times of the n items originally on test are from a continuous population with c.d.f. F(x) and p.d.f. f(x), then the joint p.d.f. of $X_{1:m:n}^{(R_1,R_2,...,R_m)},\ X_{2:m:n}^{(R_1,R_2,...,R_m)}, ..., X_{m:m:n}^{(R_1,R_2,...,R_m)}$ is given by (cf. Balakrishnan and Sandhu [11] and Saran and Pushkarna [28])

$$f_{X_{1:m:n},...,X_{m:m:n}}(x_1, x_2, ..., x_m) = A(n, m-1) \prod_{i=1}^{m} f(x_i)[1 - F(x_i)]^{R_i},\ 0 \leq x_1 < x_2 < ... < x_m < \infty, \quad (1)$$

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where $A(n, m-1) = n(n - R_1 - 1)(n - R_1 - R_2 - 2)\cdots(n - R_1 - R_2 - ... - R_{m-1} - m + 1)$.

Here, note that all the factors in $A(n, m-1)$ are positive integers. Also it may be observed that the different factors in $A(n, m-1)$ represent the number of units still on test immediately preceding the first, second, ..., m^{th} observed failures, respectively. Similarly, for convenience in notation, let us define

$$A(p, q) = p(p - R_1 - 1)(p - R_1 - R_2 - 2)...(p - R_1 - R_2 - ... - R_q - q),$$

for $q = 0, 1, ..., p-1$, with all the factors being positive integers.

Progressive censoring and associated inferential procedures have been extensively studied in the literature for a number of distributions by several authors. Cohen ([12], [13], [14], [15] and [16]), Mann ([20], [21]), Cohen and Whitten [17], Viveros and Balakrishnan [31], Balakrishnan and Sandhu [11], Aggarwala and Balakrishnan [1] and Balakrishnan and Aggarwala [4] have derived recurrence relations for single and product moments of progressively Type-II right censored order statistics from exponential, Pareto and power function distributions and their truncated forms.

Saran and Pande [27], Saran and Pushkarna ([28], [29]), Saran et al. [26] and Pushkarna et al. [25] have derived recurrence relations for single and product moments of the corresponding progressively Type-II right censored order statistics from half logistic, Burr, left truncated logistic, Frechet and a general class of doubly truncated continuous distributions.

Mahmoud et al. [19] derived some new recurrence relations for single and product moments of progressively Type-II right censored order statistics from the linear exponential distribution and also obtained maximum likelihood estimators (MLEs) of the location and scale parameters. Balakrishnan et al. [5] and Balakrishnan and Saleh ([7], [8], [9], [10]) have established several recurrence relations for single and product moments of progressively Type-II right censored order statistics from logistic, half-logistic, log-logistic, generalized half logistic and generalized logistic distributions and utilized them to derive the best linear unbiased estimators of the location and scale parameters.

In this paper, we derive some recurrence relations satisfied by the single and product moments of progressively Type-II right censored order statistics from odds generalized exponential-Pareto distribution. These relations enable the recursive computation of moments for all sample sizes and all possible progressive censoring schemes. Then we use these results to compute the means, variances and covariances of progressively Type-II right censored order statistics for some specific values of the parameters, which will be utilized to derive the best linear unbiased estimators (BLUEs) of location and scale parameters of the location-scale odds generalized exponential-Pareto distribution as well as their variances and covariances. Tables of these quantities are presented for different sample sizes up to n = 8 and some selected progressive censoring schemes, corresponding to particular values of the parameters. Further, for the special case $R_1 = R_2 = ... = R_m = 0$, the derived results would reduce to the general recurrence relations for the usual order statistics from the odds generalized exponential-Pareto distribution. Also, we briefly discuss the best linear unbiased predictors (BLUPs) of the censored failure times by making use of the results developed on the BLUEs. Finally, two numerical examples, one with real data and another with simulated data, are presented to illustrate the estimation and prediction methods discussed here.

2 Odds Generalized Exponential-Pareto Distribution

The pdf and cdf of odds generalized exponential-Pareto distribution are given by

$$f(x) = \frac{\lambda\theta}{\alpha^\theta} x^{\theta-1} e^{-\lambda\left[\left(\frac{x}{\alpha}\right)^\theta - 1\right]}, \quad x \geq \alpha, \quad \lambda,\ \theta > 0, \tag{2}$$

$$F(x) = 1 - e^{-\lambda\left[\left(\frac{x}{\alpha}\right)^\theta - 1\right]}. \tag{3}$$

And the characterizing differential equation for odds generalized exponential-Pareto distribution is given by

$$\alpha^\theta f(x) = \lambda\theta x^{\theta-1}(1 - F(x)). \tag{4}$$

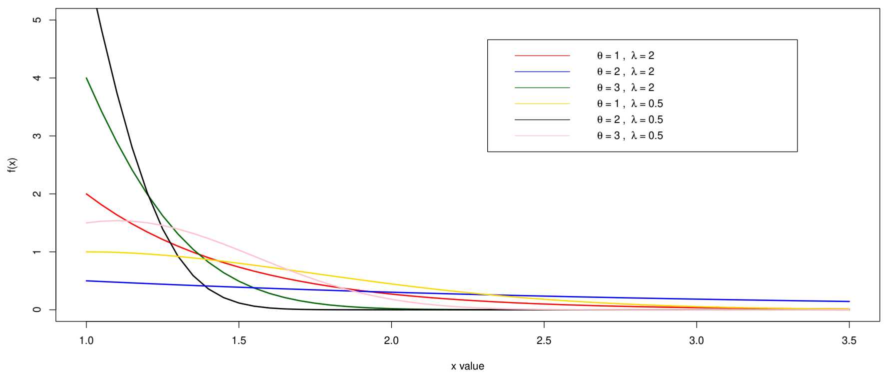

Figure 1: p.d.f. of odds generalized exponential-Pareto distribution

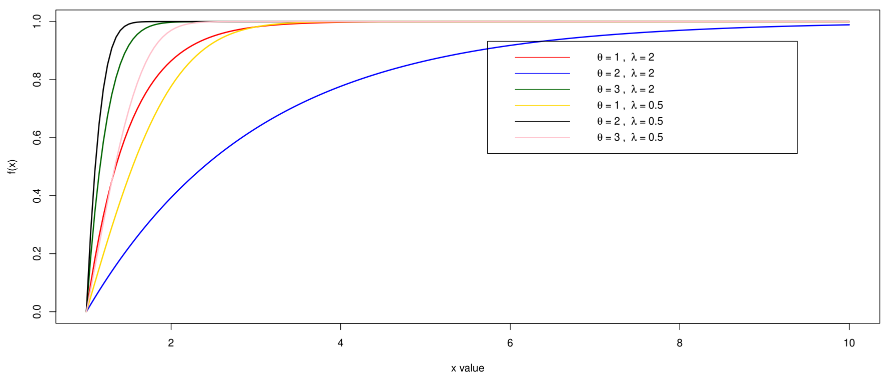

Figure 2: c.d.f. of odds generalized exponential-Pareto distribution

More details on this distribution can be found in Maiti and Pramanik [22]. The graphs of the p.d.f. and c.d.f. of odds generalized exponential-Pareto distribution as given in (2) and (3) for $\alpha = 1,\ \lambda = 2,\ 0.5$ and for different values of $\theta = 1.0,\ 2.0,\ 3.0$ are shown in Figures 1 and 2, respectively.

