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# Fixed Point Theorems Of Kannan Type With An Application To Control Theory*

Ahmed Chaouki Aouine†, Abdelkrim Aliouche‡

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**Abstract**

We prove unique fixed point theorems for a self-mapping in complete metric spaces and that the fixed point problem is well-posed. Examples are provided to illustrate the validity of our results and we give some remarks about the papers [1], [2] and [16]. Afterwards, we apply our result to study the possibility of optimally controlling the solution of an ordinary differential equation via dynamic programming.

## 1 Introduction and Preliminaries

Fixed point theory is attractive to many researchers since 1922 with the famous Banach's fixed point theorem called Banach contraction principle, see [3]. This theorem provided a technique for solving a variety of applied problems in mathematical sciences and engineering. Subsequently, the superb result of Banach was extended and generalized by several authors using various contractive conditions in different spaces.

On the other hand, Connell [7] gave an example of a metric space $(X, d)$ such that $(X, d)$ is not complete and every contraction on $X$ has a fixed point. Thus, Banach's fixed point theorem cannot characterize the metric completeness of $(X, d)$. A mapping $T$ on a metric space $(X, d)$ is called Kannan if there exists $\alpha \in [0, 1/2)$ such that

$$d(Tx, Ty) \le \alpha(d(x, Tx) + d(y, Ty))$$

for all $x, y \in X$. In the year 1968, Kannan [14] proved that if $(X, d)$ is complete, then $T$ has a unique fixed point in $X$. Kannan [15] provided examples which show that Kannan's fixed point theorem is independent of the Banach contraction principle and Kannan mapping need not be continuous. Kannan's theorem is also very interesting because Subrahmanyam [30] demonstrated that Kannan's theorem characterizes the metric completeness, that is a metric space $(X, d)$ is complete if and only if every Kannan mapping on $X$ has a fixed point. Several authors generalized Kannan's fixed point theorem, see [2], [6], [8], [10], [11], [12], [13], [17], [19], [22] and [24].

Suzuki [31] categorized the theorems which ensure the existence of a fixed point of a mapping $T$ into the following four types.

- **($\mathbf{T}_1$)** Leader type [18]: $T$ has a unique fixed point and $\{T^n x\}$ converges to the fixed point for all $x \in X$. Such a mapping is called a Picard operator.
- **($\mathbf{T}_2$)** Unnamed type: $T$ has a unique fixed point and $\{T^n x\}$ does not necessarily converge to the fixed point.
- **($\mathbf{T}_3$)** Subrahmanyam [30]: $T$ may have more than one fixed point and $\{T^n x\}$ converges to a fixed point for all $x \in X$. Such a mapping is called a weakly Picard operator.
- **($\mathbf{T}_4$)** Caristi type [4], [5]: $T$ may have more than one fixed point and $\{T^n x\}$ does not necessarily converge to a fixed point.

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†Department of Mathematics and computer sciences, larbi Ben M'hidi University, Oum El Bouaghi, 04000, Algeria and Department of Mathematics and computer sciences, Mohamed-Cherif Messaadia University, Souk-Ahas, 41000, Algeria

‡Department of Mathematics and computer sciences, Laboratory of dynamical systems and control, larbi Ben M'hidi University, Oum El Bouaghi, 04000, Algeria

We find the relationship between the class of mappings we considered and metric completeness in the following:

We know that most of the theorems such as Banach's [3], and Kannan's [15] belong to ($\mathrm{T}_1$). Also, very recently, Suzuki [32] characterized ($\mathrm{T}_1$) ([32, Theorem 3]: Let $(X,d)$ be a complete metric space and let $T$ be a strong Leader mapping on $X$ (see definition 2 [32]). Then ($\mathrm{T}_1$) holds.). Subrahmanyam's theorem [30] belongs to ($\mathrm{T}_3$), and Caristi's theorem [4], [5] belong to ($\mathrm{T}_4$). On the other hand, to our best knowledge, there are no theorems belonging to ($\mathrm{T}_2$).

