Applied Mathematics E-Notes, 21(2021), 421-425 © ISSN 1607-2510
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On The Shafer-Type Inequality For The Inverse Tangent Function*

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Received 6 June 2020

Abstract

We give a simple generalization of double-sided Shafer-type inequality and sharpen the left one on an infinite interval. The right inequality is also sharpened on a small finite interval.

1 Introduction and Preliminaries

In 1966, Robert E. Shafer [2] proposed the inequality

$$\frac{3}{1+2\sqrt{1+x^2}} < \frac{\arctan x}{x};\ x>0 \tag{1}$$

as a problem in the journal *American Mathematical Monthly.* Here arctan denotes inverse tangent function. In the subsequent year, 1967, Shafer, Grinstein, Marsh and Konhauser [3] proved inequality (1) independently by different methods. Shafer, further in 1977-78 [4, 5] proved that, for $x>0$ the inequality

$$\frac{8}{3+\sqrt{25+\frac{80}{3}x^2}} < \frac{\arctan x}{x}. \tag{2}$$

Of course the inequality (2) is sharper than (1). This inequality (2) was extended to

$$\frac{8}{3+\sqrt{25+\frac{80}{3}x^2}} < \frac{\arctan x}{x} < \frac{8}{3+\sqrt{25+\frac{256}{\pi^2}x^2}};\ x>0 \tag{3}$$

by L. Zhu [6] in 2008, and named as Shafer-type inequality. In the above inequality (3), author of [6] pointed that the numbers $80/3$ and $256/\pi^2$ are the best constants. In this paper, we present generalization of inequalities listed above by finding necessary and sufficient conditions. One of the particular cases will refine the lower bound in (3) on an infinite interval whereas the upper bound on a small finite interval.

For proving our main result we basically employ the same method used in [6] with parameters and we require following series expansions for this purpose.

$$\cot t = \frac{1}{t} - \sum_{k=1}^{+\infty} \frac{2^{2k}}{(2k)!}|B_{2k}|t^{2k-1};\ |t|<\pi \tag{4}$$

where B_{2k} are the even indexed Bernoulli numbers. For expansion (4), we refer to [1, 1.411]. From this one can easily obtain the following:

$$\csc^2 t = -\frac{d(\cot t)}{dt} = \frac{1}{t^2} + \sum_{k=1}^{+\infty} \frac{2^{2k}}{(2k)!}(2k-1)|B_{2k}|t^{2k-2};\ |t|<\pi \tag{5}$$

where csc denotes the cosecant function. And using the identity $\cot^2 t = \csc^2 t - 1$ we have

$$\cot^2 t = \frac{1}{t^2} - \frac{2}{3} + \sum_{k=2}^{+\infty} \frac{2^{2k}}{(2k)!}(2k-1)|B_{2k}|t^{2k-2};\ |t|<\pi. \tag{6}$$

*Mathematics Subject Classifications: 26D05, 26D20, 33B10.

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2 Main Results

First of all, it is to be noted that for any function of the type $a/(b+\sqrt{c+dx^2})$ to be sharp bound for $\arctan x/x$, it's limit as x tends to 0 should be 1. This forces the equality $c=(a-b)^2$ which enables us to formulate the following.

Theorem 1 *Let $a>b>0$, $c=(a-b)^2$ and the function $f_{a,b,c}(x)=\frac{\left(\frac{ax}{\arctan x}-b\right)^2-c}{x^2}$ be defined on $(0,+\infty)$. Then*

I. $f_{a,b,c}(x)$ is strictly decreasing on $(0,+\infty)$ if and only if $3a\geqslant 8b$ and

II. $f_{a,b,c}(x)$ is strictly increasing on $(0,+\infty)$ if and only if $a\leqslant 2b$.

Proof. Let $\arctan x=t;\ x>0$. Then $x=\tan t;\ t\in(0,\pi/2)$ and

$$f_{a,b,c}(x)=\frac{\left(\frac{ax}{\arctan x}-b\right)^2-c}{x^2}=\frac{\left(\frac{a\tan t}{t}-b\right)^2-c}{\tan^2 t}:=g_{a,b,c}(t).$$

