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On The Sum-Connectivity Index Of Trees*

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Abstract

The sum-connectivity index of a graph G is defined as the sum of weights $1/\sqrt{d_u+d_v}$ over all edges uv of G, where d_u and d_v are the degrees of the vertices u and v in G, respectively. In this paper, we give a relation between the sum-connectivity index and diameter in trees.

1 Introduction

Let G be a simple graph with vertex set $V = V(G)$ and edge set $E(G)$. The integers $n = n(G) = |V(G)|$ and $m = m(G) = |E(G)|$ are the *order* and the *size* of the graph G, respectively. The *open neighborhood* of vertex v is $N_G(v) = N(v) = \{u \in V(G) \mid uv \in E(G)\}$ and the *degree* of v is $d_G(v) = d_v = |N(v)|$. A pendant vertex is a vertex of degree one. The distance between two vertices is the number of edges in a shortest path connecting them and the diameter $D(G)$ of G is the distance between any two furthest vertices in G. A diametral path is a shortest path in G connecting two vertices whose distance is $D(G)$. A unicyclic graph is a connected graph containing exactly one cycle. A subgraph G' of a graph G is a graph whose set of vertices is a subset of $V(G)$ and set of edges is a subset of $E(G)$.

Topological indices have been used and have been shown to give a high degree of predictability of pharmaceutical properties. The sum-connectivity index of a graph G was proposed in [11], defined as

$$\mathcal{SCI}(G) = \sum_{uv\in E(G)} \frac{1}{\sqrt{d_u+d_v}}.$$

The applications of the sum-connectivity index have been investigated in [6, 7]. Some basic mathematical properties of the sum-connectivity index have been established in [1, 2, 3, 4, 6, 7, 9, 10, 11].

In [11], it was shown that for a graph G with $n \geq 5$ vertices and without isolated vertices, $\mathcal{SCI}(G) \geq \frac{n-1}{n}$ with equality if and only if G is the star. In [9], sum-connectivity index of molecular trees are characterized. In [4], the authors obtained relations between the sum-connectivity index and matching number and in [3], the authors obtained relations between Randić index and the general sum-connectivity index. In this paper, we consider the relationship between the sum-connectivity index and the diameter.

Main Results

In this section, we obtain relationship between the sum-connectivity index and the diameter. An edge x_1x_2 is called local maximum if its weight $\frac{1}{\sqrt{d_{x_1}+d_{x_2}}}$ is maximum in its neighborhood, i.e.,

$$\frac{1}{\sqrt{d_{x_1}+d_{x_2}}} \geq \frac{1}{\sqrt{d_{x_i}+d_u}}$$

for any edge x_iu for $i = 1, 2$. Here, we proving the following lemma that will need to obtain the main result.

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Lemma 1 *Let x_1x_2 be a local maximum edge in graph G. Then*

$$\mathcal{SCI}(G) - \mathcal{SCI}(G - x_1x_2) > 0.$$

Proof. By the definition of sum-connectivity index, we have

$$\begin{aligned}\mathcal{SCI}(G) - \mathcal{SCI}(G - x_1x_2) &= \frac{1}{\sqrt{d_{x_1}+d_{x_2}}} + \sum_{u\in N_G(x_1)\setminus\{x_2\}}\left(\frac{1}{\sqrt{d_{x_1}+d_u}} - \frac{1}{\sqrt{d_{x_1}+d_u-1}}\right)\\ &+ \sum_{v\in N_G(x_2)\setminus\{x_1\}}\left(\frac{1}{\sqrt{d_{x_2}+d_v}} - \frac{1}{\sqrt{d_{x_2}+d_v-1}}\right)\\ &\geq \frac{1}{\sqrt{d_{x_1}+d_{x_2}}} + (d_{x_1}-1)\left(\frac{1}{\sqrt{d_{x_1}+d_{x_2}}} - \frac{1}{\sqrt{d_{x_1}+d_{x_2}-1}}\right)\\ &+ (d_{x_2}-1)\left(\frac{1}{\sqrt{d_{x_1}+d_{x_2}}} - \frac{1}{\sqrt{d_{x_1}+d_{x_2}-1}}\right)\\ &= \frac{d_{x_1}+d_{x_2}-1}{\sqrt{d_{x_1}+d_{x_2}}} - \frac{d_{x_1}+d_{x_2}-2}{\sqrt{d_{x_1}+d_{x_2}-1}}\\ &= \frac{(d_{x_1}+d_{x_2}-1)^{3/2} + 2\sqrt{d_{x_1}+d_{x_2}} - (d_{x_1}+d_{x_2})^{3/2}}{\sqrt{d_{x_1}+d_{x_2}}\sqrt{d_{x_1}+d_{x_2}-1}} > 0.\end{aligned}$$

■

If x_1x_2 is a pendant edge of G , i.e., at least one of vertices x_1, x_2 has degree one, we can see that it is a local maximum edge. Thus, by Lemma 1, we get the next result.

Corollary 2 *If x_1x_2 is a pendant edge in the graph G, then*

$$\mathcal{SCI}(G) - \mathcal{SCI}(G - x_1x_2) > 0.$$

Now we obtain a relation between the sum-connectivity index and diameter of trees.

Theorem 3 *Let T be a tree of order $n \geq 4$ and diameter $D(T)$. Then*

$$SCI(T) - D(T) \geq \frac{2}{\sqrt{3}} - \frac{n+1}{2}$$

with equality if and only if T is a path P_n.

Proof. If T is a path, we have $\mathcal{SCI}(T) = \frac{n}{2} + \frac{2}{\sqrt{3}} - \frac{3}{2}$ and $D(T) = n - 1$. Now we assume that T is not a path, then $D(T) \leq n - 2$ and there are at least three pendent vertices in T. Assume $P = u_0, u_1, \ldots, u_D$ be a longest path in T. Then at least one pendent vertex, say v_1, is not contained in P. Now we start an operation on T, i.e., we continually delete pendent vertices which are not contained in P until the resulting tree is P. Assume that $v_1, v_2, \ldots, v_k$ are the vertices in the order they were deleted. We have

$$\mathcal{SCI}(T) > \mathcal{SCI}(T - v_1) > \cdots > \mathcal{SCI}(T - \bigcup_{i=1}^{k} v_k) = \mathcal{SCI}(P) = \frac{D(T)}{2} + \frac{1}{\sqrt{6}}$$

by Corollary 2 and

$$D(T) = D(T - v_1) = \cdots = D(T - \bigcup_{i=1}^{k} v_k) = D(P).$$

Thus, we have

$$\mathcal{SCI}(T) - D(T) > \mathcal{SCI}(P) - D(P)$$

implying that

$$\frac{2}{\sqrt{3}} - \frac{1}{2} - \frac{D(T)+1}{2} \geq \frac{2}{\sqrt{3}} - \frac{1}{2} - \frac{n-1}{2} = \frac{2}{\sqrt{3}} - \frac{n}{2} > \frac{2}{\sqrt{3}} - \frac{n+1}{2}.$$

■

This result seems true for any connected graph G of order n, and we propose it as a conjecture as follows: Let G be a connected graph of order $n \geq 4$ and diameter $D(G)$. Then

$$\mathcal{SCI}(G) - D(G) \geq \frac{2}{\sqrt{3}} - \frac{n+1}{2}.$$

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