For more details, readers interested in equivalence of completeness and fixed point problem are referred to Nicolae [20]. Motivated by the above, Khojasteh et al. [16] established two new types of fixed point theorems of single-valued and multivalued mappings which belong to ($\mathrm{T}_3$). We begin by the following definition and theorem.

**Definition 1 ( [25])** *Let $(X,d)$ be a metric space and $T : X \to X$ a mapping. The fixed point problem of $T$ is said to be well-posed if*

*i) $T$ has a unique fixed point $z$ in $X$,*

*ii) for any sequence $\{y_n\}$ in $X$ such that $\lim_{n\to\infty} d(Ty_n, y_n) = 0$, we have $\lim_{n\to\infty} d(y_n, z) = 0$.*

The next theorem was shown by Khojasteh et al. [16].

**Theorem 2** *Let $(X,d)$ be a complete metric space and $T$ a mapping from $X$ into itself satisfying the following condition*

$$d(Tx,Ty) \le \frac{d(y,Tx)+d(x,Ty)}{d(x,Tx)+d(y,Ty)+1} d(x,y) \tag{1}$$

*for all $x, y \in X$. Then*

*(i) $T$ has at least one fixed point $z \in X$,*

*(ii) $\{T^n x\}$ converges to a fixed point for all $x \in X$, and*

*(iii) if $z$ and $w$ are distinct fixed points of $T$, then $d(z,w) \ge \frac{1}{2}$.*

Inspired by Theorem 2, it is our purpose in this paper to prove unique fixed point theorems for a self-mapping in complete metric spaces and that the fixed point problem is well-posed. Examples are furnished to illustrate the validity of our results and we give some remarks about the papers [1], [2] and [16]. Afterwards, we apply our Theorem 4 to study the possibility of optimally controlling the solution of an ordinary differential equation via dynamic programming.

## 2 Main Results

The next lemma plays a crucial role in the proof of our main theorems.

**Lemma 3** *Let $(X,d)$ be a metric space and $\{x_n\}$ a sequence in $X$ such that*

$$d(x_n, x_{n+1}) \le \beta_n d(x_{n-1}, x_n) \tag{2}$$

*for all $n \in \mathbb{N}^*$, where*

$$\beta_n = \frac{d(x_{n-1},x_n)+d(x_n,x_{n+1})}{d(x_{n-1},x_n)+d(x_n,x_{n+1})+1}.$$

*Then $\{x_n\}$ is a Cauchy sequence.*

**Proof.** As in the proof of Lemma 2.3 of Rhoades [27], assume that $x_{n-1} \neq x_n$ for each $n \geq 1$ and set $t_n = d(x_{n-1}, x_n)$. Therefore

$$\beta_n = \frac{t_n + t_{n+1}}{t_n + t_{n+1} + 1}. \tag{3}$$

Since $0 < \beta_n < 1$, we deduce from (2) that

$$t_{n+1} \leq \beta_n t_n < t_n \text{ for any } n \in \mathbb{N}. \tag{4}$$

We will prove that for all $n \geq 1$, $\beta_n < \beta_{n-1}$. Using (3) , we obtain that $\beta_n < \beta_{n-1}$ is equivalent to

$$\frac{t_n + t_{n+1}}{t_n + t_{n+1} + 1} < \frac{t_{n-1} + t_n}{t_{n-1} + t_n + 1}$$

The above inequality yields $t_{n+1} < t_{n-1}$ which is fulfilled by (4). Consequently

$$t_{n+1} < \beta_1 t_n \text{ for every } n \in \mathbb{N}.$$

Thus, $\{x_n\}$ is a Cauchy sequence in $X$. ■

**Theorem 4** *Let $(X, d)$ be a complete metric space and $T$ a mapping from $X$ into itself satisfying the following condition*

$$d(Tx, Ty) \leq \frac{d(x, Ty) + d(y, Tx)}{d(x, Tx) + d(y, Ty) + 1} \max\{d(x, Tx), d(y, Ty)\} \tag{5}$$

