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On The Zero Bounds Of Polynomials And Related Analytic Functions*

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Abstract

In this paper, by relaxing the hypothesis of well-known Eneström-Kakeya theorem, we obtain a result which is applicable to a larger class of polynomials and generalizes several well-known results concerning the location of zeros of polynomials. In addition to this, we also obtain a similar result for analytic functions.

1 Introduction

For a polynomial $P(z)$ of degree n, the Fundamental Theorem of Algebra states that $P(z)$ has exactly n zeros (counting multiplicity) in the complex plane, but it does not provide any information about the location of zeros. On the other hand, N. H. Abel and E. Galois proved that there is no general method for the exact computation of zeros of polynomials of degree 5 or greater. In view of this and significant applications of zero bounds in scientific disciplines such as stability theory, mathematical biology, communication theory and computer engineering, it became interesting to identify the suitable regions in the complex plane which contain the zeros of a given polynomial. In this direction, the classical result due to Cauchy [3] states that:

Theorem 1 *Let $P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ be a polynomial of degree n. Then all the zeros of $P(z)$ lie in the disk $|z| \le 1 + \max_{0\le j\le n-1} |a_j|$.*

The following result on the location of zeros of a polynomial with restricted coefficients is known as Eneström-Kakeya Theorem (see [5], [6]) which states that:

Theorem 2 *Let $P(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ be a polynomial of degree n with real coefficients satisfying $a_n \ge a_{n-1} \ge \cdots \ge a_1 \ge a_0 \ge 0$. Then all the zeros of $P(z)$ lie in $|z| \le 1$.*

Aziz and Zargar [2] relaxed the hypothesis of Theorem 2 by proving the following results:

Theorem 3 *Let $P(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ be a polynomial of degree n with real coefficients such that for some $k \ge 1$, $ka_n \ge a_{n-1} \ge \cdots \ge a_1 \ge a_0$. Then all the zeros of $P(z)$ lie in*

$$|z + k - 1| \le \frac{ka_n - a_0 + |a_0|}{|a_n|}.$$

W. M. Shah and A. Liman [9] extended Theorem 3 to the polynomials with complex coefficients by proving that:

Theorem 4 *Let $P(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ be a polynomial of degree n with complex coefficients such that for some real β, $|\arg a_j - \beta| \le \alpha \le \pi/2$, $j = 0, 1, 2, \ldots, n$ and $k \ge 1$, $k|a_n| \ge |a_{n-1}| \ge \cdots \ge |a_1| \ge |a_0|$. Then all the zeros of $P(z)$ lie in*

$$|z + k - 1| \le \frac{1}{|a_n|}\left\{(k|a_n| - |a_0|)(\sin\alpha + \cos\alpha) + |a_0| + 2\sin\alpha \sum_{j=0}^{n-1} |a_j|\right\}.$$

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Recently Rather et al. [7] relaxed the hypothesis of Theorem 3 and proved that:

Theorem 5 *Let $P(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ be a polynomial of degree n with real coefficients such that for some $k_j \geq 1$, $j = 0, 1, 2, \ldots, r$ where $1 \leq r \leq n-1$,*

$$k_0 a_n \geq k_1 a_{n-1} \geq k_2 a_{n-2} \geq \cdots \geq k_r a_{n-r} \geq a_{n-r-1} \geq \cdots \geq a_1 \geq a_0 \geq 0.$$

Then all the zeros of $P(z)$ lie in

$$\left| z + k_0 - 1 - (k_1 - 1)\frac{a_{n-1}}{a_n} \right| \leq \frac{1}{a_n}\left(k_0 a_n - (k_1 - 1)a_{n-1} + 2\sum_{j=1}^{r}(k_j - 1)a_{n-j} \right).$$

Since the results discussed above are applicable to a small class of polynomials, so it is interesting to look for the results applicable to the larger class of polynomials. In this paper, we extend theorem 5 to the polynomials with complex coefficients and thereby, obtain a result with relaxed hypothesis that gives zero bounds of the polynomials with complex coefficients. More precisely, we prove