After simple calculations we have

$$g_{a,b,c}(t)=\frac{a^2-2abt\cot t+(b^2-c)t^2\cot^2 t}{t^2}.$$

Making use of (4) and (5), we write $g(t)$ as

$$\begin{aligned}g_{a,b,c}(t)&=\left(a^2-2ab+2ab\sum_{k=1}^{+\infty}\frac{2^{2k}}{(2k)!}|B_{2k}|t^{2k}+(b^2-c)-\frac{2}{3}(b^2-c)t^2\right.\\&\left.+(b^2-c)\sum_{k=2}^{+\infty}\frac{2^{2k}}{(2k)!}(2k-1)|B_{2k}|t^{2k}\right)\Big/t^2\\&=\frac{\left[(a-b)^2-c\right]+\left[\frac{2ab}{3}-\frac{2(b^2-c)}{3}\right]t^2+\sum_{k=2}^{+\infty}\frac{2^{2k}}{(2k)!}\left[2ab+(2k-1)(b^2-c)\right]|B_{2k}|t^{2k}}{t^2}\\&=\frac{2ab}{3}-\frac{2(b^2-c)}{3}+\sum_{k=2}^{+\infty}\frac{2^{2k}}{(2k)!}\left[2ab+(2k-1)(b^2-c)\right]|B_{2k}|t^{2k-2}\end{aligned}$$

since $(a-b)^2-c=0$. It has derivative

$$g'_{a,b,c}(t)=\sum_{k=2}^{+\infty}\frac{2^{2k}}{(2k)!}(2k-2)\left[2ab+(2k-1)(b^2-c)\right]|B_{2k}|t^{2k-3}.$$

Now $g_{a,b,c}(t)$ is strictly decreasing if and only if the derivative $g'_{a,b,c}(t)<0$. This means

$$2ab+(2k-1)(b^2-c)<0 \text{ i.e., } 2k(2ab-a^2)+a^2<0.$$

After rearrangement of terms we get

$$\frac{b}{a}<\frac{(2k-1)}{4k}:=h(k),$$

where it is not difficult to show that $h(k)$ is increasing for $k\geqslant 2$. Whence

$$\frac{b}{a}\leqslant\inf\{h(k):k\geqslant 2\}=\frac{3}{8}.$$

By making similar argument, we can say that $g(t)$ is strictly increasing if and only if

$$\frac{b}{a}>\frac{(2k-1)}{4k}=h(k).$$

$$\text{i. e., } \frac{b}{a} \geqslant \sup\{h(t) : t \geqslant 2\} = f(+\infty^-) = \frac{1}{2}.$$

Thus our assertion is proved. ■

Corollary 1 *If $a > b > 0$ are such that $c = (a-b)^2$ and $3a \geqslant 8b$, then for $x > 0$ the inequalities*

$$\frac{a}{b + \sqrt{c + \frac{2a}{3}(a-b)x^2}} < \frac{\arctan x}{x} < \frac{a}{b + \sqrt{c + \frac{4a^2}{\pi^2}x^2}} \tag{7}$$

hold with the best possible constants $\frac{2a}{3}(a-b)$ and $\frac{4a^2}{\pi^2}$.

Proof. By Theorem 1, for $3a \geqslant 8b$ we have

$$f_{a,b,c}(0^+) > f_{a,b,c}(x) = \frac{\left(\frac{ax}{\arctan x} - b\right)^2 - c}{x^2} > f_{a,b,c}(+\infty^-).$$

The limits $f_{a,b,c}(0^+) = g_{a,b,c}(0^+) = \frac{2a}{3}(a-b)$ and $f_{a,b,c}(+\infty^-) = g_{a,b,c}(\frac{\pi}{2}^-) = \frac{4a^2}{\pi^2}$ give inequalities (7). ■

Corollary 2 *If $a > b > 0$ are such that $c = (a-b)^2$ and $a \leqslant 2b$, then for $x > 0$ the inequalities*

$$\frac{a}{b + \sqrt{c + \frac{4a^2}{\pi^2}x^2}} < \frac{\arctan x}{x} < \frac{a}{b + \sqrt{c + \frac{2a}{3}(a-b)x^2}} \tag{8}$$

hold with the best possible constants $\frac{4a^2}{\pi^2}$ and $\frac{2a}{3}(a-b)$.