*for all $x, y \in X$. Then*

*a) $T$ has a unique fixed point $z \in X$,*

*b) The fixed point problem of $T$ is well- posed, and*

*c) $T$ is continuous at $z$.*

**Proof.** a) Let $x_0$ be an arbitrary point in $X$. We define a sequence $\{x_n\}$ in $X$ by $x_{n+1} = Tx_n$, $n \in \mathbb{N}$. Employing the inequality (5), we have

$$\begin{aligned}
d(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n) \\
&\leq \frac{d(x_{n-1}, x_{n+1})}{d(x_{n-1}, x_n) + d(x_n, x_{n+1}) + 1} \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} \\
&\leq \frac{d(x_{n-1}, x_n) + d(x_n, x_{n+1})}{d(x_{n-1}, x_n) + d(x_n, x_{n+1}) + 1} \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} \\
&= \frac{d(x_{n-1}, x_n) + d(x_n, x_{n+1})}{d(x_{n-1}, x_n) + d(x_n, x_{n+1}) + 1} d(x_{n-1}, x_n) \\
&= \beta_n d(x_{n-1}, x_n),
\end{aligned}$$

where

$$\beta_n = \frac{d(x_{n-1}, x_n) + d(x_n, x_{n+1})}{d(x_{n-1}, x_n) + d(x_n, x_{n+1}) + 1}.$$

Applying Lemma 3, we deduce that $\{x_n\}$ is a Cauchy sequence. Since $(X, d)$ is complete, there exists a point $z \in X$ such that $\lim_{n \to \infty} x_n = z$. We assert that $z$ is a fixed point of $T$. If $Tz \neq z$ using the inequality (5) we get

$$d(Tx_n, Tz) \leq \frac{d(x_n, Tz) + d(z, x_{n+1})}{d(x_n, x_{n+1}) + d(z, Tz) + 1} \max\{d(x_n, x_{n+1}), d(z, Tz)\}. \tag{6}$$

Taking the limit as $n \to \infty$ in (6), we obtain

$$d(Tz,z) \leq \frac{d^2(Tz,z)}{d(Tz,z)+1} < d(Tz,z).$$

Therefore, $z$ is a fixed point of $T$. For the uniqueness, assume that $w \neq z$ is an other fixed point of $T$. From the inequality (5) we find

$$d(z,w) \leq d(Tz,Tw) \leq 0.$$

Hence, $z$ is unique.

b) Let $\{y_n\}$ be a sequence in $X$ such that $\lim_{n\to\infty} d(Ty_n, y_n) = 0$. We have

$$d(y_n,z) \leq d(y_n,Ty_n) + d(Ty_n,Tz).$$

Utilizing the inequality (5) we get

$$d(Ty_n,Tz) \leq \frac{d(y_n,Tz)+d(z,Ty_n)}{d(y_n,Ty_n)+d(z,Tz)+1} \max\{d(y_n,Ty_n), d(z,Tz)\}. \tag{7}$$

Therefore

$$d(Ty_n,Tz) \leq \frac{d(y_n,z)+d(z,Tz)+d(z,y_n)+d(y_n,Ty_n)}{d(y_n,Ty_n)+1} d(y_n,Ty_n).$$

Thus, $\lim_{n\to\infty} d(Ty_n,Tz) = 0$ and so $\lim_{n\to\infty} d(y_n,z) = 0$, that is the fixed point problem of $T$ is well-posed.

c) Let $\{y_n\}$ be a sequence in $X$ such that $\lim_{n\to\infty} y_n = z$. Suppose that

$$\limsup_{n\to\infty} d(Ty_n,Tz) = \limsup_{n\to\infty} d(Ty_n,z) = l > 0.$$

Letting $n \to \infty$ in (7), we obtain $l \leq \dfrac{l}{l+1} l < l$. Hence, $\lim_{n\to\infty} d(Ty_n,Tz) = 0$ and so $T$ is continuous at $z$. ■

The following example supports our Theorem 4.

