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On The E. Study Maps For The Dual Quaternions*

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Abstract

In this study, the generalization of the well-known E. Study map for unit dual quaternion vectors in $H_{\mathbb{D}}^3$ is obtained, where $\mathbb{D}$ and $H_{\mathbb{D}}$ are sets of dual numbers and dual quaternions, respectively.

1 Introduction

Dual numbers introduced by W. K. Clifford are tools for geometrical investigations. Eduard Study (1903) defined directed lines in $\mathbb{R}^3$ by using dual unit numbers and introduced the mapping using his name: E. Study mapping.

In this paper, firstly the properties of dual numbers and dual quaternions will be examined. After that dual quaternion functions and their properties will be introduced. Then by considering unit dual quaternions in $H_{\mathbb{D}}^3$, the generalization of E. Study map will be given, where $\mathbb{D}$ and $H_{\mathbb{D}}$ are sets of dual numbers and dual quaternions, respectively.

1.1 Dual Numbers

A dual number is defined by $z = a + \varepsilon b$, where $a, b \in \mathbb{R}$ and $\varepsilon \neq 0$ with the property that $\varepsilon^2 = 0$. The real numbers a and b are called the real part and dual part of z and denoted by $Re\,(z) = a$ and $Du\,(z) = b$, respectively. The set of dual numbers is denoted by

$$\mathbb{D} = \left\{ z = a + \varepsilon b \mid a, b \in \mathbb{R},\ \varepsilon^2 = \varepsilon^3 = ... = 0,\ \varepsilon \neq 0 \right\}.$$

Let $z_1 = a_1 + \varepsilon b_1$ and $z_2 = a_2 + \varepsilon b_2 \in \mathbb{D}$. Then, the following properties of the dual numbers are summarized:

- **The addition** of two dual numbers z_1 and z_2 is given by

$$z_1 + z_2 = (a_1 + \varepsilon b_1) + (a_2 + \varepsilon b_2) = (a_1 + a_2) + \varepsilon\,(b_1 + b_2).$$

- **The multiplication** of two dual numbers z_1 and z_2 is defined by

$$z_1 z_2 = (a_1 + \varepsilon b_1)(a_2 + \varepsilon b_2) = (a_1 a_2) + \varepsilon\,(a_1 b_2 + a_2 b_1).$$

- **The conjugate** of z is denoted by $\overline{z}$ and defined as usual way

$$\overline{z} = \overline{a + \varepsilon b} = a - \varepsilon b.$$

- **The norm** of z is defined by

$$\|z\| = \sqrt{z\overline{z}} = \sqrt{a^2} = |a|.$$

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- **The dual circle** is given by with the following properties

$$\|z\| = r \Leftrightarrow a = \pm r, \text{ where } a = Re\,(z)\,.$$

If $\|z\| = 1$, then z is called unit dual number, hence the set containing the elements that satisfy $\|z\| = 1$ is called Galilean unit circle on the dual plane. Moreover, Galilean cosine and Galilean sine for all real numbers θ are given as follow

$$\cos_g \theta = 1 \text{ and } \sin_g \theta = \theta.$$

- **The Euler's formula** for dual number is given by

$$e^{\varepsilon\theta} = 1 + \varepsilon\theta$$

for any real numbers θ. Thus, the set of dual numbers $\mathbb{D}$ is isomorphic to Galilean plane (or Isotropic plane) G^2.

- **The dual function** $f : \mathbb{D} \to \mathbb{D}$ is defined by

$$f\,(z) = \ f\,(x + \varepsilon y) = U\,(x, y) + \varepsilon V\,(x, y)$$

where U and V are real functions of two variables. Then, the Taylor polynomial of an dual variable function f has just two terms such that:

$$f\,(x + \varepsilon y) = f\,(x) + \varepsilon y f'\,(x)\,, \quad f^{'}\,(x) = \frac{df}{dx}. \tag{1}$$

The dual function $f = U + \varepsilon V$ is analytic if and only if $f'\,(z) = U_x + \varepsilon V_y$, [8].

Example 1 *For any $z = a + \varepsilon b \in \mathbb{D}$, the following properties can be given:*

1) $\cos\,(a + \varepsilon b) = \cos x - \varepsilon y \sin x$,

2) $\sin\,(a + \varepsilon b) = \sin x + \varepsilon y \cos x$,

3) $\sqrt{x + \varepsilon y} = \sqrt{x} + \varepsilon \frac{y}{2\sqrt{x}}$.