The c.d.f. of the location-scale parameter odds generalized exponential-Pareto distribution is given by

$$F(x) = 1 - e^{-\lambda\left[\left(\frac{x-\mu}{\sigma\alpha}\right)^{\theta}-1\right]}, \quad x \geq \left(\frac{\alpha-\mu}{\sigma}\right), \quad \lambda,\ \theta > 0. \tag{5}$$

3 Recurrence Relations for Single Moments

In this section, we shall establish several recurrence relations for single moments of progressively Type-II right censored order statistics from odds generalized exponential-Pareto distribution satisfying the characterizing differential equation (4). Using (1), we have

$$\begin{aligned} \mu_{r:m:n}^{(R_1,R_2,\ldots,R_m)^{(k)}} &= E[X_{r:m:n}^{(R_1,R_2,\ldots,R_m)}]^k \\ &= A(n,m-1)\int \underset{\alpha\leq x_1<x_2<\ldots<x_m<\infty}{\int\cdots\int} \int x_r^k \prod_{t=1}^{m} f(x_t)[1-F(x_t)]^{R_t}dx_t. \end{aligned} \tag{6}$$

Theorem 1 *For $k \geq 0$,*

$$\alpha^{\theta}\mu_{1:1:1}^{(0)^{(k+1)}} = \frac{\lambda\theta}{\theta+k+1}\left[-\alpha^{\theta+k+1} + \mu_{1:1:1}^{(0)^{(\theta+k+1)}}\right]. \tag{7}$$

Proof. From (6), for $n = m = r = 1$, we obtain

$$\alpha^{\theta} {\mu_{1:1:1}^{(0)}}^{(k+1)} = A(n,0) \int_{\alpha}^{\infty} \alpha^{\theta} x_1^{k+1} f(x_1) \left[1 - F(x_1)\right]^{n-1} dx_1,$$

using (4), we have

$$\alpha^{\theta} {\mu_{1:1:1}^{(0)}}^{(k+1)} = n\lambda\theta \int_{\alpha}^{\infty} x_1^{\theta+k} \left[1 - F(x_1)\right]^{n} dx_1.$$

Solving the integral on the R.H.S. of the above equation by taking $(1 - F(x_1))^n$ for differentiation and the rest of the integrand for integration, and then after some simplification, it leads to the required result (7). ■

Theorem 2 *For $n \geq 2$ and $k \geq 0$,*

$$\alpha^{\theta} \mu_{1:1:n}^{(n-1)(k+1)} = \frac{n\lambda\theta}{\theta + k + 1} \left[-\alpha^{\theta+k+1} + \mu_{1:1:n}^{(n-1)(\theta+k+1)}\right]. \tag{8}$$

Proof. Proceeding in a similar manner as in Theorem 1, we can easily establish the relation (8). ■

Theorem 3 *For $2 \leq m \leq n-1$, $k \geq 0$ and $R_1 \geq 0$,*

$$\begin{aligned} \alpha^{\theta} \mu_{1:m:n}^{(R_1,R_2,\ldots,R_m)(k+1)} = \frac{\lambda\theta}{\theta + k + 1} \Big[& (n - R_1 - 1)\mu_{1:m-1:n}^{(R_1+R_2+1,R_3,\ldots,R_m)(\theta+k+1)} - n\alpha^{\theta+k+1} \\ & + (R_1 + 1)\mu_{1:m:n}^{(R_1,R_2,\ldots,R_m)(\theta+k+1)} \Big]. \end{aligned} \tag{9}$$

Proof. The relation in (9) may be proved by following exactly the same steps as those used in proving Theorem 4, which is presented next. ■

Theorem 4 *For $2 \leq r \leq n-1$, $m < n$, $k \geq 0$ and $R_r \geq 0$,*

$$\begin{aligned} \alpha^{\theta} \mu_{r:m:n}^{(R_1,R_2,\ldots,R_m)(k+1)} = \frac{\lambda\theta}{\theta + k + 1} \Big[& (n - S_r - r)\mu_{r:m-1:n}^{(R_1,\ldots,R_{r-1},R_r+R_{r+1}+1,R_{r+2},\ldots,R_m)(\theta+k+1)} \\ & - (n - S_{r-1} - r + 1)\mu_{r-1:m-1:n}^{(R_1,\ldots,R_{r-2},R_{r-1}+R_r+1,R_{r+1},\ldots,R_m)(\theta+k+1)} \\ & + (R_r + 1)\mu_{r:m:n}^{(R_1,\ldots,R_m)(\theta+k+1)} \Big], \end{aligned} \tag{10}$$

where $S_i = R_1 + R_2 + \ldots + R_i$, $1 \leq i \leq m$.

Proof. Using (6), we have

$$\begin{aligned} & \alpha^{\theta} \mu_{r:m:n}^{(R_1,R_2,\ldots,R_m)(k+1)} \\ & = A(n, m-1) \int \underset{\alpha \leq x_1 < x_2 < \ldots < x_{r-1} < x_{r+1} < \ldots < x_m < \infty}{\int \cdots \int} I(x_{r-1}, x_{r+1}) \prod_{u=1, u \neq r}^{m} [1 - F(x_u)]^{R_u} f(x_u) dx_u, \end{aligned} \tag{11}$$

where

$$I(x_{r-1}, x_{r+1}) = \int_{x_{r-1}}^{x_{r+1}} \alpha^{\theta} x_r^{k+1} [1 - F(x_r)]^{R_r} f(x_r) dx_r. \tag{12}$$

Using the characterizing differential equation (4), we have

$$I(x_{r-1}, x_{r+1}) = \lambda\theta \int_{x_{r-1}}^{x_{r+1}} x_r^{k+\theta} [1 - F(x_r)]^{R_r+1} dx_r. \tag{13}$$

Upon substituting the resultant expression for $I(x_{r-1}, x_{r+1})$ from (13) in (11) and simplifying, it leads to Theorem 4 ■

Next, we state another result on single moments which can easily be estblished on similar lines.