**Example 5** *Let $X = \{0,1,2\}$ and $d : X \times X \to \mathbb{R}_+$ defined by*

$$\begin{aligned} d(0,1) &= d(1,0) = 1,\ d(1,2) = d(2,1) = 2, \\ d(0,2) &= d(2,0) = 3, \\ d(0,0) &= d(1,1) = d(2,2) = 0. \end{aligned}$$

*$(X,d)$ is a complete metric space. Define $T : X \to X$ by:*

$$T(0) = 0,\ T(1) = 0,\ T(2) = 1.$$

*1) The cases $x = y$ and $(x,y) = (0,1)$ are obvious.*

*2) For the case $(x,y) = (0,2)$, we have*

$$\begin{aligned} d(T(0),T(2)) &= d(0,1) = 1 \\ &< \frac{d(0,T(2))+d(2,T(0))}{d(2,T(2))+1} d(2,T(2)) \\ &= \frac{d(0,1)+d(2,0)}{d(2,1)+1} d(2,1) \\ &= \frac{4}{3} \times 2 = \frac{8}{3}. \end{aligned}$$

*3) For the case* $(x, y) = (1, 2)$*, we get*

$$\begin{aligned} d(T(1), T(2)) &= d(0,1) = 1 \\ &< \frac{d(1,T(2)) + d(2,T(1))}{d(1,T(1)) + d(2,T(2)) + 1} \max\{d(1,T(1)), d(2,T(2))\} \\ &= \frac{d(1,1) + d(2,0)}{d(1,0) + d(2,1) + 1} \max\{d(1,0), d(2,1)\} \\ &= \frac{3}{4} \times 2 = \frac{3}{2}. \end{aligned}$$

*Hence, $T$ satisfies all the conditions of Theorem 4 and $T$ has a unique fixed point* $0$*.*
*Kannan's fixed point theorem is not applicable because*

$$\begin{aligned} d(T(0), T(2)) &= 1 \\ &> \alpha(d(0,T(0)) + d(2,T(2))) \\ &= 2\alpha \end{aligned}$$

*for any* $\alpha \in [0, 1/2)$*.*

In a similar manner, we can prove the subsequent theorem. We omit the proof.

**Theorem 6** *Let $(X, d)$ be a complete metric space and $T$ a mapping from $X$ into itself such that the inequality*

$$d(Tx, Ty) \leq N(x,y) \max\{d(x,Tx), d(y,Ty)\}$$

*is fulfilled for all* $x, y \in X$*, where*

$$N(x,y) = \frac{\max\{d(x,y), d(x,Tx) + d(y,Ty), d(x,Ty) + d(y,Tx)\}}{d(x,Tx) + d(y,Ty) + 1}$$

*Then*

*a) $T$ has a unique fixed point $z \in X$,*

*b) The fixed point problem of $T$ is well- posed, and*

*c) $T$ is continuous at $z$.*

The following example illustrates our Theorem 6.

**Example 7** *Let $X = \{0, 1, 2, 3\}$ and $d : X \times X \to \mathbb{R}_+$ defined by:*

$$\begin{aligned} d(0,1) &= 1,\ d(1,2) = d(0,2) = 2, \\ d(1,3) &= d(0,3) = 3,\ d(2,3) = 5. \\ d(x,x) &= 0 \text{ for all } x \in X \text{ and } d(x,y) = d(y,x) \text{ for all } x, y \in X. \end{aligned}$$

*$(X, d)$ is a complete metric space. Define $T : X \to X$ by:*

$$T(0) = 0,\ T(1) = 0,\ T(2) = 1,\ T(3) = 2.$$

*1) The cases $x = y$ and $(x, y) = (0, 1)$ are obvious.*

*2) For the case* $(x, y) = (0, 2)$*, we have*

$$\begin{aligned} d(T(0), T(2)) &= d(0,1) = 1 \\ &< \frac{\max\{d(0,2), d(2,T(2)), d(0,T(2)) + d(2,T(0))\}}{d(2,T(2)) + 1} d(2,T(2)) \\ &= \frac{\max\{d(0,2), d(2,1), d(0,1) + d(2,0)\}}{d(2,1) + 1} d(2,1) \\ &= \frac{\max\{2, 2, 1+2\}}{2+1} \times 2 = 1 \times 2 = 2. \end{aligned}$$