Let examine dual vectors. The set $\mathbb{D}^3 = \left\{ \vec{A} = (A_1, A_2, A_3) \middle| A_i = a_i + \varepsilon b_i \in \mathbb{D}, i = 1, 2, 3 \right\}$ is a module over the ring $\mathbb{D}$ which is called a $\mathbb{D}-$module or dual space. Moreover, a dual vector can be written in the following form:

$$\vec{A} = \vec{a} + \varepsilon\vec{b};\ \vec{a}, \vec{b} \in \mathbb{R}^3,\ \ \varepsilon^2 = 0.$$

Let $\vec{A} = (A_i) = \vec{a}_1 + \varepsilon\vec{b}_1$ and $\vec{B} = (B_i) = \vec{a}_2 + \varepsilon\vec{b}_2 \in \mathbb{D}^3$.

- **The inner product** of two vectors $\vec{A} = \vec{a}_1 + \varepsilon\vec{b}_1$ and $\vec{B} = \vec{a}_2 + \varepsilon\vec{b}_2$ is expressed by:

$$\begin{aligned} <\vec{A}, \vec{B}> &= \sum_{i=1}^{3} A_i B_i, \\ &= <\vec{a}_1, \vec{a}_2> + \varepsilon \left(<\vec{a}_1, \vec{b}_2> + <\vec{a}_2, \vec{b}_1> \right). \end{aligned}$$

- **The norm** of a dual vector $\vec{A} = \vec{a} + \varepsilon\vec{b} \in \mathbb{D}^3$ is defined by

$$\begin{aligned} \left\|\vec{A}\right\| &= \sqrt{<\vec{A}, \vec{A}>}, \\ &= \sqrt{<\vec{a}, \vec{a}> + 2\varepsilon <\vec{a}, \vec{b}>}, \\ &= \|\vec{a}\| + \varepsilon \frac{<\vec{a}, \vec{b}>}{\|\vec{a}\|}. \end{aligned}$$

- The set of the unit dual vectors are called **unit dual sphere** and unit dual sphere is denoted by

$$S^2 = \left\{ \vec{A} \in \mathbb{D}^3 \middle| \left\| \vec{A} \right\| = 1 + \varepsilon 0 \right\}.$$

- **The cross product** of $\vec{A} = \vec{a}_1 + \varepsilon \vec{b}_1$ and $\vec{B} = \vec{a}_2 + \varepsilon \vec{b}_2 \in \mathbb{D}^3$ is given by

$$\vec{A} \Lambda \vec{B} = \vec{a}_1 \Lambda \vec{a}_2 + \varepsilon \left(\vec{a}_1 \Lambda \vec{b}_2 + \vec{a}_2 \Lambda \vec{b}_1 \right).$$

- If $\vec{A}$ and $\vec{B}$ are unit dual vectors in $\mathbb{D}^3$, dual then number $\Phi = \varphi + \varepsilon \varphi^*$ is called **dual angle** and defined by (Figure 1)

$$< \vec{A}, \vec{B} >= \cos \Phi = \cos \left(\varphi + \varepsilon \varphi^* \right).$$

The detailed information can be found in papers [2, 4, 6, 7].

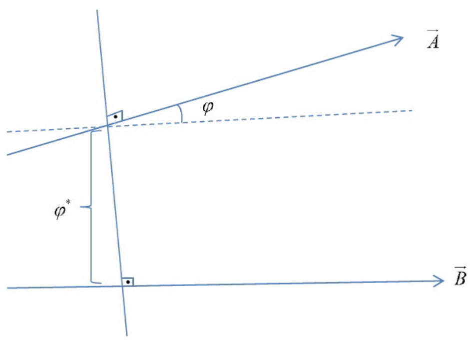

Figure 1: Dual Angle.

Theorem 1 ([11], *E. Study Map*) *There exists a one-to-one correspondence between the unit points of the dual sphere in $\mathbb{D}^3$ and the oriented lines in $\mathbb{R}^3$.*

1.2 Dual Quaternion

The set of dual quaternions can be written in the form:

$$H_{\mathbb{D}} = \left\{ q = a + bi + cj + dk \mid a, b, c, d \in \mathbb{R},\ i^2 = j^2 = k^2 = ijk = 0 \right\}.$$

To simplify the expressions of dual quaternions we can write $q = a + \mathbf{q}$, where $\mathbf{q} = bi + cj + dk$ denotes the vector part of dual quaternion.