Proof. By Theorem 1, for $a \leqslant 2b$ we have

$$f_{a,b,c}(0^+) < f_{a,b,c}(x) = \frac{\left(\frac{ax}{\arctan x} - b\right)^2 - c}{x^2} < f_{a,b,c}(+\infty^-).$$

The limits as in the proof of Corollary 1 give inequalities (8). ■

Remark 1 *Shafer's inequality (1) is a particular case of Corollary 1 where $a = 3$ and $b = 1$. Shafer-type inequality (3) is also a particular case of Corollary 1 where $a = 8$ and $b = 3$.*

All the inequalities in Corollaries 1 and 2 are sharp enough to satisfy our needs. It is observed that among all the inequalities of type (7) the sharpest one is obtained by taking $3a = 8b$. Thus the Shafer-type inequality (3) is sharpest of type (7). Similarly the sharpest inequality of type (8)(obtained by taking $a = 2b$) is given by

$$\frac{2}{1 + \sqrt{1 + \frac{16}{\pi^2}x^2}} < \frac{\arctan x}{x} < \frac{2}{1 + \sqrt{1 + \frac{4}{3}x^2}};\ x > 0. \tag{9}$$

Now it becomes important to compare inequalities (3) and (9). We compare them in the following propositions. First we compare lower bounds.

Proposition 1 *For $\lambda = \frac{\pi\sqrt{180-15\pi^2}}{5\pi^2-48} \approx 13.174333$, the inequalities*

$$\frac{2}{1 + \sqrt{1 + \frac{16}{\pi^2}x^2}} \leqslant \frac{8}{3 + \sqrt{25 + \frac{80}{3}x^2}};\ x \in (0, \lambda] \tag{10}$$

and

$$\frac{8}{3 + \sqrt{25 + \frac{80}{3}x^2}} \leqslant \frac{2}{1 + \sqrt{1 + \frac{16}{\pi^2}x^2}};\ x \in [\lambda, +\infty) \tag{11}$$

are true with equalities at $x = \lambda$.

Proof. Let

$$F(x) = \sqrt{25 + \frac{80}{3}x^2} - \sqrt{16 + \frac{256}{\pi^2}x^2} - 1.$$

The derivative of $F(x)$ is

$$F'(x) = 16x \frac{5\pi^2\sqrt{16 + \frac{256}{\pi^2}x^2} - 48\sqrt{25 + \frac{80}{3}x^2}}{3\pi^2\sqrt{16 + \frac{256}{\pi^2}x^2}\sqrt{25 + \frac{80}{3}x^2}}.$$

From this, $F'(x) > 0$ implies

$$25\pi^4\left(16 + \frac{256}{\pi^2}x^2\right) > 2304\left(25 + \frac{80}{3}x^2\right)$$

i.e.,

$$x > \sqrt{\frac{57600 - 400\pi^4}{6400\pi^2 - 61440}} \approx 3.286451 = \mu(\text{say}),$$

whereas $F'(x) < 0$ implies $x < \mu$ which help us to conclude that $F(x)$ is decreasing in $(0, \mu)$ and increasing in $(\mu, +\infty)$. Since $F(x)$ is negative in $(0, \mu)$ as $F(0^+) = 0 > F(x);\ x \in (0, \mu)$, increasing in $(\mu, +\infty)$ and $F(+\infty^-) = +\infty$, there exists a unique solution of $F(x) = 0$ in $(\mu, +\infty)$. This solution can be found with the help of Maple as $\lambda = \pi\sqrt{180 - 15\pi^2}/(5\pi^2 - 48) \approx 13.174333$. It is now clear that $F(x)$ is decreasing in $(0, \mu)$ and increasing in (μ, λ) and we have $F(x) < F(\lambda^-) = 0;\ x \in (\mu, \lambda)$ too. Thus $F(x) < 0;\ x \in (0, \lambda)$ gives first inequality (10). Similarly $F(x)$ being increasing in $(\lambda, +\infty) \subset (\mu, +\infty)$ we have $F(x) > F(\lambda) = 0$. This gives second inequality (11) and the proof is complete. ∎

Next we compare upper bounds of (3) and (9).