Theorem 5 *For $2 \le m \le n,\ k \ge 0$ and $R_m \ge 0$,*

$$\alpha^{\theta}\mu_{m:m:n}^{(R_1,R_2,\ldots,R_m)^{(k+1)}} = \frac{\lambda\theta}{\theta+k+1}\Bigg[-(n-S_{m-1}-r+1)\mu_{m-1:m-1:n}^{(R_1,R_2,\ldots,R_{m-2},R_{m-1}+R_m+1)^{(\theta+k+1)}} +(R_m+1)\mu_{m:m:n}^{(R_1,\ldots,R_m)^{(\theta+k+1)}}\Bigg].$$

Remark 1 *It may be mentioned that if $R_1 = R_2 = \ldots = R_{k-1} = 0$, i.e. there is no censoring before the time of the k^{th} failure, then the first k progressively Type-II right censored order statistics are simply the first k usual order statistics. Thus, for the special case $R_1 = R_2 = \ldots = R_m = 0$, so that $m = n$ in which case the progressively censored order statistics become the usual order statistics $X_{1:n}, X_{2:n}, \ldots, X_{n:n}$, the recurrence relations established in Section 3 would reduce to the corresponding recurrence relations for the single moments of usual order statistics from the odds generalized exponential-Pareto distribution satisfying the characterizing differential equation (4).*

4 Recurrence Relations for Product Moments

Using (1) we can write the product moments of progressively Type-II right censored order statistics as follows:

$$\mu_{r,s:m:n}^{(R_1,R_2,\ldots,R_m)^{(k_1,k_2)}} = E\left[\left\{X_{r:m:n}^{(R_1,R_2,\ldots,R_m)}\right\}^{k_1}\left\{X_{s:m:n}^{(R_1,R_2,\ldots,R_m)}\right\}^{k_2}\right]$$
$$= A(n,m-1)\int\underset{0\le x_1<x_2<\ldots<x_m<\infty}{\int\cdots\int}\int x_r^{k_1}x_s^{k_2}\prod_{t=1}^{m} f(x_t)[1-F(x_t)]^{R_t}dx_t, \tag{14}$$

where $1 \le r < s \le m \le n$ and $k_1,\ k_2 \ge 0$. In this Section, we shall derive various recurrence relations for the product moments of progressively Type-II right censored order statistics from odds generalized exponential-Pareto distribution with p.d.f. f(x) and c.d.f. F(x) satisfying the characterizing differential equation (4).

Theorem 6 *For $2 \le r < s \le m < n,\quad k,t \ge 0$ and $R_r \ge 0$,*

$$\alpha^{\theta}\mu_{r,s:m:n}^{(R_1,R_2,\ldots,R_m)^{(k+1,t)}} = \frac{\lambda\theta}{\theta+k+1}\Bigg[(n-S_r-r)\mu_{r,s-1:m-1:n}^{(R_1,\ldots,R_{r-1},R_r+R_{r+1}+1,R_{r+2},\ldots,R_m)^{(\theta+k+1,t)}} -(n-S_{r-1}-r+1)\mu_{r-1,s-1:m-1:n}^{(R_1,\ldots,R_{r-2},R_{r-1}+R_r+1,R_{r+1},\ldots,R_m)^{(\theta+k+1,t)}} +(R_r+1)\mu_{r,s:m:n}^{(R_1,\ldots,R_m)^{(\theta+k+1,t)}}\Bigg]. \tag{15}$$

Proof. From (14), let us consider for $2 \le r < s \le m < n,\quad k,t \ge 0$ and $R_r \ge 0$,

$$\alpha^{\theta}\mu_{r,s:m:n}^{(R_1,R_2,\ldots,R_m)^{(k+1,t)}}$$
$$= A(n,m-1)\int\underset{0\le x_1<x_2<\ldots<x_{r-1}<x_{r+1}<\ldots x_m<\infty}{\int\cdots\int} x_s^t I(x_{r-1},x_{r+1})\prod_{u=1,u\ne r}^{m} f(x_u)[1-F(x_u)]^{R_u}dx_u, \tag{16}$$

where $I(x_{r-1}, x_{r+1})$ is the same as given in equation (12). Now putting the value of $I(x_{r-1}, x_{r+1})$ into the equation (16) and then simplifying, it leads to (15). ■

Theorem 7 *For $2 \le s \le m \le n - R_1,\quad k,t \ge 0$ and $R_1 \ge 0$,*

$$\alpha^{\theta}\mu_{1,s:m:n}^{(R_1,R_2,\ldots,R_m)^{(k+1,t)}} = \frac{\lambda\theta}{\theta+k+1}\Bigg[(n-S_1-1)\mu_{1,s-1:m-1:n}^{(R_1+R_2+1,R_3,\ldots,R_m)^{(\theta+k+1,t)}} -n\alpha^{\theta+k+1}\mu_{s-1:m-1:n-R_1-1}^{(R_2,R_3,R_4,\ldots,R_m)^{(k)}}+(R_1+1)\mu_{1,s:m:n}^{(R_1,R_2,\ldots,R_m)^{(\theta+k+1,t)}}\Bigg]. \tag{17}$$

Proof. The relation in (17) may be proved by following exactly the same steps as those used in proving Theorem 6. ■

Theorem 8 *For* $1 \leq r < s < m < n,\ k, t \geq 0$ *and* $R_s \geq 0,$

$$\begin{aligned}\alpha^{\theta}\mu_{r,s:m:n}^{(R_1,R_2,\ldots,R_m)^{(k,t+1)}} &= \frac{\lambda\theta}{\theta+t+1}\Big[(n-S_s-s)\mu_{r,s:m-1:n}^{(R_1,\ldots,R_{s-1},R_s+R_{s+1}+1,R_{s+2},\ldots,R_m)^{(k,t+\theta+1)}} \\ &\quad -(n-S_{s-1}-s+1)\mu_{r,s-1:m-1:n}^{(R_1,\ldots,R_{s-2},R_{s-1}+R_s+1,R_{s+1},\ldots,R_m)^{(k,t+\theta+1)}} \\ &\quad +(R_s+1)\mu_{r,s:m:n}^{(R_1,\ldots,R_m)^{(k,t+\theta+1)}}\Big]. \qquad (18)\end{aligned}$$

Proof. The relation in (18) may be proved by following the similar steps as those used in proving (15). ■

Finally, we state another result on product moments which can be established on similar lines.