For $q_1 = a_1 + b_1 i + c_1 j + d_1 k = a_1 + \mathbf{q_1}$, $q_2 = a_2 + b_2 i + c_2 j + d_2 k = a_2 + \mathbf{q_2} \in H_{\mathbb{D}}$, the following operations are defined by:

- **The addition** of two dual quaternions q_1, q_2;

$$q_1 + q_2 = (a_1 + a_2) + (b_1 + b_2) i + (c_1 + c_2) j + (d_1 + d_2) k = (a_1 + a_2) + \mathbf{q}_1 + \mathbf{q}_2.$$

- **The multiplication** of two dual quaternions q_1, q_2;

$$\begin{aligned} q_1 . q_2 &= (a_1 . a_2) + (a_1 b_2 + a_2 b_1) i + (a_1 c_2 + a_2 c_1) j + (a_1 d_2 + a_2 d_1) k \\ &= a_1 a_2 + a_1 \mathbf{q}_2 + a_2 \mathbf{q}_1. \end{aligned}$$

- **The conjugate** of a dual quaternion $q = a + bi + cj + dk$ represented by

$$\overline{q} = a - bi - cj - dk = a - \mathbf{q}.$$

- **The norm** of q is calculated as follows:

$$\|q\| = \sqrt{q\overline{q}} = \sqrt{a^2} = |a|\,.$$

Every unit dual quaternion $q = a + bi + cj + dk$ can be expressed in the trigonometric (polar) form as follows:

$$q = \cos_g \theta + \omega \sin_g \theta,$$

where $\cos_g \theta = 1,\ \ \sin_g \theta = \theta$ and ω is the pure unit dual quaternion such that $\omega^2 = 0$. On the other hand, **the Euler's formula** for the pure unit dual quaternion ω can be written as follows:

$$e^{\omega\theta} = 1 + \omega\theta.$$

Further information can be found in [3] and [5].

2 Dual Quaternion Functions

In this original section, the dual quaternion functions will be given and the properties of these functions will be examined.

Lemma 2 *Let $q = a + bi + cj + dk$ be a dual quaternion in $H_{\mathbb{D}}$. Then,*

$$q^n = a^n + na^{n-1}\left(bi + cj + dk\right), \tag{2}$$

where $n \in \mathbb{Z}$.

Proof. Let use mathematical induction for all $n \in \mathbb{Z}$. The basis step is $n = 1$. Here the left side of the equation (2) is q^1, and the right side is $a^1 + 1.a^0\left(bi + cj + dk\right) = a + bi + cj + dk = q$, so both sides are equal and the equation (2) is true for $n = 1$. Also for $n = 2$, the following equation can be given:

$$q^2 = qq = a^2 + 2abi + 2acj + 2adk.$$

For the inductive step, we assume that the equation (2) is true for $n = k$, that is, $q^k = a^k + ka^{k-1}\left(bi + cj + dk\right)$. Multiplying both sides by q, the following equation can be obtained:

$$\begin{aligned} q^{k+1} &= q^k q = \left(a^k + ka^{k-1}\left(bi + cj + dk\right)\right)\left(a + bi + cj + dk\right) \\ &= a^{k+1} + \left(k+1\right)a^k\left(bi + cj + dk\right). \end{aligned}$$

Thus the equation (2) holds for $n = k + 1$, and this completes the proof. ■

Theorem 3 *If the function $f : H_{\mathbb{D}} \to H_{\mathbb{D}}$ defined by*

$$f\left(q\right) = f_1\left(x, y, z, t\right) + f_2\left(x, y, z, t\right)i + f_3\left(x, y, z, t\right)j + f_4\left(x, y, z, t\right)k,$$

for $q = x + yi + zj + tk \in H_{\mathbb{D}}$ and $f_i : H_{\mathbb{D}} \to \mathbb{R}$ has continuous derivatives of all orders at $q_0 = 0$, then

$$f\left(q\right) = f\left(x\right) + \left(yi + zj + tk\right)f'\left(x\right). \tag{3}$$

Proof. Using by Maclaurin series expansion of the function $f(q)$ about $q_0 = 0$, the following equation holds:

$$f(q) = f(0) + \frac{f'(0)}{1!}q + \frac{f''(0)}{2!}q^2 + ... + \frac{f^{(n)}(0)}{n!}q^n + ... \quad .$$