*3) For the case* $(x, y) = (0, 3)$*, we get*

$$\begin{aligned} d(T(0), T(3)) &= d(0,2) = 2 \\ &< \frac{\max\{d(0,3), d(3,T(3)), d(0,T(3)) + d(3,T(0))\}}{d(3,T(3)) + 1} d(3,T(3)) \\ &= \frac{\max\{d(0,3), d(3,2), d(0,2) + d(3,0)\}}{d(3,2) + 1} d(3,2) \\ &= \frac{\max\{3, 5, 2+3\}}{5+1} \times 5 = \frac{5}{6} \times 5 = \frac{25}{6} \end{aligned}$$

*4) For the case* $(x, y) = (1, 2)$*, we obtain*

$$\begin{aligned} d(T(1), T(2)) &= d(0,1) = 1 \\ &< \frac{\begin{array}{c}\max\{d(1,2), d(1,T(1)) + d(2,T(2)), \\ d(1,T(2)) + d(2,T(1))\}\end{array}}{d(1,T(1)) + d(2,T(2)) + 1} \max\{d(1,T(1)), d(2,T(2))\} \\ &= \frac{\max\{d(1,2), d(1,0) + d(2,1), d(1,1) + d(2,0)\}}{d(1,0) + d(2,1) + 1} \max\{d(1,0), d(2,1)\} \\ &= \frac{\max\{2, 1+2, 2\}}{1+2+1} \times 2 \\ &= \frac{3}{4} \times 2 = \frac{3}{2}. \end{aligned}$$

*5) For the case* $(x, y) = (1, 3)$*, we find*

$$\begin{aligned} d(T(1), T(3)) &= d(0,2) = 2 \\ &< \frac{\begin{array}{c}\max\{d(1,3), d(1,T(1)) + d(3,T(3)), \\ d(1,T(3)) + d(3,T(1))\}\end{array}}{d(1,T(1)) + d(3,T(3)) + 1} \max\{d(1,T(1)), d(3,T(3))\} \\ &= \frac{\max\{d(1,3), d(1,0) + d(3,2), d(1,2) + d(3,0)\}}{d(1,0) + d(3,2) + 1} \max\{d(1,0), d(3,2)\} \\ &= \frac{\max\{3, 1+5, 2+3\}}{1+5+1} \times 5 = \frac{6}{7} \times 5 = \frac{30}{7}. \end{aligned}$$

*6) For the case $(x,y)=(2,3)$, we have*

$$\begin{aligned} d(T(2),T(3)) &= d(1,2)=2 \\ &< \frac{\begin{array}{c}\max\{d(2,3),d(2,T(2))+d(3,T(3)),\\ d(2,T(3))+d(3,T(2))\}\end{array}}{d(2,T(2))+d(3,T(3))+1}\max\{d(2,T(2)),d(3,T(3))\} \\ &= \frac{\max\{d(2,3),d(2,1)+d(3,2),d(2,2)+d(3,1)\}}{d(2,1)+d(3,2)+1}\max\{d(2,1),d(3,2)\} \\ &= \frac{\max\{5,2+5,3\}}{2+5+1}\times 5=\frac{7}{8}\times 5=\frac{35}{8}. \end{aligned}$$

*Hence, $T$ satisfies all the conditions of Theorem 6 and $T$ has a unique fixed point $0$.*

**Remark 8** *Theorem 4 is not applicable because*

$$\begin{aligned} d(T(2),T(3)) &= d(1,2)=2 \\ &> \frac{d(2,2)+d(3,1)\}}{d(2,1)+d(3,2)+1}\max\{d(2,1),d(3,2)\} \\ &= \frac{3}{8}\times 5=\frac{15}{8}. \end{aligned}$$

*This shows that Theorem 6 is a genuine generalization of Theorem 4. Also, Kannan's fixed point theorem is not applicable because $d(T(0),T(2))=1>2\alpha$ for every $\alpha\in[0,1/2)$.*