Theorem 9 *For* $1 \leq r < m < n,\ k,\ t \geq 0$ *and* $R_m \geq 0,$

$$\begin{aligned}\alpha^{\theta}\mu_{r,m:m:n}^{(R_1,R_2,\ldots,R_m)^{(k,t+1)}} &= \frac{\lambda\theta}{\theta+t+1}\Big[-(n-S_{m-1}-m+1)\mu_{r-1,m-1:m-1:n}^{(R_1,\ldots,R_{m-2},R_{m-1}+R_m+1)^{(k,t+\theta+1)}} \\ &\quad +(R_m+1)\mu_{r,m:m:n}^{(R_1,\ldots,R_m)^{(k,t+\theta+1)}}\Big]. \qquad (19)\end{aligned}$$

Remark 2 *For the special case* $R_1 = R_2 = \ldots = R_m = 0$*, the recurrence relations established in Section 4 reduce to the corresponding recurrence relations for the product moments of usual order statistics from the odds generalized exponential-Pareto distribution satisfying the characterizing differential equation* (4).

5 Numerical Results

The recurrence relations obtained in the preceding Sections 3 and 4 allow us to evaluate the means, variances and covariances of progressively Type-II right censored order statistics from odds generalized exponential-Pareto distribution for all sample sizes 'n' and all censoring schemes $(R_1,\ R_2,\ \ldots,\ R_m),\ m < n$. These quantities can be used for various inferential purposes; for example, they are useful in determining BLUEs of location/scale parameters and BLUPs of censored failure times. In this Section, we compute means, variances and covariances of progressively Type-II right censored order statistics from odds generalized exponential-Pareto distribution for some specific values of parameters, viz. $\alpha = 1,\ \lambda = 2,\ \theta = 2$, for sample sizes up to $n = 8$ and for different choices of m and progressive censoring schemes $(R_1,\ R_2,\ \ldots,\ R_m),\ m < n$. These values are presented in the following Tables 1 and 2.

Table 1: First four single moments of progressively Type-II right censored order statistics from odds generalized exponential-Pareto distribution

S.No.	n	m	Censoring Scheme	$\mu_{r:m:n}^{(R_1,R_2,\ldots,R_m)^{(k)}}$, $r = 1, 2, \ldots, m$ and $k = 1$		
1	4	2	1,1	1.059163	1.167175	
2	5	2	1,2	1.047804	1.122423	
3	5	2	2,1	1.047804	1.156746	
4	5	2	0, 3	1.047804	1.104598	
5	5	2	3, 0	1.047804	1.251405	
6	6	2	2, 2	1.040112	1.115192	
7	7	2	3, 2	1.034556	1.109974	
8	4	3	0, 0, 1	1.059163	1.133118	1.235289
9	5	3	0, 0, 2	1.047804	1.104598	1.175899
10	5	3	0, 1, 1	1.047804	1.104598	1.208894

11	5	3	2, 0, 0	1.047804	1.156746	1.346063	
12	6	3	1, 1, 1	1.040112	1.097265	1.202131	
13	6	3	1,2,0	1.040112	1.097265	1.293977	
14	6	3	1,0,2	1.040112	1.097265	1.168972	
15	7	3	0, 2, 2	1.034556	1.073450	1.146498	
16	7	3	1, 2, 1	1.034556	1.080926	1.187074	
17	5	4	0, 0, 0, 1	1.047804	1.104598	1.175899	1.274883
18	5	4	1, 0, 0, 0	1.047804	1.122423	1.225391	1.406400
19	6	4	1, 0, 1, 0	1.040112	1.097265	1.168972	1.356480
20	7	4	1, 2, 0, 0	1.034556	1.080926	1.187074	1.372562
21	8	4	1,3,0,0	1.030353	1.069389	1.176467	1.363248
22	8	4	2,0,0,2	1.030353	1.076890	1.132307	1.202054
23	8	4	1,1,2,0	1.030353	1.069389	1.125141	1.318097

S.No.	n	m	Censoring Scheme	$\mu_{r:m:n}^{(R_1,R_2,\ldots,R_m)^{(k)}}$, $r=1,2,\ldots,m$ and $k=2$			
1	4	2	1,1	1.125000	1.375000		
2	5	2	1,2	1.100000	1.266667		
3	5	2	2,1	1.100000	1.350000		
4	5	2	0, 3	1.100000	1.225000		
5	5	2	3, 0	1.100000	1.600000		
6	6	2	2, 2	1.083333	1.250000		
7	7	2	3, 2	1.071429	1.238095		
8	4	3	0, 0, 1	1.125000	1.291667	1.541667	
9	5	3	0, 0, 2	1.100000	1.225000	1.391667	
10	5	3	0, 1, 1	1.100000	1.225000	1.475000	
11	5	3	2, 0, 0	1.100000	1.350000	1.850000	
12	6	3	1, 1, 1	1.083333	1.208333	1.45833	
13	6	3	1,2,0	1.083333	1.208333	1.708333	
14	6	3	1,0,2	1.083333	1.208333	1.375000	
15	7	3	0, 2, 2	1.071429	1.154762	1.321429	
16	7	3	1, 2, 1	1.071429	1.171429	1.421429	
17	5	4	0, 0, 0, 1	1.100000	1.225000	1.391667	1.641667
18	5	4	1, 0, 0, 0	1.100000	1.266667	1.516667	2.016667
19	6	4	1, 0, 1, 0	1.083333	1.208333	1.375000	1.875000
20	7	4	1, 2, 0, 0	1.071429	1.171429	1.421429	1.921429
21	8	4	1,3,0,0	1.062500	1.145833	1.395833	1.895833
22	8	4	2,0,0,2	1.062500	1.162500	1.287500	1.454167
23	8	4	1,1,2,0	1.062500	1.145833	1.270833	1.770833

S.No.	n	m	Censoring Scheme	$\mu_{r:m:n}^{(R_1,R_2,\ldots,R_m)^{(k)}}$, $r=1,2,\ldots,m$ and $k=3$		
1	4	2	1,1	1.198593	1.636284	
2	5	2	1,2	1.157171	1.437777	
3	5	2	2,1	1.157171	1.590950	
4	5	2	0, 3	1.157171	1.364283	
5	5	2	3, 0	1.157171	2.095724	
6	6	2	2, 2	1.130014	1.408812	
7	7	2	3, 2	1.110845	1.388339	
8	4	3	0, 0, 1	1.198593	1.481873	1.945106
9	5	3	0, 0, 2	1.157171	1.364283	1.658258
10	5	3	0, 1, 1	1.157171	1.364283	1.817618
11	5	3	2, 0, 0	1.157171	1.590950	2.600498
12	6	3	1, 1, 1	1.130014	1.335751	1.786550
13	6	3	1,2,0	1.130014	1.335751	2.306234