By using Lemma 2, the following equation can be obtained:

$$\begin{aligned} f(q) &= f(0) + \frac{f'(0)}{1!}(x + yi + zj + tk) + \frac{f''(0)}{2!}\left(x^2 + 2xyi + 2xzj + 2xtk\right) + ... \\ &\quad + \frac{f^{(n)}(0)}{n!}\left(x^n + nx^{n-1}yi + nx^{n-1}zj + nx^{n-1}tk\right) + ... \\ &= \left[f(0) + \frac{f'(0)}{1!}x + ... + \frac{f^{(n)}(0)}{n!}x^n + ...\right] + \left[f'(0) + \frac{f''(0)}{2!}2x + ... + \frac{f^{(n)}(0)}{n!}nx^{n-1} + ...\right] yi \\ &\quad + \left[f'(0) + \frac{f''(0)}{2!}2x + ... + \frac{f^{(n)}(0)}{n!}nx^{n-1} + ...\right] zj \\ &\quad + \left[f'(0) + \frac{f''(0)}{2!}2x + ... + \frac{f^{(n)}(0)}{n!}nx^{n-1} + ...\right] tk. \end{aligned}$$

Maclaurin expansion for f' about $q_0 = 0$ in above equality can be given:

$$f(q) = f(x) + (yi + zj + tk) f'(x) = f(x) + \mathbf{q} f'(x) \tag{4}$$

That is the generalization of the equation (1) for the unit dual quaternion functions. ■

Now, by using the last equation the following theorem can be given:

Theorem 4 *Let* $q = x + yi + zj + tk = x + \mathbf{q}$, $q_1 = x_1 + y_1 i + z_1 j + t_1 k$ *and* $q_2 = x_2 + y_2 i + z_2 j + t_2 k \in H_{\mathbb{D}}$. *Then, by using the equation (3), the following properties can be given:*

1) $\sqrt{q} = \sqrt{x} + \mathbf{q}\frac{1}{2\sqrt{x}}$,

2) $|q| = |x| + \mathbf{q}\frac{x}{|x|}$, $x \neq 0$,

3) $|q| = q$, *if* $x > 0$, $|q| = -q$, *if* $x < 0$,

4) $q_1 > q_2 \Leftrightarrow x_1 > x_2$, $q_1 < q_2 \Leftrightarrow x_1 < x_2$,

5) $\ln(q_1 q_2) = \ln q_1 + \ln q_2$,

6) $e^q = e^x(1 + \mathbf{q}) \neq 0$, $e^{-q} = e^{-x}(1 - \mathbf{q})$,

7) ${q_1}^{q_2} = {x_1}^{x_2}\left[1 + \left(\frac{x_2 y_1}{x_1} + {x_1}^{y_2}\right)i + \left(\frac{x_2 z_1}{x_1} + {x_1}^{z_2}\right)j + \left(\frac{x_2 t_1}{x_1} + {x_1}^{t_2}\right)k\right]$,

8) $\cos q = \cos x - \mathbf{q}\sin x$,

9) $\sin q = \sin x + \mathbf{q}\cos x$,

10) $\cos(-q) = \cos q$, $\sin(-q) = -\sin q$,

11) $\cos^2 q + \sin^2 q = 1$,

12) $\sin(q_1 q_2) = \sin q_1 \cos q_2 + \cos q_1 \sin q_2$,

13) $\cos(q_1 q_2) = \cos q_1 \cos q_2 - \sin q_1 \sin q_2$,

14) $\tan q = \tan x - \mathbf{q}\ \frac{1}{1+x^2}$, *15)* $\arcsin q = \arcsin x + \mathbf{q}\ \frac{1}{\sqrt{1-x^2}}$,

16) $\arccos q = \arccos x - \mathbf{q}\ \frac{1}{\sqrt{1-x^2}}$,

17) $\arctan q = \arctan x + \mathbf{q}\ \frac{1}{1+x^2}$,

18) $\sinh q = \sinh x + \mathbf{q} \cosh x$,

19) $\cosh q = \cosh x + \mathbf{q} \sinh x$,

20) $\cosh^2 q - \sinh^2 q = 1$,

21) $\sinh q = \frac{e^q - e^{-q}}{2}$,

22) $\cosh q = \frac{e^q + e^{-q}}{2}$,

23) $\tanh q = \tanh x - \mathbf{q}\ \frac{1}{cosh^2 x}$,

24) $arcsinh\, q = arcsinh\, x + \mathbf{q}\ \frac{1}{\sqrt{1+x^2}}$,

25) $arccosh\, q = arccosh\, x - \mathbf{q}\ \frac{1}{\sqrt{x^2-1}}$,

26) $arctanh\, q = arctanh\, x + \mathbf{q}\ \frac{1}{1-x^2}$.