Theorem 6 *Let $P(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ be a polynomial of degree n with complex coefficients such that for some real β, $|\arg a_j - \beta| \leq \alpha \leq \pi/2$, $j = 0, 1, 2, \ldots, n$ and $k_j \geq 1$, $j = 0, 1, \ldots, r$ where $1 \leq r \leq n-1$,*

$$k_0|a_n| \geq k_1|a_{n-1}| \geq k_2|a_{n-2}| \geq \cdots \geq k_r|a_{n-r}| \geq |a_{n-r-1}| \geq \cdots \geq |a_1| \geq |a_0|. \tag{1}$$

Then all the zeros of $P(z)$ lie in

$$\begin{aligned}\left| z + k_0 - 1 - (k_1 - 1)\frac{a_{n-1}}{a_n} \right| \quad \leq \quad & \frac{1}{|a_n|}\Bigg\{ (k_0|a_n| - |a_0|)(\cos\alpha + \sin\alpha) + 2\sin\alpha\left(\sum_{j=1}^{r} k_j|a_{n-j}| + \sum_{j=r+1}^{n}|a_{n-j}| \right) \\ & (k_1 - 1)|a_{n-1}| + 2\sum_{j=1}^{r}(k_j - 1)|a_{n-j}| + |a_0| \Bigg\}.\end{aligned}$$

Applying Theorem 6 to the polynomial $P(tz)$, we obtain the following result:

Corollary 7 *Let $P(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ be a polynomial of degree n with complex coefficients such that for some real β, $|\arg a_j - \beta| \leq \alpha \leq \pi/2$, $j = 0, 1, 2, \ldots, n$, $t > 0$ and $k_j \geq 1$, $j = 0, 1, \ldots, r$ where $1 \leq r \leq n-1$,*

$$k_0 t^n|a_n| \geq k_1 t^{n-1}|a_{n-1}| \geq k_2 t^{n-2}|a_{n-2}| \geq \cdots \geq k_r t^{n-r}|a_{n-r}| \geq t^{n-r-1}|a_{n-r-1}| \geq \cdots \geq t|a_1| \geq |a_0|.$$

Then all the zeros of $P(z)$ lie in

$$\begin{aligned}\left| z + (k_0 - 1)t - (k_1 - 1)\frac{a_{n-1}}{a_n} \right| \quad \leq \quad & \frac{t}{|a_n|}\Bigg\{ \left(k_0|a_n| - \frac{|a_0|}{t^n} \right)(\cos\alpha + \sin\alpha) + \frac{|a_0|}{t^n} + 2\sum_{j=1}^{r}(k_j - 1)\frac{|a_{n-j}|}{t^j} \\ & -(k_1 - 1)|a_{n-1}| + 2\sin\alpha\left(\sum_{j=1}^{r} k_j\frac{|a_{n-j}|}{t^j} + \sum_{j=r+1}^{n}\frac{|a_{n-j}|}{t^j} \right) \Bigg\}.\end{aligned}$$

Taking $r = 1$ in Theorem 6, we get:

Corollary 8 *Let $P(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ be a polynomial of degree n with complex coefficients such that for some real β, $|\arg a_j - \beta| \leq \alpha \leq \pi/2$, $j = 0, 1, 2, \ldots, n$ and $k_0, k_1 \geq 1$,*

$$k_0|a_n| \geq k_1|a_{n-1}| \geq |a_{n-2}| \geq |a_{n-3}| \geq \cdots \geq |a_1| \geq |a_0|.$$

Then all the zeros of $P(z)$ lie in

$$\left|z+k_0-1-(k_1-1)\frac{a_{n-1}}{a_n}\right| \leq \frac{1}{|a_n|}\Bigg\{(k_0|a_n|-|a_0|)(\cos\alpha+\sin\alpha)+2\sin\alpha\Bigg(k_1|a_{n-1}|+\sum_{j=2}^{n}|a_{n-j}|\Bigg) +(k_1-1)|a_{n-1}|+|a_0|\Bigg\}.$$