14	6	3	1,0,2	1.130014	1.335751	1.627994	
15	7	3	0, 2, 2	1.110845	1.245027	1.531651	
16	7	3	1, 2, 1	1.110845	1.272984	1.718137	
17	5	4	0, 0, 0, 1	1.157171	1.364283	1.658258	2.136339
18	5	4	1, 0, 0, 0	1.157171	1.437777	1.897298	2.952098
19	6	4	1, 0, 1, 0	1.130014	1.335751	1.627994	2.645354
20	7	4	1, 2, 0, 0	1.110845	1.272984	1.718137	2.747558
21	8	4	1,3,0,0	1.096596	1.230269	1.671445	2.693880
22	8	4	2,0,0,2	1.096596	1.258129	1.470437	1.770950
23	8	4	1,1,2,0	1.096596	1.230269	1.441233	2.429806

S.No.	n	m	Censoring Scheme	$\mu_{r:m:n}^{(R_1,R_2,\ldots,R_m)^{(k)}}$, $r=1,2,\ldots,m$ and $k=4$			
1	4	2	1,1	1.281250	1.968750		
2	5	2	1,2	1.220000	1.642222		
3	5	2	2,1	1.220000	1.895000		
4	5	2	0, 3	1.220000	1.526250		
5	5	2	3, 0	1.220000	2.820000		
6	6	2	2, 2	1.180556	1.597222		
7	7	2	3, 2	1.153061	1.565760		
8	4	3	0, 0, 1	1.281250	1.711806	2.482639	
9	5	3	0, 0, 2	1.220000	1.526250	1.990139	
10	5	3	0, 1, 1	1.220000	1.526250	2.263750	
11	5	3	2, 0, 0	1.220000	1.895000	3.745000	
12	6	3	1, 1, 1	1.180556	1.482639	2.211806	
13	6	3	1,2,0	1.180556	1.482639	3.190972	
14	6	3	1,0,2	1.180556	1.482639	1.940972	
15	7	3	0, 2, 2	1.153061	1.345522	1.785998	
16	7	3	1, 2, 1	1.153061	1.387347	2.098061	
17	5	4	0, 0, 0, 1	1.220000	1.526250	1.990139	2.810972
18	5	4	1, 0, 0, 0	1.220000	1.642222	2.400556	4.417222
19	6	4	1, 0, 1, 0	1.180556	1.482639	1.940972	3.815972
20	7	4	1, 2, 0, 0	1.153061	1.387347	2.098061	4.019490
21	8	4	1,3,0,0	1.132813	1.323785	2.021701	3.917535
22	8	4	2,0,0,2	1.132813	1.365313	1.687188	2.171910
23	8	4	1,1,2,0	1.132813	1.323785	1.641493	3.412326

Table 2: Variances and covariances of progressively Type-II right censored order statistics from odds generalized exponential-Pareto distribution

m	s	r	$\sigma_{r,s:m:4}^{(1,1)}$	$\sigma_{r,s:m:5}^{(2,1)}$	$\sigma_{r,s:m:5}^{(1,2)}$	$\sigma_{r,s:m:5}^{(0,3)}$	$\sigma_{r,s:m:5}^{(3,0)}$
2	1	1	0.003174	0.002106	0.002106	0.002106	0.002016
	2	1	0.002917	0.001981	0.001931	0.002009	0.001811
		2	0.012702	0.006832	0.011939	0.004863	0.033986
	s	r	$\sigma_{r,s:m:6}^{(2,2)}$	$\sigma_{r,s:m:7}^{(3,2)}$			
	1	1	0.0015	0.001123			
	2	1	0.001409	0.001054			
		2	0.006347	0.006053			
m	s	r	$\sigma_{r,s:m:4}^{(0,0,1)}$	$\sigma_{r,s:m:5}^{(0,0,2)}$	$\sigma_{r,s:m:5}^{(0,1,1)}$	$\sigma_{r,s:m:5}^{(2,0,0)}$	$\sigma_{r,s:m:6}^{(1,1,1)}$
3	1	1	0.003174	0.002106	0.002106	0.002106	0.0015

	2	1	0.00299	0.002009	0.002009	0.001931	0.001429
		2	0.007709	0.004863	0.004863	0.011939	0.004343
	3	1	0.00277	0.001898	0.001853	0.00169	0.001317
		2	0.007146	0.004597	0.00449	0.010517	0.004006
		3	0.015727	0.008928	0.013576	0.038113	0.013215
	s	r	$\sigma^{(1,2,0)}_{r,s:m:6}$	$\sigma^{(1,0,2)}_{r,s:m:6}$	$\sigma^{(0,2,2)}_{r,s:m:7}$	$\sigma^{(1,2,1)}_{r,s:m:7}$	
	1	1	0.0015	0.0015	0.001123	0.001123	
	2	1	0.001429	0.001429	0.001086	0.001079	
		2	0.004343	0.004343	0.002466	0.003028	
	3	1	0.001238	0.001349	0.001022	0.000992	
		2	0.00377	0.004103	0.002323	0.002786	
		3	0.033957	0.008505	0.006971	0.012284	
m	s	r	$\sigma^{(0,0,0,1)}_{r,s:m:5}$	$\sigma^{(1,0,0,0)}_{r,s:m:5}$	$\sigma^{(1,0,1,0)}_{r,s:m:6}$	$\sigma^{(1,2,0,0)}_{r,s:m:7}$	$\sigma^{(1,3,0,0)}_{r,s:m:8}$
4	1	1	0.002106	0.002106	0.0015	0.001123	0.000872
	2	1	0.002009	0.001981	0.001429	0.001079	0.000843
		2	0.004863	0.006832	0.004343	0.003028	0.002241
	3	1	0.001893	0.001831	0.001349	0.000992	0.000774
		2	0.004597	0.006327	0.004103	0.002786	0.002058
		3	0.008928	0.015084	0.008505	0.012284	0.011758
	4	1	0.001764	0.001619	0.001183	0.000871	0.000678
		2	0.004275	0.005608	0.003603	0.002449	0.001804
		3	0.008308	0.013403	0.007479	0.010864	0.010386
		4	0.01634	0.038706	0.034963	0.010864	0.037388
	s	r	$\sigma^{(1,1,2,0)}_{r,s:m:8}$	$\sigma^{(2,0,0,2)}_{r,s:m:8}$			
	1	1	0.000872	0.000872			
	2	1	0.000843	0.000838			
		2	0.002241	0.002809			
	3	1	0.000804	0.0008			
		2	0.002139	0.002683			
		3	0.00489	0.005382			
	4	1	0.000699	0.000757			
		2	0.001861	0.002541			
		3	0.004264	0.005098			
		4	0.033455	0.009232			

6 BLUEs of μ and σ

Suppose we obtain a progressively Type-II censored data from the location-scale parameter odds generalized exponential-Pareto distribution with c.d.f. as given in (5).