3 Dual Quaternion n-Space

In this original section, firstly dual quaternion n-space will be defined. After that the generalization of E. Study map for unit dual quaternion vectors will be introduced.

The set $H_{\mathbb{D}}^n = \{\vec{q} = (q_1, q_2, ..., q_n) : q_r = a_r + b_r i + c_r j + d_r k \in H_{\mathbb{D}}, 1 \leq r \leq n\}$ is a vector space on $\mathbb{R}$. This set is called dual quaternion n-space. A quaternion vector can be written in the following form:

$$\vec{q} = \vec{a} + \vec{b}i + \vec{c}j + \vec{d}k, \quad \vec{a}, \vec{b}, \vec{c}, \vec{d} \in \mathbb{R}^n$$

Also, $dim H_{\mathbb{D}}^n = 4n$ and $H_{\mathbb{D}}^n \cong \mathbb{R}^{4n}$.

The inner product of dual quaternion $n-$vectors $\vec{q} = (q_1, q_2, ..., q_n) \in H_{\mathbb{D}}^n$ and $\vec{p} = (p_1, p_2, ..., p_n) \in H_{\mathbb{D}}^n$ is defined by

$$< \vec{q}, \vec{p} >= \sum_{r=1}^{n} q_r p_r,$$

where $q_r = a_r + b_r i + c_r j + d_r k$ and $p_r = x_r + y_r i + z_r j + t_r k \in H_{\mathbb{D}}$.

Let $\vec{q} = \vec{a} + \vec{b}i + \vec{c}j + \vec{d}k$, $\vec{p} = \vec{x} + \vec{y}i + \vec{z}j + \vec{t}k \in H_{\mathbb{D}}^n$ and $\vec{a}, \vec{b}, \vec{c}, \vec{d}\ \vec{x}, \vec{y}, \vec{z}, \vec{t} \in \mathbb{R}^n$, then the following equation holds:

$$< \vec{q}, \vec{p} >=< \vec{a}, \vec{x} > +(< \vec{a}, \vec{y} > + < \vec{b}, \vec{x} >)i + (< \vec{a}, \vec{z} > + < \vec{c}, \vec{x} >)\, j + (< \vec{a}, \vec{t} > + < \vec{d}, \vec{x} >)k.$$

The norm of $\vec{q} \in H_{\mathbb{D}}^n$ is defined by

$$\|\vec{q}\| = \sqrt{< \vec{q}, \vec{q} >}.$$

Considering the first property of Theorem 4, the following equation can be calculated:

$$\|\vec{q}\| = \|\vec{a}\| + \frac{1}{\|\vec{a}\|} \left(< \vec{a}, \vec{b} > i + < \vec{a}, \vec{c} > j + < \vec{a}, \vec{d} > k \right).$$

The unit dual quaternion $(n-1)$**-sphere** is described as follows:

$$\mathbb{S}^{n-1} = \{ \vec{q} \in H_{\mathbb{D}}^n | \, \|\vec{q}\| = 1 \}.$$

Then, the following properties can be given:

$$\|\vec{q}\| = 1 \Longleftrightarrow \|\vec{a}\| = 1 \text{ and } < \vec{a}, \vec{b} >=< \vec{a}, \vec{c} >=< \vec{a}, \vec{d} >= 0. \tag{5}$$

Theorem 5 *Let $\vec{q}, \vec{p}, \vec{p_1}, \vec{p_2} \in H_{\mathbb{D}}^n$, and $p, q \in H_{\mathbb{D}}$. Then the inner product has the following properties:*

1) $< \vec{q}, \vec{p_1} + \vec{p_2} >=< \vec{q}, \vec{p_1} > + < \vec{q}, \vec{p_2} >$,

2) $< \vec{q}, \vec{p} >= \overline{< \vec{p}, \vec{q} >}$,

3) $< \vec{q}, \vec{q} >\geq 0$, *and* $< \vec{q}, \vec{q} >= 0 \Longleftrightarrow \vec{q}$ *is pure dual quaternion n -vector,*

4) $< \vec{q}q, \vec{p}p >= q < \vec{q}, \vec{p} > p$,

5) $\| < \vec{q}, \vec{p} > \| \leq \|\vec{q}\| \, \|\vec{p}\|$.