Remark 1 *For $k_1=1$, Corollary 8 reduces to Theorem 4.*

For $\alpha=\beta=0$, Theorem 6 reduces to Theorem 5. Now we turn to the study of the zeros of a class of analytic functions. The following result was proved by A. Aziz and Shah [1]:

Theorem 9 *Let $f(z)=\sum_{j=0}^{\infty}a_jz^j\not\equiv 0$ be analytic in $|z|\leq t$. If there exist $k\geq 1$ and $t>0$ such that*

$$ka_0\geq ta_1\geq t^2a_2\geq t^3a_3\geq\ldots,\qquad a_j>0 \text{ and } j=0,1,2,3\ldots,$$

then $f(z)$ does not vanish in the region

$$\left|z-\frac{(k-1)t}{2k-1}\right|<\frac{kt}{2k-1}.$$

Shah and Liman [9] generalized Theorem 9 and proved that:

Theorem 10 *Let $f(z)=\sum_{j=0}^{\infty}a_jz^j\not\equiv 0$ be analytic in $|z|\leq t$. If there exist $k\geq 1$ and $t>0$ such that*

$$k\,|a_0|\geq t\,|a_1|\geq t^2\,|a_2|\geq t^3\,|a_3|\geq\ldots,$$

and there exists β such that $|\arg a_j-\beta|\leq\alpha\leq\pi/2$ for $j=0,1,2,\ldots,$ then $f(z)$ does not vanish in the region

$$\left|z-\frac{(k-1)t}{M^2-(k-1)^2}\right|<\frac{Mt}{M^2-(k-1)^2},$$

where

$$M=k(\sin\alpha+\cos\alpha)+\frac{2\sin\alpha}{|a_0|}\sum_{j=1}^{\infty}t^j|a_j|.$$

Here we propose to relax the hypothesis of Theorem 10. More precisely, we prove the following result:

Theorem 11 *Let $f(z)=\sum_{j=0}^{\infty}a_jz^j\not\equiv 0$ be analytic in $|z|\leq t$. If there exists $k_1,k_2\geq 1$ and $t>0$ such that*

$$k_1|a_0|\geq k_2t|a_1|\geq t^2|a_2|\geq t^3|a_3|\geq\ldots$$

and there exists β such that $|\arg a_j-\beta|\leq\alpha\leq\pi/2$ for $j=0,1,2,\ldots,$ then $f(z)$ does not vanish in the region

$$\left|z-\frac{(k_1-1)t}{B^2-(k_1-1)^2}\right|<\frac{Bt}{B^2-(k_1-1)^2},$$

where

$$B=k_1(\sin\alpha+\cos\alpha)+2t(k_2-1)\frac{|a_1|}{|a_0|}+\frac{2\sin\alpha}{|a_0|}\left(k_2t|a_1|+\sum_{j=2}^{\infty}t^j|a_j|\right).$$

Remark 2 *On setting $k_2 = 1$, Theorem 11 reduces to Theorem 10.*

For $\alpha = \beta = 0$, Theorem 11 yields the following result which includes Theorem 9 as a special case.

Corollary 12 *Let $f(z) = \sum_{j=0}^{\infty} a_j z^j \not\equiv 0$ be analytic in $|z| \leq t$. If there exists $k_1, k_2 \geq 1$ and $t > 0$ such that*

$$k_1 a_0 \geq k_2 t a_1 \geq t^2 a_2 \geq t^3 a_3 \geq \ldots,$$

then $f(z)$ does not vanish in the region

$$\left| z - \frac{(k_1 - 1)t}{B^{*2} - (k_1 - 1)^2} \right| < \frac{B^* t}{B^{*2} - (k_1 - 1)^2},$$

where $B^* = k_1 + 2t(k_2 - 1)a_1/a_0$.

2 Computations and Analysis

In this section, some examples are given to show how the two main theorems are used in practice. First we give two examples of polynomials not only to show how Theorem 6 is used but also to show that Theorem 6 gives better information about the location of the zeros than Theorem 1. It is worth mentioning that all existing Eneström-Kakeya type results are not applicable for these polynomials.