In this section, we make use of means, variances and covariances of progressively Type-II right censored order statistics as determined by using the recurrence relations given in Sections 3 and 4 for deriving the BLUEs of the location and scale parameters μ and σ as well as the variances and covariance of these estimates.

Let $Y_{1:m:n} \leq Y_{2:m:n} \leq ... \leq Y_{m:m:n}$ be a progressively Type-II right censored sample from the location-scale parameter odds generalized exponential-Pareto distribution (5), and let $X_{i:m:n} = \frac{(Y_{i:m:n}-\mu)}{\sigma}$, $i = 1, 2, ..., m$, be the corresponding progressively Type-II right censored order statistics from the location-scale parameter odds generalized exponential-Pareto distribution.
Let us denote $E(X_{i:m:n})$ by μ_i, $Var(X_{i:m:n})$ by $\sigma_{i,i}$ and $Cov(X_{i:m:n}, X_{j:m:n})$ by $\sigma_{i,j}$; furthermore, let

$$Y = (Y_{1:m::n}, Y_{2:m:n}, ..., Y_{m:m:n})^T,$$

$$\mu = (\mu_1, \mu_2, ..., \mu_m)^T ,$$
$$\mathbf{1} = (\underbrace{1, 1, ..., 1})^T$$

and

$$\sum = (\sigma_{i,j}) , \ 1 \le i, \ j \le m.$$

Then, the BLUEs of μ and σ are obtained by minimizing the generalized variance $Q(\delta) = (Y - A\delta)^T \sum^T (Y - A\delta)$ with respect to δ, where $\delta = (\mu, \sigma)^T$, A is $m \times 2$ matrix $(\mathbf{1}, \mu)$, $\mathbf{1}$ is $m \times 1$ vector with components all 1's, μ is the mean vector of $\mathbf{X}$, and $\sum$ is the variance-covariance matrix of $\mathbf{X}$. The minimization leads to the expressions for the BLUE's of μ and σ as (see Anrold et al. (1992) and Balakrishnan and Cohen (1991))

$$\mu^* = \left\{ \frac{\mu^T \sum^{-1} \mu \mathbf{1}^T \sum^{-1} - \mu^T \sum^{-1} \mathbf{1} \mu^T \sum^{-1}}{(\mu^T \sum^{-1} \mu)(\mathbf{1}^T \sum^{-1} \mathbf{1}) - (\mu^T \sum^{-1} \mathbf{1})^2} \right\} Y \ = \sum_{r=1}^{m} a_r Y_{r:m:n} \tag{20}$$

and

$$\sigma^* = \left\{ \frac{1^T \sum^{-1} \mathbf{1} \mu^T \sum^{-1} - \mathbf{1}^T \sum^{-1} \mu \mathbf{1}^T \sum^{-1}}{(\mu^T \sum^{-1} \mu)(\mathbf{1}^T \sum^{-1} \mathbf{1}) - (\mu^T \sum^{-1} \mathbf{1})^2} \right\} Y \ = \sum_{r=1}^{m} b_r Y_{r:m:n}, \tag{21}$$

and the variances and covariance of these BLUEs are given by

$$Var(\mu^*) = \sigma^2 \left\{ \frac{\mu^T \sum^{-1} \mu}{(\mu^T \sum^{-1} \mu)(\mathbf{1}^T \sum^{-1} \mathbf{1}) - (\mu^T \sum^{-1} \mathbf{1})^2} \right\} = \sigma^2 V_1,$$

$$Var(\sigma^*) = \sigma^2 \left\{ \frac{\mathbf{1}^T \sum^{-1} \mathbf{1}}{(\mu^T \sum^{-1} \mu)(\mathbf{1}^T \sum^{-1} \mathbf{1}) - (\mu^T \sum^{-1} \mathbf{1})^2} \right\} = \sigma^2 V_2$$

and

$$Cov(\mu^*, \ \sigma^*) = \sigma^2 \left\{ \frac{-\mu^T \sum^{-1} \mathbf{1}}{(\mu^T \sum^{-1} \mu)(\mathbf{1}^T \sum^{-1} \mathbf{1}) - (\mu^T \sum^{-1} \mathbf{1})^2} \right\} = \sigma^2 V_3.$$

The coefficients of the BLUEs in (20) and (21) satisfy the conditions $\sum_{r=1}^{m} a_r = 1$ and $\sum_{r=1}^{m} b_r = 0$ respectively.

The coefficients of the BLUEs for μ and σ, and variances and covariance of these estimates are presented in Tables 3, 4 and 5, respectively, for various sample sizes up to n = 8 and for different choices of m and progressive censoring schemes.

7 Best Linear Unbiased Predictors (BLUPs)

Based on observations on m progressively Type-II right censored order statistics $Y_{1:m:n}^{(R_1,...,R_m)}, ..., Y_{m:m:n}^{(R_1,...,R_m)}$, we discuss the prediction of times to failure of the last $R_m(\ge 1)$ units still surviving at the observation $Y_{m:m:n}^{(R_1,...,R_m)}$. Of course, one can discuss the prediction of other censored failure times in a similar manner as well. Doganaksoy and Balakrishnan [18] established that the BLUEs remain unchanged if the BLUPs of future failures are treated as observed values.

The BLUP of $Y_{m+1:m+1:n}^{(R_1,...,R_{m-1},0,R_m-1)}$ from any location-scale family of distributions is given by

$$Y_{m+1:m+1:n}^{(R_1,...,R_{m-1},0,R_m-1)^*} = \mu^* + \mu_{m+1:m+1:n}\sigma^* + w^T\Sigma^{-1}\left(Y - \mu^*\mathbf{1} - \sigma^*\mu\right)$$

and its variance is given by

$$\sigma^2 \left\{ \sigma_{m+1,m+1:m+1:n} - \omega^T\Sigma^{-1}\omega + \lambda_1^2 1^T\Sigma^{-1}1 + \lambda_2^2\mu^T\Sigma^{-1}\mu + 2\lambda_1\lambda_2\mu^T\Sigma^{-1}1 \right\},$$