3.1 The Generalization of E. Study Map

Let consider the dual quaternion 3-space

$$\begin{aligned} H_{\mathbb{D}}^3 &= \{ \vec{q} = (q_1, q_2, q_3) | \, q_1, q_2, q_3 \in H_{\mathbb{D}} \} \\ &= \left\{ \vec{q} = \vec{a} + \vec{b}i + \vec{c}j + \vec{d}k \right| \left. \vec{a}, \vec{b}, \vec{c}, \vec{d} \in \mathbb{R}^3 \right\}. \end{aligned}$$

The cross product of $\vec{q} = \vec{a} + \vec{b}i + \vec{c}j + \vec{d}k$ and $\vec{p} = \vec{x} + \vec{y}i + \vec{z}j + \vec{t}k$ is given by

$$\vec{q} \Lambda \vec{p} = \vec{a} \Lambda \vec{x} + (\vec{a} \Lambda \vec{y} + \vec{b} \Lambda \vec{x}) i + (\vec{a} \Lambda \vec{z} + \vec{c} \Lambda \vec{x}) \, j + (\vec{a} \Lambda \vec{t} + \vec{d} \Lambda \vec{x}) k.$$

Theorem 6 (Generalized E. Study Map) *A point of the unit dual quaternion sphere $\mathbb{S}^2$ in $H_{\mathbb{D}}^3$ corresponds to three parallel lines which do not lie on the same plane in $\mathbb{R}^3$. Conversely, three parallel lines which do not lie on same plane in $\mathbb{R}^3$ correspond six different unit dual quaternions.*

Proof. Let $\vec{q} = \vec{a} + \vec{b} + + \vec{c}j + \vec{d}k$ be a point on the unit dual quaternion sphere $\mathbb{S}^2$. That is, $\|\vec{q}\| = 1 \Longleftrightarrow \|\vec{a}\| = 1$ and $< \vec{a}, \vec{b} >=< \vec{a}, \vec{c} >=< \vec{a}, \vec{d} >= 0$, where $\vec{a}, \vec{b}, \vec{c}, \vec{d} \in \mathbb{R}^3$. On the other hand, using by E. Study Map, the vector pairs (or the orientable lines) $(\vec{a}, \vec{b})$, $(\vec{a}, \vec{c})$, $(\vec{a}, \vec{d})$ or the unit dual quaternion vectors $\vec{B} = \vec{a} + \epsilon \vec{b}$, $\vec{C} = \vec{a} + \epsilon \vec{c}$ and $\vec{D} = \vec{a} + \epsilon \vec{d} \in \mathbb{D}^3$ correspond to each of the oriented parallel three lines in $\mathbb{R}^3$, respectively. The points of these lines are

$$X = \vec{a} \Lambda \vec{b},$$

$$Y = \vec{a} \Lambda \vec{c},$$

$$Z = \vec{a} \Lambda \vec{d},$$

respectively. Then,

$$\det \left(\overrightarrow{OX}, \overrightarrow{OY}, \overrightarrow{OZ} \right) = 0.$$

Let n_1, n_2, n_3 be the normal vectors of the planes of the lines $(\vec{a}, \vec{b})$ and $(\vec{a}, \vec{c})$ or $(\vec{a}, \vec{b})$ and $(\vec{a}, \vec{d})$ or $(\vec{a}, \vec{c})$, $(\vec{a}, \vec{d})$, respectively. Then the following properties can be given:

$$n_1 =< \vec{a}, \vec{b} - \vec{c} > \vec{a} + \vec{b} - \vec{c},$$

$$n_2 =< \vec{a}, \vec{b} - \vec{d} > \vec{a} + \vec{b} - \vec{d},$$

$$n_3 =< \vec{a}, \vec{c} - \vec{d} > \vec{a} + \vec{c} - \vec{d},$$

$$n_1 \neq n_2 \neq n_3,$$

$$< n_1, \vec{a} >=< n_2, \vec{a} >=< n_3, \vec{a} >= 0.$$

That is, these three parallel lines do not lie on the same plane in $\mathbb{R}^3$.