**Remark 9** *Khojasteh et al. [16] gave the following example.*
*Let $X=[0,2-\sqrt{3}]$ be endowed with Euclidean metric and $T:X\to X$ defined by*

$$Tx=\begin{cases} 0 & \text{if } x\in[0,2-\sqrt{3}), \\ 2-\sqrt{3} & \text{if } x=2-\sqrt{3}. \end{cases}$$

*The authors claimed that $T$ satisfies all the conditions of theorem 2, but this example is false because*

$$d(0,2-\sqrt{3})=2-\sqrt{3}=0.27<\frac{1}{2}.$$

*If we replace $2-\sqrt{3}$ in the above example by $1$, we get $d(0,1)=1>\frac{1}{2}$, however, the inequality (1) does not hold for all $x,y\in[0,1]$. Indeed, for $x=\frac{1}{2}$ and $y=1$, we obtain*

$$d(T(\frac{1}{2}),T(1))=d(0,1)=1>\frac{d(1,0)+d(\frac{1}{2},1)}{d(\frac{1}{2},0)+1}d(\frac{1}{2},1)=\frac{1}{2}$$

The next example verifies Theorem 2.

**Example 10** *Let $X=\{0\}\cup[\frac{3}{4},1]$ be endowed with Euclidean metric and $T:X\to X$ defined by*

$$Tx=\begin{cases} 0 & \text{if } x=0, \\ 1 & \text{if } x\in[\frac{3}{4},1]. \end{cases}$$

*For $x=0$ and $y\in[\frac{3}{4},1]$, we find*

$$d(T(0),Ty)=1,$$

$$\frac{d(y,T(0))+d(0,Ty)}{d(y,Ty)+1}d(0,y)=\frac{y(y+1)}{2-y}.$$

*Since for all* $y\in[\frac{3}{4},1]$, $1<\frac{21}{20}\leq\frac{y(y+1)}{2-y}\leq 2$, *the inequality (1) holds. Note that the ratio*

$$1<\frac{7}{5}\leq r=\frac{d(y,T(0))+d(0,Ty)}{d(y,Ty)+1}\leq 2.$$

*The other cases are obvious. Hence,* $T$ *satisfies all the conditions of Theorem 2 and* $T$ *has two fixed points* $0$ *and* $1$*. Furthermore,* $d(0,1)=1>\frac{1}{2}$.

We end this section by giving some remarks about the papers [1] and [2]. The subsequent theorem was proved by [2].

**Theorem 11** *Let* $\{A_i\}_{i=1}^{p}$, $p>1$, $p\in\mathbb{N}$, *be nonempty closed subsets of a complete metric space* $(X,d)$ *and* $T:\cup_{i=1}^{p}A_i\to\cup_{i=1}^{p}A_i$, *then* $T$ *has a unique fixed point* $z\in\cap_{i=1}^{p}A_i$, *if*

$$\phi(d(Tx,Ty))\leq\phi(\alpha d(x,Tx)+\beta d(y,Ty))-\psi(d(x,Tx),d(y,Ty)) \tag{8}$$

*where,* $x\in A_i, y\in A_{i+1}$, $i=1,2...,p$, $\alpha,\beta\in(0,1)$ *such that* $\alpha+\beta\leq 1$*; or*

$$\phi(d(Tx,Ty))\leq\phi(\alpha d(x,Ty)+\beta d(y,Tx))-\psi(d(x,Ty),d(y,Tx)) \tag{9}$$

*where,* $x\in A_i, y\in A_{i+1}$, $i=1,2...,p$, $\beta\in(0,1)$, $\alpha\leq\frac{1}{2}$ *such that* $\alpha+\beta\leq 1$.

$\phi:\mathbb{R}_+\to\mathbb{R}_+$ *is an altering distance function and* $\psi:\mathbb{R}_+^2\to\mathbb{R}_+$ *is a continuous function with* $\psi(t,s)=0$ *if and only if* $t=s=0$.