Example 1 *Let $P(z) = 3z^4 + 2.8z^3 + 2.6z^2 + 3.2z + 1$. By taking $\beta = \alpha = 0$ and $r = 2$ with $k_0 = 16/15$, $k_1 = 8/7$ and $k_2 = 16/13$ in Theorem 6, it follows that all the zeros of $P(z)$ lie in the disc $|z - 1/15| \leq 1.6$. Whereas, if we use Theorem 1, it follows that all the zeros of $P(z)$ lie in the disc $|z| < 2.06$. Thus, Theorem 6 gives better bound, with 40% improvement in the area over Theorem 1.*

Example 2 *Let*

$$\begin{aligned} P(z) &= 2\sqrt{2}(1+i)z^5 + 1.9\sqrt{2}(1+i)z^4 + 3.5(1/\sqrt{2} + i/\sqrt{2})z^3 + 2\sqrt{2}(1+i)z^2 \\ &\quad + \sqrt{2}(1+i)z + 1/\sqrt{2} + i/\sqrt{2}. \end{aligned}$$

By taking $\beta = \pi/4, \alpha = 0$ and $r = 2$ with $k_0 = 1$, $k_1 = 20/19$ and $k_2 = 8/7$ in Theorem 6, it follows that all the zeros of $P(z)$ lie in the disc $|z - 1/20| \leq 1.3$. Whereas, if we use Theorem 1, it follows that all the zeros of $P(z)$ lie in the disc $|z| < 2$. Thus, Theorem 6 gives better bound, with 58% improvement in the area over Theorem 1.

Next, we give an example to demonstrate how Theorem 11 is used in practice for obtaining the zero-free region of the given analytic function.

Example 3 *Consider the function*

$$f(z) = 1 + \left(\frac{1}{0!} + \frac{1}{1!}\right)z + \left(\frac{1}{1!} + \frac{1}{2!}\right)z^2 + \left(\frac{1}{2!} + \frac{1}{3!}\right)z^3 + \ldots + \left(\frac{1}{n!} + \frac{1}{(n+1)!}\right)z^{n+1} + \ldots.$$

Clearly $f(z)$ is analytic in $|z| \leq 1$. Therefore, by taking $t = 1$, $k_1 = 2$, $k_2 = 1$ and $\alpha = \beta = 0$ in Theorem 11, it follows that $f(z)$ does not vanish in the region $|z - 1/3| < 2/3$, which is true as $f(z)$ has only one zero in the whole complex plane which is $z = -1$ and it lies outside the disc $|z - 1/3| < 2/3$.

3 Lemma

For the proof of these theorems, we require the following lemma which is due to Govil and Rahman [4].

Lemma 13 *If for some real β,*

$$|\arg a_j - \beta| \leq \alpha \leq \pi/2, \qquad a_j \neq 0,$$

then, for any positive real numbers t_1 and t_2,

$$|t_1 a_j - t_2 a_{j-1}| \leq |t_1|a_j| - t_2|a_{j-1}||\cos\alpha + (t_1|a_j| + t_2|a_{j-1}|)\sin\alpha.$$

4 Proof of the Theorems

Proof of Theorem 6. Consider the polynomial

$$\begin{aligned}
F(z) &= (1-z)P(z) \\
&= -a_n z^{n+1} + (a_n - a_{n-1})z^n + \cdots + (a_{n-r} - a_{n-r-1})z^{n-r} + \cdots + (a_1 - a_0)z + a_0 \\
&= -a_n z^{n+1} - (k_0 - 1)a_n z^n + (k_0 a_n - k_1 a_{n-1})z^n + (k_1 - 1)a_{n-1}z^n + (k_1 a_{n-1} - k_2 a_{n-2})z^{n-1} \\
&\quad -(k_1 - 1)a_{n-1}z^{n-1} + (k_2 - 1)a_{n-2}z^{n-1} + \cdots + (k_{r-1}a_{n-r+1} - k_r a_{n-r})z^{n-r+1} \\
&\quad -(k_{r-1} - 1)a_{n-r+1}z^{n-r+1} + (k_r - 1)a_{n-r}z^{n-r+1} + (k_r a_{n-r} - a_{n-r-1})z^{n-r} \\
&\quad -(k_r - 1)a_{n-r}z^{n-r} + (a_{n-r-1} - a_{n-r-2})z^{n-r-1} + \cdots + (a_1 - a_0)z + a_0,
\end{aligned}$$