Let us take $(\vec{a},\vec{b})$, $(\vec{a},\vec{c})$ and $(\vec{a},\vec{d})$ three parallel lines that not lie on the same plane in $\mathbb{R}^3$. From E. Study Map (Theorem 1), we conclude that three points of the unit dual sphere in $\mathbb{D}^3$. Also, we can write $<\vec{a},\vec{a}>=1$ and $<\vec{a},\vec{b}>=<\vec{a},\vec{c}>=<\vec{a},\vec{d}>=0$, because the vectors $\vec{b},\vec{c},\vec{d}$ are the moment vectors of the unit vector $\vec{a}$ with respect to the origin of reference frame. Consequently, that three parallel lines define six different unit dual quaternion as follows:

- $\vec{q_1}=\vec{a}+\vec{b}i+\vec{c}j+\vec{d}k$, • $\vec{q_2}=\vec{a}+\vec{b}i+\vec{d}j+\vec{c}k$, • $\vec{q_3}=\vec{a}+\vec{c}i+\vec{b}j+\vec{d}k$,
- $\vec{q_4}=\vec{a}+\vec{c}i+\vec{d}j+\vec{b}k$, • $\vec{q_5}=\vec{a}+\vec{d}i+\vec{b}j+\vec{c}k$, • $\vec{q_6}=\vec{a}+\vec{d}i+\vec{c}j+\vec{b}k$.

■

Example 2 *Let us consider the point*

$$\vec{q}=\left(\frac{1}{\sqrt{3}}+i+j-3k,\frac{1}{\sqrt{3}}+2i+2k,\frac{1}{\sqrt{3}}-3i-j+k\right)\in\mathbb{H}^3_{\mathbb{D}}.$$

Thus, we obtain

$$\vec{q}=\vec{a}+\vec{b}i+\vec{c}j+\vec{d}k,$$

where $\vec{a}=\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)$, $\vec{b}=(1,2,-3)$, $\vec{c}=(1,0,-1)$ *and* $\vec{d}=(-3,2,1)\in\mathbb{R}^3$. *These vectors satisfy the equation* (5). *Also, we can write for these parallel line* $(\vec{a},\vec{b}),(\vec{a},\vec{c})$ *and* $(\vec{a},\vec{d})$ *that not lie on the same plane in* $\mathbb{R}^3$*:*

$$\begin{array}{ll}(\vec{a},\vec{b}): & \sqrt{3}x+5=\sqrt{3}y-4=\sqrt{3}z-1,\\ (\vec{a},\vec{c}): & \sqrt{3}x+1=\sqrt{3}y-2=\sqrt{3}z+1,\\ (\vec{a},\vec{d}): & \sqrt{3}x+1=\sqrt{3}y+4=\sqrt{3}z-5.\end{array}\tag{6}$$

Note that the unit dual quaternions

$$\begin{aligned}
\vec{q_1}&=\vec{q},\\
\vec{q_2}&=\left(\frac{1}{\sqrt{3}}+i-3j+k,\frac{1}{\sqrt{3}}+2i+2j,\frac{1}{\sqrt{3}}-3i+j-k\right),\\
\vec{q_3}&=\left(\frac{1}{\sqrt{3}}+i+j-3k,\frac{1}{\sqrt{3}}+2j+2k,\frac{1}{\sqrt{3}}-i-3j+k\right),\\
\vec{q_4}&=\left(\frac{1}{\sqrt{3}}+i-3j+k,\frac{1}{\sqrt{3}}+2j+2k,\frac{1}{\sqrt{3}}-i+j-3k\right),\\
\vec{q_5}&=\left(\frac{1}{\sqrt{3}}-3i+j+k,\frac{1}{\sqrt{3}}+2i+2j,\frac{1}{\sqrt{3}}+i-3j-k\right),\\
\vec{q_6}&=\left(\frac{1}{\sqrt{3}}-3i+j+k,\frac{1}{\sqrt{3}}+2i+2k,\frac{1}{\sqrt{3}}+i-j-3k\right)
\end{aligned}$$

corresponds to the three parallel lines given by the equation (6).