**Remark 12** *The inequalities (8) and (9) imply that*

$$d(Tx,Ty)\leq\alpha d(x,Tx)+\beta d(y,Ty), \tag{10}$$

$$d(Tx,Ty)\leq\alpha d(x,Ty)+\beta d(y,Tx). \tag{11}$$

**I)** *If* $\alpha+\beta<1$, *employing the inequality (10) we get a special case of Theorem 7 of [22].*

**II)** *If* $\alpha+\beta<1$, *using the inequality (11) we get a special case of Theorem 8 of [22].*

**III)** *If* $\alpha+\beta=1$, *in the light of (10) or (11), Theorem 11 becomes false in general except additional conditions are added: the continuity of* $T$ *and the compactness of the metric space* $(X,d)$*, see [10]. Besides, the two inequalities of Example 3.1 in [2] imply (10) and (11) and this example is not true because we cannot take* $\psi(t,s)=0$*. A Kannan type mapping* $T:X\to X$ *such that for all* $x,y\in X$

$$d(Tx,Ty)\leq\frac{1}{2}(d(x,Tx)+d(y,Ty))$$

*in a complete metric space* $(X,d)$ *may not have a fixed point, see [23] and [11] Example 1.4. Therefore, we cannot take* $\psi(t,s)=0$ *and* $\alpha=\beta=\frac{1}{2}$ *in remark 2.1 of [2] and so Corollary 1 of [1] is also incorrect. Even a continuous Kannan type mapping such that for all* $x,y\in X$, $x\neq y$

$$d(Tx,Ty)<\frac{1}{2}(d(x,Tx)+d(y,Ty))$$

*in complete but noncompact metric space* $(X,d)$ *may not have a fixed point, see [11, Example 1.5.].*

# 3 Application to Control Theory

In this section, inspired by the papers of Pathak and Shahzad [21] and Rhoades et al. [26], we investigate the possibility of optimally controlling the solution of the ordinary differential equation (14) via dynamic programming.

Let $A$ be a compact subset of $\mathbb{R}^m$ and for each given $a \in A$, $F_a : \mathbb{R}^n \to \mathbb{R}^n$ is a mapping such that

$$F_a(x) = f(x, a) \text{ for all } x \in \mathbb{R}^n,$$

where $f : \mathbb{R}^n \times A \to \mathbb{R}^n$ is a given bounded continuous function which satisfies the following contractive condition

$$|f(x,a)| \leq C \quad \text{for some} \quad C > 0, \tag{12}$$

$$|f(x,a) - f(y,a)| \leq \frac{|x - f(y,a)| + |y - f(x,a)|}{|x - f(x,a)| + |y - f(y,a)| + 1} \max\{|x - f(x,a)|, |y - f(y,a)|\}, \tag{13}$$

where

$$0 \leq \frac{|x - f(y,a)| + |y - f(x,a)|}{|x - f(x,a)| + |y - f(y,a)| + 1} < 1 \quad \text{for all } x, y \in X.$$

Now, we will study the possibility of optimally controlling the solution $\mathbf{x}(.)$ of the ordinary differential equation

$$\begin{cases} x'(s) = f(x(s), \alpha(s)), & t < s < T, \\ x(t) = x. \end{cases} \tag{14}$$

Here $x'(s) = \frac{dx(s)}{ds}, T > 0$, is a fixed terminal time and $x \in \mathbb{R}^n$ is a given initial point, taken on by our solution $\mathbf{x}(.)$ at the starting time $t \geq 0$. At later times $t < s < T$, $\mathbf{x}(\cdot)$ evolves according to the ODE (14).The function $\alpha(.)$ appearing in (14) is a control, that is some appropriate for adjusting parameters from the set $A$ as time evolves there by affecting the dynamics of the system modelled by (14). Let us write

$$Ad = \{\alpha : [0, T] \to A \text{ such that } \alpha(.) \text{ is measurable}\}$$

to denote the set of admissible controls. Since $F_a(x) = f(x, a)$ for all $x \in \mathbb{R}^n$, employing (12) and (13) we obtain

$$|F_a(x) - F_a(y)| \leq \frac{|x - F_a(y)| + |y - F_a(x))|}{|x - F_a(x)| + |y - F_a(y))| + 1} \max\{|x - F_a(x)|, |y - F_a(y)|\}$$