which implies

$$\begin{aligned}
|F(z)| &\geq |z|^n \Bigg[|(z + k_0 - 1)a_n - (k_1 - 1)a_{n-1}| - \Bigg(|k_0 a_n - k_1 a_{n-1}| + \frac{|k_1 a_{n-1} - k_2 a_{n-2}|}{|z|} \\
&\quad + \frac{|k_1 - 1||a_{n-1}|}{|z|} + \frac{|k_2 - 1||a_{n-2}|}{|z|} + \cdots + \frac{|k_{r-1}a_{n-r+1} - k_r a_{n-r}|}{|z|^{r-1}} \\
&\quad + \frac{|k_{r-1} - 1||a_{n-r+1}|}{|z|^{r-1}} + \frac{|k_r - 1||a_{n-r}|}{|z|^{r-1}} + \frac{|k_r a_{n-r} - a_{n-r-1}|}{|z|^r} \\
&\quad + \frac{|k_r - 1||a_{n-r}|}{|z|^r} + \frac{|a_{n-r-1} - a_{n-r-2}|}{|z|^{r+1}} + \cdots + \frac{|a_1 - a_0|}{|z|^{n-1}} + \frac{|a_0|}{|z|^n}\Bigg)\Bigg].
\end{aligned}$$

Let $|z| > 1$ so that $1/|z| < 1$. Then we have

$$\begin{aligned}
|F(z)| &\geq |z|^n \Big[|(z + k_0 - 1)a_n - (k_1 - 1)a_{n-1}| - \Big(|k_0 a_n - k_1 a_{n-1}| + |k_1 a_{n-1} - k_2 a_{n-2}| \\
&\quad + |k_1 - 1||a_{n-1}| + |k_2 - 1||a_{n-2}| + \cdots + |k_{r-1}a_{n-r+1} - k_r a_{n-r}| \\
&\quad + |k_{r-1} - 1||a_{n-r+1}| + |k_r - 1||a_{n-r}| + |k_r a_{n-r} - a_{n-r-1}| \\
&\quad + |k_r - 1||a_{n-r}| + |a_{n-r-1} - a_{n-r-2}| + \cdots + |a_1 - a_0| + |a_0|\Big)\Big].
\end{aligned}$$

Applying Lemma 13, we have for $|z| > 1$,

$$\begin{aligned}
|F(z)| &\geq |a_n||z|^n \Bigg[\left|z + k_0 - 1 - (k_1 - 1)\frac{a_{n-1}}{a_n}\right| - \frac{1}{|a_n|}\Bigg\{\Big(\Big|k_0|a_n| - k_1|a_{n-1}|\Big| + \Big|k_1|a_{n-1}| - k_2|a_{n-2}|\Big| \\
&\quad + \cdots + \Big|k_{r-1}|a_{n-r+1}| - k_r|a_{n-r}|\Big| + \Big|k_r|a_{n-r}| - |a_{n-r-1}|\Big| + \Big||a_{n-r-1}| - |a_{n-r-2}|\Big| \\
&\quad + \cdots + \Big||a_1| - |a_0|\Big|\Big)\cos\alpha + \Big(k_0|a_n| + k_1|a_{n-1}| + k_1|a_{n-1}| + k_2|a_{n-2}| + \cdots + k_{r-1}|a_{n-r+1}| \\
&\quad + k_r|a_{n-r}| + k_r|a_{n-r}| + |a_{n-r-1}| + |a_{n-r-1}| + |a_{n-r-2}| + \cdots + |a_1| + |a_0|\Big)\sin\alpha \\
&\quad - (k_1 - 1)|a_{n-1}| + 2\sum_{j=1}^{r}(k_j - 1)|a_{n-j}| + |a_0|\Bigg\}\Bigg],
\end{aligned}$$