Table 3: Coefficients of the BLUEs of location parameter

S.No.	n	m	Censoring Scheme	$a_i, i = 1, 2, ..., m$			
1	4	2	1, 1	10.805950	-9.805950		
2	5	2	1, 2	15.042050	-14.042050		
3	5	2	2, 1	10.618040	-9.61804		
4	5	2	0, 3	19.44926	-18.44926		
5	5	2	3, 0	6.146378	-5.146378		
6	6	2	2, 2	14.85344	-13.85344		
7	7	2	3, 2	14.71755	-13.71755		
8	4	3	0, 0, 1	8.079980	-1.838169	-5.241810	
9	5	3	0, 0, 2	10.169140	-1.777174	-7.391970	
10	5	3	0, 1, 1	9.996353	-3.848794	-5.147559	
11	5	3	2, 0, 0	5.676797	-1.833391	-2.843406	
12	6	3	1, 1, 1	9.874548	-3.792754	-5.081794	
13	6	3	1, 2, 0	9.488743	-5.667589	-2.821154	
14	6	3	1, 0, 2	10.045720	-1.750421	-7.295299	
15	7	3	0, 2, 2	14.020600	-5.790736	-7.229864	
16	7	3	1, 2, 1	11.777880	-5.739782	-5.038098	
17	5	4	0, 0, 0, 1	7.024789	-1.121289	-1.306854	-3.596646
18	5	4	1, 0, 0, 0	5.487852	-1.107954	-1.363946	-2.015953
19	6	4	1, 0, 1, 0	6.786662	-1.075428	-2.729666	-1.981568
20	7	4	1, 2, 0, 0	7.814908	-3.513118	-1.317460	-1.984329
21	8	4	1, 1, 2, 0	9.244409	-2.289104	-4.004055	-1.951251
22	8	4	1, 3, 0, 0	8.955894	-4.677955	-1.303474	-1.974466
23	8	4	2, 0, 0, 2	8.292759	-1.085876	-1.231713	-4.975170

where

$$Y = \left(Y_{1:m+1:n}^{(R_1,...,R_{m-1},0,R_m-1)}, ..., Y_{m:m+1:n}^{(R_1,...,R_{m-1},0,R_m-1)}\right)^T,$$

$$E(Y) = \mu 1 + \sigma\mu = (\mu + \sigma\mu_{1:m+1:n}, ..., \mu + \sigma\mu_{m:m+1:n})^T$$

$$Var(Y) = \sigma^2\Sigma = \sigma^2 \begin{pmatrix} \sigma_{1,1:m+1:n} & \cdots & \sigma_{1,m:m+1:n} \\ \cdots & \cdots & \cdots \\ \sigma_{m,1:m+1:n} & \cdots & \sigma_{m,m:m+1:n} \end{pmatrix},$$

$$E(Y_{m+1:m+1:n}^{(R_1,...,R_{m-1},0,R_m-1)}) = \mu + \sigma\mu_{m+1:m+1:n},$$

$$Var\left(Y_{m+1:m+1:n}^{(R_1,...,R_{m-1},0,R_m-1)}\right) = \sigma^2\sigma_{m+1,m+1:m+1:n},$$

$$Cov\left(Y_{m+1:m+1:n}^{(R_1,...,R_{m-1},0,R_m-1)}, Y\right) = \sigma^2\omega = \sigma^2\left(\sigma_{m+1,1:m+1:n}, ..., \sigma_{m+1,m:m+1:n}\right)^T,$$

$$\lambda_1 = \frac{\mu^T\Sigma^{-1}\mu - \mu^T\Sigma^{-1}\mu w^T\Sigma^{-1}1 - \mu_{m+1:m+1:n}\mu^T\Sigma^{-1}1 + \mu^T\Sigma^{-1}1w^T\Sigma^{-1}\mu}{\Delta}$$

and

$$\lambda_2 = \frac{-\mu^T\Sigma^{-1}1 + \mu^T\Sigma^{-1}1w^T\Sigma^{-1}1 + \mu_{m+1:m+1:n}1^T\Sigma^{-1}1 + 1^T\Sigma^{-1}1w^T\Sigma^{-1}\mu}{\Delta}$$

with $\Delta = \left(\mu^T\Sigma^{-1}\mu\right)\left(1^T\Sigma^{-1}1\right) - \left(\mu^T\Sigma^{-1}1\right)^2$. Also, $\mu_{i:m+1:n}$ and $\sigma_{i,j:m+1:n}$ denote respectively the mean and covariance of the progressively Type-II right censored order statistics from the standard $(\mu = 0, \sigma = 1)$ distribution, and μ^* and σ^* are the BLUEs of μ and σ based on the progressively Type-II censoed sample Y. The BLUPs and their variances can therefore be readily computed from the means, variances and covariances of the progressively Type-II right censoed order statistics produced in Section 5. It is also illustrated in the next section with two numerical examples one using real data and another one using simulated data set.

Table 4: Coefficients of the BLUEs of scale parameter

S.No.	n	m	Censoring Scheme	$b_i, i = 1, 2, ..., m$			
1	4	2	1, 1	-9.258205	9.258205		
2	5	2	1, 2	-13.40141	13.40141		
3	5	2	2, 1	-9.179234	9.179234		
4	5	2	0, 3	-17.60754	17.60754		
5	5	2	3, 0	-4.911583	4.911583		
6	6	2	2, 2	-13.31918	13.31918		
7	7	2	3, 2	-13.25936	13.25936		
8	4	3	0, 0, 1	-6.686697	1.739272	4.947425	
9	5	3	0, 0, 2	-8.752319	1.698793	7.053526	
10	5	3	0, 1, 1	-8.587998	3.676425	4.911573	
11	5	3	2, 0, 0	-4.465068	1.752332	2.712736	
12	6	3	1, 1, 1	-8.533840	3.648865	4.884975	
13	6	3	1, 2, 0	-8.163935	5.423240	2.711611	
14	6	3	1, 0, 2	-8.697977	1.684909	7.013068	
15	7	3	0, 2, 2	-12.587010	5.599336	6.987674	
16	7	3	1, 2, 1	-10.419380	5.550225	4.869155	
17	5	4	0, 0, 0, 1	-5.753108	1.073185	1.249299	3.430625
18	5	4	1, 0, 0, 0	-4.286544	1.060581	1.303556	1.922407
19	6	4	1, 0, 1, 0	-5.566479	1.036335	2.626133	1.904011
20	7	4	1, 2, 0, 0	-6.590625	3.398970	1.274528	1.917127
21	8	4	1, 1, 2, 0	-8.004078	2.223580	3.887424	1.893073
22	8	4	1, 3, 0, 0	-7.724730	4.543236	1.265933	1.915561
23	8	4	2, 0, 0, 2	-7.079321	1.055248	1.196429	4.827644