for all $x, y \in \mathbb{R}^n$, $a \in A$. Applying Theorem 4, we deduce that for each control $\alpha(.) \in Ad$, the ODE (14) has a unique continuous solution $\mathbf{x} = \mathbf{x}^{\alpha(\cdot)}(.)$, existing on the time interval $[t, T]$ and solving the ODE for almost everywhere time $t < s < T$. We call $\mathbf{x}(.)$ the response of the system to the control $\alpha(.)$ and $\mathbf{x}(s)$ the state of the system at time $s$.

Our goal is to find a control $\alpha^*(.)$ which optimally steers the system. We must first introduce a cost criterion. Given $x \in \mathbb{R}^n$ and $0 \leq t \leq T$, let us define for each admissible control $\alpha(.) \in Ad$ the corresponding cost

$$P_{x,t}(\alpha(.)) := \int_t^T h(\mathbf{x}(s), \alpha(s))ds + g(\mathbf{x}(T)), \tag{15}$$

where $\mathbf{x} = \mathbf{x}^{\alpha(\cdot)}(.)$ solves the ODE (14) and

$$h : \mathbb{R}^n \times A \to \mathbb{R}, g : \mathbb{R}^n \to \mathbb{R}$$

are given functions. We call $h$ the running cost per unit time and $g$ the terminal cost and will hereafter suppose

$$\begin{cases} |H_a(x)|, \ |g(x)| \leq C \quad \text{for some} \quad C > 0, \\ |H_a(x) - H_a(y)| \leq \dfrac{|x - H_a(y)| + |y - H_a(x))|}{|x - H_a(x)| + |y - H_a(y))| + 1} \max\{|x - H_a(x)|, |y - H_a(y)|\}, \\ |g(x) - g(y)| \leq \dfrac{|x - g(y)| + |y - g(x))|}{|x - g(x)| + |y - g(y))| + 1} \max\{|x - g(x)|, |y - g(y)|\}, \\ \text{for all} \quad x, y \in \mathbb{R}^n, \, a \in A, \end{cases}$$

where $H_a : \mathbb{R}^n \to \mathbb{R}^n$ is a mapping such that

$$H_a(x) = h(x, a) \text{ for all } x \in \mathbb{R}^n.$$

Given $x \in \mathbb{R}^n$ and $0 \leq t \leq T$, we would like to find if possible a control $\alpha^*(.)$ which minimizes the cost functional (15) among all other admissible controls.

To investigate the above problem we shall apply the method of dynamic programming. We now turn our attention to the value function $u(x,t)$ defined by

$$u(x,t) := \inf_{\alpha(.)\in Ad} P_{x,t}\left(\alpha(.)\right), \, x \in \mathbb{R}^n, \, 0 \leq t \leq T.$$

The idea is: having defined $u(x,t)$ as the least cost given we start at the position $x$ at time $t$, we want to study $u$ as a function of $x$ and $t$. We are therefore embedding our given control problem (14) and (15) into the larger class of all such problems, as $x$ and $t$ vary. This idea then can be used to show that $u$ solves a certain Hamilton–Jacobi type PDE, and finally to show conversely that a solution of this PDE helps us to synthesize an optimal feedback control.

Let us fix $x \in \mathbb{R}^n$ and $0 \leq t \leq T$. Following the technique of Evan [9, p. 553–554], the subsequent theorem gives the optimality conditions in the form (16).

**Theorem 13** *For each $\xi > 0$ so small that $t + \xi \leq T$, we have*

$$u(x,t) := \inf_{\alpha(.)\in Ad} \left\{ \int_t^{t+\xi} h(\mathbf{x}(s), \alpha(s))ds + u(\mathbf{x}(t+\xi), t+\xi) \right\}, \tag{16}$$

*where $\mathbf{x} = \mathbf{x}^{\alpha(.)}$ solves the ODE (14) for the control $\alpha(.)$.*

**Proof.** It follows as in Theorem 1 [p. 553] of Evan [9]. ■

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