which in view of (1), yields

$$\begin{aligned} |F(z)| &\geq |a_n||z|^n\Bigg[\left|z+k_0-1-(k_1-1)\frac{a_{n-1}}{a_n}\right|-\frac{1}{|a_n|}\Bigg\{(k_0|a_n|-|a_0|)(\cos\alpha+\sin\alpha) \\ &\quad +2\sin\alpha\Bigg(\sum_{j=1}^{r}k_j|a_{n-j}|+\sum_{j=r+1}^{n}|a_{n-j}|\Bigg)-(k_1-1)|a_{n-1}|+2\sum_{j=1}^{r}(k_j-1)|a_{n-j}|+|a_0|\Bigg\}\Bigg] \\ &> 0 \end{aligned}$$

if

$$\begin{aligned} \left|z+k_0-1-(k_1-1)\frac{a_{n-1}}{a_n}\right| &> \frac{1}{|a_n|}\Bigg\{(k_0|a_n|-|a_0|)(\cos\alpha+\sin\alpha)+2\sin\alpha\Bigg(\sum_{j=1}^{r}k_j|a_{n-j}|+\sum_{j=r+1}^{n}|a_{n-j}|\Bigg) \\ &\quad -(k_1-1)|a_{n-1}|+2\sum_{j=1}^{r}(k_j-1)|a_{n-j}|+|a_0|\Bigg\}. \end{aligned}$$

This shows that those zeros of $F(z)$ whose modulus is greater than 1 lie in

$$\begin{aligned} \left|z+k_0-1-(k_1-1)\frac{a_{n-1}}{a_n}\right| &\leq \frac{1}{|a_n|}\Bigg\{(k_0|a_n|-|a_0|)(\cos\alpha+\sin\alpha)+2\sin\alpha\Bigg(\sum_{j=1}^{r}k_j|a_{n-j}|+\sum_{j=r+1}^{n}|a_{n-j}|\Bigg) \\ &\quad -(k_1-1)|a_{n-1}|+2\sum_{j=1}^{r}(k_j-1)|a_{n-j}|+|a_0|\Bigg\}. \end{aligned}$$

But those zeros of $F(z)$ whose modulus is less than or equal to 1 already lie in this region. Hence it follows that all the zeros of $F(z)$ and therefore of $P(z)$ lie in

$$\begin{aligned} \left|z+k_0-1-(k_1-1)\frac{a_{n-1}}{a_n}\right| &\leq \frac{1}{|a_n|}\Bigg\{(k_0|a_n|-|a_0|)(\cos\alpha+\sin\alpha)+2\sin\alpha\Bigg(\sum_{j=1}^{r}k_j|a_{n-j}|+\sum_{j=r+1}^{n}|a_{n-j}|\Bigg) \\ &\quad -(k_1-1)|a_{n-1}|+2\sum_{j=1}^{r}(k_j-1)|a_{n-j}|+|a_0|\Bigg\}. \end{aligned}$$

This completes the proof of Theorem 6. ■

Proof of Theorem 11. Since $f(z)=\sum_{j=0}^{\infty}a_jz^j$ is analytic in $|z|\leq t$ and it is easy to observe that $\lim_{j\to\infty}t^ja_j=0$. Now consider the function

$$\begin{aligned} F(z) &= (z-t)f(z) \\ &= -ta_0+(a_0-ta_1)z+(a_1-ta_2)z^2+(a_2-ta_3)z^3+\ldots \\ &= -ta_0+(k_1a_0-tk_2a_1)z-((k_1-1)a_0-(k_2-1)ta_1)z \\ &\quad +((k_2a_1-ta_2)-(k_2-1)a_1)z^2+(a_2-ta_3)z^3+\ldots \\ &= -ta_0-((k_1-1)a_0-(k_2-1)ta_1)z+\phi(z), \end{aligned} \tag{2}$$

where

$$\phi(z)=(k_1a_0-tk_2a_1)z+((k_2a_1-ta_2)-(k_2-1)a_1)z^2+\sum_{j=3}^{\infty}(a_{j-1}-ta_j)z^j.$$