Proposition 2 *For* $\xi = \frac{\pi\sqrt{15\pi^2 - 144}}{48 - 4\pi^2} \approx 0.741376$, *the inequalities*

$$\frac{2}{1 + \sqrt{1 + \frac{4}{3}x^2}} \leqslant \frac{8}{3 + \sqrt{25 + \frac{256}{\pi^2}x^2}};\ x \in (0, \xi] \tag{12}$$

and

$$\frac{8}{3 + \sqrt{25 + \frac{256}{\pi^2}x^2}} \leqslant \frac{2}{1 + \sqrt{1 + \frac{4}{3}x^2}};\ x \in [\xi, +\infty) \tag{13}$$

are true with equalities at $x = \xi$.

Proof. Let

$$G(x) = \sqrt{25 + \frac{256}{\pi^2}x^2} - \sqrt{16 + \frac{64}{3}x^2} - 1.$$

The derivative of $G(x)$ is

$$G'(x) = 16x \frac{48\sqrt{1 + \frac{4}{3}x^2} - \pi^2\sqrt{25 + \frac{256}{\pi^2}x^2}}{3\pi^2\sqrt{1 + \frac{4}{3}x^2}\sqrt{25 + \frac{256}{\pi^2}x^2}}.$$

Now $G'(x) > 0$ implies

$$2304\left(1 + \frac{4}{3}x^2\right) > \pi^4\left(25 + \frac{256}{\pi^2}x^2\right)$$

i.e.,

$$x > \sqrt{\frac{25\pi^4 - 2304}{3072 - 256\pi^2}} \approx 0.490526 := \rho,$$

whereas $G'(x) < 0$ implies $x < \rho$. Hence we conclude that $G(x)$ is decreasing in $(0, \rho)$ and increasing in $(\rho, +\infty)$. Since $G(x)$ is negative in $(0, \rho)$ as $G(0^+) = 0 > G(x)$; $x \in (0, \rho)$, increasing in $(\rho, +\infty)$ and $G(+\infty^-) = +\infty$, there exists a unique solution of $G(x) = 0$ in $(\rho, +\infty)$. This solution can be found with the help of Maple as $\xi = \pi\sqrt{15\pi^2 - 144}/(48 - 4\pi^2) \approx 0.741376$. From this it is clear that $G(x)$ is decreasing in $(0, \rho)$ and increasing in (ρ, ξ) and we have $G(x) < G(\xi^-) = 0$; $x \in (\rho, \xi)$ too. Thus $G(x) < 0$; $x \in (0, \xi)$ gives first inequality (12). Similarly $G(x)$ being increasing in $(\xi, +\infty) \subset (\rho, +\infty)$ we have $G(x) > G(\xi) = 0$. This gives second inequality (13). ■

2.1 Figures

A graphical comparison of lower bounds, as well as upper bounds of (3) and (9), can be seen in the following figures. Considering the sharpness of both the types of bounds and limitations for taking infinite intervals, comparison is done only in the small intervals where the points of intersections of corresponding bounds lie.

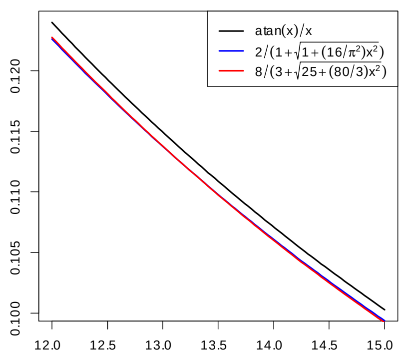

Figure 1: Graphs of lower bounds of (3) and (9) for $x \in (12, 15)$.

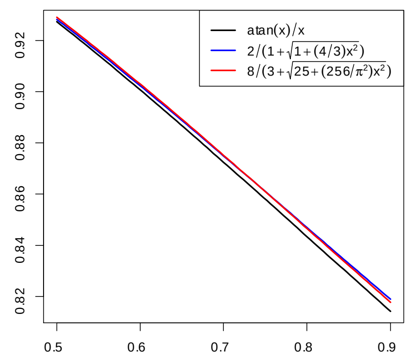

Figure 2: Graphs of upper bounds of (3) and (9) for $x \in (0.5, 0.9)$.

Acknowledgment. The author would like to thank the anonymous referee for his/her careful reading of the manuscript and constructive comments. This small piece of work is dedicated to the memory of my late sister Ranjana.

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