8 Illustrative Examples

Example 1 *Consider the data, produced in Nelson [23], giving the log-times to breakdown of an insulating fluid in an accelerated test at 28kv.*

i	1	2	3
$y_{i:m:n}$	4.2319	4.6848	4.7031
R_i	0	0	2

In this case, we have $n = 5,\ m = 3$*. The correlation between above censored values and the corresponding expected values in Table 1 for* $n = 5,\ m = 3$ *with censoring scheme* $(0, 0, 2)$ *is* 0.850108*, which indicates a quite high degree of correlation. Hence assuming the data have come from odds generalized exponential-Pareto distribution as given in* (5)*, with specific values of parameters, viz.* $\alpha = 1,\ \lambda = 2,\ \theta = 2$*, based on the progressively Type-II right censored sample* $y_{1:3:5},\ y_{2:3:5},\ y_{3:3:5}$ *presented above, we find the BLUEs of* μ *and* σ *to be*

$$\begin{aligned}\mu^* &= 4.2319 \times 10.169140 + 4.6848 \times (-1.777174) + 4.7031 \times (-7.391970)\\ &= -0.0560953,\\ \sigma^* &= 4.2319 \times (-8.752319) + 4.6848 \times 1.698793 + 4.7031 \times 7.053526\\ &= 4.093005\end{aligned}$$

respectively, and their standard errors to be

$$SE(\mu^*) = \sigma^* \sqrt{\frac{\mu^T \Sigma^{-1} \mu}{(\mu^T \Sigma^{-1} \mu)(1^T \Sigma^{-1} 1) - (\mu^T \Sigma^{-1} 1)^2}} = 2.846996,$$

Table 5: Variances and covariance of the BLUEs when $\mu = 0$ and $\sigma = 1$

S.no.	n	m	Censoring Scheme	$Var(\mu^*)$	$var(\sigma^*)$	$Cov(\mu^*, \sigma^*)$
1	4	2	1, 1	0.973789	0.860719	-0.914020
2	5	2	1, 2	0.986913	0.893795	-0.938200
3	5	2	2, 1	0.947471	0.858004	-0.900628
4	5	2	0, 3	1.010458	0.915163	-0.960630
5	5	2	3, 0	0.865161	0.783330	-0.822228
6	6	2	2, 2	0.969164	0.892138	-0.929135
7	7	2	3, 2	0.956628	0.890970	-0.922674
8	4	3	0, 0, 1	0.479645	0.420352	-0.447539
9	5	3	0, 0, 2	0.483826	0.435652	-0.458110
10	5	3	0, 1, 1	0.474829	0.427520	-0.449555
11	5	3	2, 0, 0	0.431041	0.387949	-0.407930
12	6	3	1, 1, 1	0.465223	0.426427	-0.444686
13	6	3	1, 2, 0	0.445339	0.408157	-0.425625
14	6	3	1, 0, 2	0.474093	0.434579	-0.453189
15	7	3	0, 2, 2	0.478834	0.444566	-0.460841
16	7	3	1, 2, 1	0.464892	0.431595	-0.447394
17	5	4	0, 0, 0, 1	0.307085	0.274851	-0.289527
18	5	4	1, 0, 0, 0	0.287728	0.257431	-0.271164
19	6	4	1, 0, 1, 0	0.292460	0.266886	-0.278665
20	7	4	1, 2, 0, 0	0.285009	0.263689	-0.273603
21	8	4	1, 1, 2, 0	0.294093	0.274922	-0.283944
22	8	4	1, 3, 0, 0	0.283963	0.265431	-0.274119
23	8	4	2, 0, 0, 2	0.303245	0.283494	-0.292782

$$SE(\sigma^*) = \sigma^* \sqrt{\frac{1^T \Sigma^{-1} 1}{(\mu^T \Sigma^{-1} \mu)\,(1^T \Sigma^{-1} 1) - (\mu^T \Sigma^{-1} 1)^2}} = 2.701575.$$

We obtain the BLUP of $y_{4:4:5}^{(0,0,0,1)}$ *to be* $y_{4:4:5}^{(0,0,0,1)^*} = 5.111442$ *(by taking* $w = (\sigma_{m+1,1:m+1:n}, ..., \sigma_{m+1,m:m+1:n})^T$ *and progressive censoring scheme* $(R_1, ..., R_{m-1}, 0, R_m - 1)$*).*

Example 2 *The following progressively Type-II right censored sample from the odds generalized exponential-Pareto distribution, with parameters, viz.* $\alpha = 1$*,* $\lambda = 2$*,* $\theta = 2$*, is simulated with* $m = 2$*,* $n = 5$*, location parameter* $\mu = 0$*, scale parameter* $\sigma = 1$ *and censoring scheme* $(0, 3) : 1.019693, 1.074034$*. We find BLUEs of* μ *and* σ *to be*

$$\begin{aligned}\mu^* &= 1.019693 \times 19.44926 + 1.074034 \times (-18.44926)\\ &= 0.01714176,\\ \sigma^* &= 1.019693 \times (-17.60754) + 1.074034 \times 17.60754\\ &= 0.9568113\end{aligned}$$

respectively, and their standard errors to be $SE(\mu^*) = 0.961801$ *and* $SE(\sigma^*) = 0.915325$*. We obtain the BLUP of* $y_{3:3:5}^{(0,0,2)}$ *to be* $y_{3:3:5}^{(0,0,2)^*} = 1.142256$ *(by taking* $w = (\sigma_{m+1,1:m+1:n}, ..., \sigma_{m+1,m:m+1:n})^T$ *and progressive censoring scheme* $(R_1, ..., R_{m-1}, 0, R_m - 1)$*). As established by Doganaksoy and Balakrishnan [18] this prediction value, now, can be used as observed value and we can predict further future values. So, by taking* $w = (\sigma_{m+2,1:m+2:n}, ..., \sigma_{m+2,m+1:m+2:n})^T$*,* $\mu_{m+2:m+2:n}$ *in place of* $\mu_{m+1:m+1:n}$ *and the progressive censoring scheme as* $(R_1, ..., R_{m-1}, 0, 0, R_m - 2)$*, we can predict the second failure time i.e. BLUP of* $y_{4:4:5}^{(0,0,0,1)}$ *to be* $y_{4:4:5}^{(0,0,0,1)^*} = 1.236965$.

9 Conclusion

In this paper, we have established several recurrence relations for the single and product moments of progressively Type-II right censored order statistics from odds generalized exponential-Pareto distribution. With the help of these relations and using R software, we have computed all the means, variances and covariances of progressively Type-II right censored order statistics for different sample sizes and all possible censoring schemes. These moments have then been used to obtain the best linear unbiased estimators (BLUEs) of location and scale parameters of location-scale odds generalized exponential-Pareto distribution (5), as well as the best linear unbiased predictors (BLUPs) of the times to failure of the surviving units in the experiment. Finally, two numerical examples, one with real data and another with simulated data, have been presented to illustrate the estimation and prediction methods discussed in this paper.

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