Since $|\arg a_j - \beta| \le \alpha \le \pi/2$, $j = 0, 1, 2, \ldots$, and by applying Lemma 13, we have, for $|z| = t$,

$$\begin{aligned}
|\phi(z)| &\le |k_1 a_0 - k_2 t a_1||z| + |k_2 a_1 - t a_2||z|^2 + (k_2-1)|a_1||z|^2 + \sum_{j=3}^{\infty} |a_{j-1} - t a_j||z|^j \\
&\le (k_1|a_0| + k_2 t|a_1|)t\sin\alpha + |k_1|a_0| - k_2 t|a_1||t\cos\alpha + (k_2|a_1| + t|a_2|)t^2\sin\alpha \\
&\quad + |k_2|a_1| - t|a_2||t^2\cos\alpha + (k_2-1)t^2|a_1| + (|a_2| + t|a_3|)t^3\sin\alpha + ||a_2| - t|a_3||t^3\cos\alpha + \ldots \\
&= (k_1|a_0| + k_2 t|a_1|)t\sin\alpha + (k_1|a_0| - k_2 t|a_1|)t\cos\alpha + (k_2|a_1| + t|a_2|)t^2\sin\alpha \\
&\quad + (k_2|a_1| - t|a_2|)t^2\cos\alpha + (k_2-1)t^2|a_1| + (|a_2| + t|a_3|)t^3\sin\alpha + (|a_2| - t|a_3|)t^3\cos\alpha + \ldots \\
&\quad \text{(By hypothesis)} \\
&= t|a_0|\left\{k_1(\sin\alpha + \cos\alpha) + t(k_2-1)\frac{|a_1|}{|a_0|} + \frac{2\sin\alpha}{|a_0|}\left(k_2 t|a_1| + \sum_{j=2}^{\infty} t^j|a_j|\right)\right\}.
\end{aligned}$$

Clearly $\phi(z)$ is analytic for $|z| \le t$ with $\phi(0) = 0$. Hence, by the Schwarz Lemma,

$$|\phi(z)| \le |a_0||z|\left\{k_1(\sin\alpha + \cos\alpha) + t(k_2-1)\frac{|a_1|}{|a_0|} + \frac{2\sin\alpha}{|a_0|}\left(k_2 t|a_1| + \sum_{j=2}^{\infty} t^j|a_j|\right)\right\} \quad \text{for } |z| \le t.$$

Therefore from (2) we have for $|z| \le t$,

$$\begin{aligned}
|F(z)| &\ge |a_0||t + (k_1-1)z| - |a_0||z|\left\{k_1(\sin\alpha + \cos\alpha) + 2t(k_2-1)\frac{|a_1|}{|a_0|} + \frac{2\sin\alpha}{|a_0|}\left(k_2 t|a_1| + \sum_{j=2}^{\infty} t^j|a_j|\right)\right\} \\
&= |a_0||t + (k_1-1)z| - |a_0||z|B > 0
\end{aligned}$$

if

$$|z|B < |t + (k_1-1)z|$$

where

$$B = k_1(\sin\alpha + \cos\alpha) + 2t(k_2-1)\frac{|a_1|}{|a_0|} + \frac{2\sin\alpha}{|a_0|}\left(k_2 t|a_1| + \sum_{j=2}^{\infty} t^j|a_j|\right), \tag{3}$$

that is, $F(z)$ and therefore $f(z)$ does not vanish in

$$|z|B < |(k_1-1)z + t|,$$

which is precisely the region

$$\left|z - \frac{(k_1-1)t}{B^2 - (k_1-1)^2}\right| < \frac{Bt}{B^2 - (k_1-1)^2},$$

where B is defined in (3). That completes the proof of Theorem 11. ∎

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