3.2 $H^4_{\mathbb{D}}$-Space

Let $\vec{q}=(q_1,q_2,q_3,q_4)=\vec{a}+\vec{b}i+\vec{c}j+\vec{d}k\in H^4_{\mathbb{D}}$ be the dual quaternion vector, where $q_1,q_2,q_3,q_4\in H_{\mathbb{D}}$ and $\vec{a},\vec{b},\vec{c},\vec{d}\in\mathbb{R}^4$. If $\vec{q}$ is the unit quaternion 4−vector, then $<\vec{a},\vec{a}>=1$, $<\vec{a},\vec{b}>=<\vec{a},\vec{c}>=<\vec{a},\vec{d}>=0$. Hence the equality $\vec{a}=\frac{\vec{b}\wedge\vec{c}\wedge\vec{d}}{\|\vec{b}\wedge\vec{c}\wedge\vec{d}\|}$ holds, i.e. the real part $\vec{a}$ of $\vec{q}$ is the unit normal vector of the subspace $sp\{\vec{b},\vec{c},\vec{d}\}$ (see [1] for more information about vector product in $\mathbb{R}^4$.)

4 Conclusions

In a spatial motion, the trajectories of the oriented lines embedded in a moving rigid body are generally ruled surfaces. i.e. A ruled surface can be described as the set of points swept by a moving straight line. Thus, the spatial geometry of ruled surfaces is important in the study of rational design problems in spatial mechanisms.

In Kinematics geometry, some results (analog to Holditch Theorem) about the total torsion and total curvature of closed ruled surfaces resp. the area and length of closed spherical curves are presented by Hering [9].

In line geometry, lines are represented by Plücker vectors [10]. These consist of a vector, $\vec{a}$, directed along the line and a vector, $\vec{a}^*$, which represents the moment of $\vec{a}$ with respect to a chosen coordinate frame. These vectors are usually defined as:

$$l = \{(\vec{a}, \vec{a}^*) : \vec{a}^* = \vec{r}\Lambda\vec{a},\ <\vec{a}, \vec{a}>= 1,\ <\vec{a}, \vec{a}^*>= 0\},$$

where $\vec{r}$ is a vector to a point on the line. There are six coordinates in l (referred to as Plücker coordinates) and two constraints leaving four independent coordinates.

The elements of $\mathbb{D}^3$ are written as $\hat{a} = \vec{a} + \varepsilon\vec{a}^*$, where $\vec{a}, \vec{a}^* \in \mathbb{R}^3$, $\varepsilon^2 = 0$. The dual representation of the line, $\hat{a} = \vec{a} + \varepsilon\vec{a}^*$, is a dual unit vector since $<\hat{a}, \hat{a}>=<\vec{a}, \vec{a}> +2\varepsilon <\vec{a}, \vec{a}^*>= 1$. Therefore, the image of Plücker quadric in $\mathbb{D}^3$ is the dual unit sphere. A ruled surface is then a spherical curve on this dual unit sphere. This representational of the space of lines is due to Study [11] and is referred as Study's model of the line space.

In this study, by considering fundamental properties of dual numbers and dual quaternions, the dual quaternion functions and their properties are examined. Then, the dual quaternion n-space were given. The elements of $H^3_{\mathbb{D}}$ are written as $\vec{q} = \vec{a} + \vec{b}\,i + \vec{c}\,j + \vec{d}\,k$, where $\vec{a}, \vec{b}, \vec{c}, \vec{d} \in \mathbb{R}^3$, $i^2 = j^2 = k^2 = 0$. The dual quaternion representation of three parallel lines which do not lie on the same plane in $\mathbb{R}^3$, $\vec{q} = \vec{a} + \vec{b}\,i + \vec{c}\,j + \vec{d}\,k$, is a unit dual quaternion vector since

$$<\vec{q}, \vec{q}>=<\vec{a}, \vec{a}> +2<\vec{a}, \vec{b}>\ i + 2<\vec{a}, \vec{c}>\ j + 2<\vec{a}, \vec{d}>\ k = 1.$$

Finally, the generalization of E. Study Map which is the main target of this study is obtained. The striking geometrical meaning of the generalization of E. Study Map is that, there exists a strictly correspondence between a curve lie on unit dual quaternion sphere $S^2 \in H^3_{\mathbb{D}}$ and three ruled surfaces parallel to each other. So this aspect of E. Study Map is efficient for analog Holditch Theorem about the theory of the ruled surfaces.

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