

A Generalized Fixed Point Theorem In An Extended Cone b -Metric Space Over Banach Algebra With Its Application To Dynamical Programming*

Kushal Roy[†], Sayantan Panja[‡], Mantu Saha[§]

Received 10 April 2020

Abstract

In this manuscript, we introduce the notion of an extended cone b -metric space over Banach algebra with underlying non-normal cone. A fixed point theorem has been proved for a generalized Lipschitzian mapping on such spaces. Several corollaries have been obtained from our given fixed point theorem. Some supporting examples have been cited to examine the validity of our established result. Moreover, our fixed point result is applied to obtain solutions for some functional equations.

1 Introduction and Preliminaries

Nowadays fixed point theory is one of the prominent area of research. In metric fixed point theory our main objective is to know whether a self-mapping in a metric space or metric-type space, has fixed point or not.

The concept of b -metric spaces had been initiated by Bakhtin [2] in 1989 which generalizes usual metric spaces and thereby he had been succeeded to generalize the Banach contraction principle over it. This concept was further improved by Czerwik [5] in his research article. Recently Kamran et al. [11] have generalized the concept of b -metric space in a totally new way. They first show that the coefficient of a b -metric space can be a function instead of just a constant.

Huang and Zhang [9] in 2007 had been able to introduce cone metric spaces as a generalization of metric spaces in which the set of real numbers are replaced by the elements of an ordered Banach space. In view of this notion they had been succeeded to define Cauchy sequence, convergence of a sequence in such spaces and obtained some fixed point theorems over it. After that several researchers have published many papers on such type of spaces (See [1], [6], [12], [15], [16]). Very recently, Liu and Xu [14] have generalized the concept of cone metric space by changing the underlying Banach space to a Banach algebra and introduced cone metric space over Banach algebra. They defined cone over a Banach algebra in a generalized way and investigated some partial orderings in terms of interior points of the underlying cone. They have also defined generalized contractive mapping in this setting where the Lipschitzian constant is a vector instead of real constant and proved some fixed point theorems. Moreover, they have given an example to show that the fixed point theorems in cone metric spaces over Banach algebras are usually not equivalent to those in case of a metric space. Though Liu and Xu have used normal cones in their article but Xu and Radenović [19] avoided the normality of cones and defined solid cones to define c -sequences on cone metric spaces over Banach algebras.

In light of the same spirit Huang and Radenović [8] have introduced the concept of cone b -metric spaces over Banach algebras by bundling together the concept of cone metric space over Banach algebra and b -metric space. Since b -metric spaces have various interesting properties than usual metric spaces it follows that cone b -metric spaces over Banach algebras have those interesting properties too. The authors proved several fixed point theorems on such spaces with the help of underlying ordered Banach algebra.

*Mathematics Subject Classifications: 47H10, 54H25.

[†]Department of Mathematics, The University of Burdwan, Purba Bardhaman-713104, West Bengal, India

[‡]Department of Mathematics, The University of Burdwan, Purba Bardhaman-713104, West Bengal, India

[§]Department of Mathematics, The University of Burdwan, Purba Bardhaman-713104, West Bengal, India

Before going to the next sections we need some preliminaries which help us to obtain our main results.

Definition 1 ([18]) *A vector space \mathcal{A} over a field K (\mathbb{R} or \mathbb{C}) is said to be an algebra if it is closed under multiplication (i.e., for all $\xi, \eta \in \mathcal{A}$, $\xi\eta \in \mathcal{A}$) and*

- (i) $(\xi\eta)\mu = \xi(\eta\mu)$, $\forall \xi, \eta, \mu \in \mathcal{A}$;
- (ii) $\xi(\eta + \mu) = \xi\eta + \xi\mu$ and $(\xi + \eta)\mu = \xi\mu + \eta\mu$, $\forall \xi, \eta, \mu \in \mathcal{A}$;
- (iii) $r(\xi\eta) = (r\xi)\eta = \xi(r\eta)$, $\forall \xi, \eta \in \mathcal{A}$, $\forall r \in K$.

A Banach space \mathcal{A} over the field K (\mathbb{R} or \mathbb{C}) is said to be a Banach algebra if

- (i) \mathcal{A} is an algebra and (ii) $\forall \xi, \eta \in \mathcal{A}$, $\|\xi\eta\| \leq \|\xi\| \cdot \|\eta\|$.

Here we shall always assume that the Banach algebra \mathcal{A} is unital, that is it has a unity element $e_{\mathcal{A}}$ such that $e_{\mathcal{A}}\xi = \xi e_{\mathcal{A}} = \xi$, for all $\xi \in \mathcal{A}$. Note that the unity element of a Banach algebra \mathcal{A} , if it exists, is unique. A non-zero element $\eta \in \mathcal{A}$ is said to be invertible if its inverse exists i.e. if there exists a non-zero element μ such that $\eta\mu = \mu\eta = e_{\mathcal{A}}$, we write $\mu = \eta^{-1}$ and we call μ is the inverse of η . One can show that in a Banach algebra \mathcal{A} , with the unity element $e_{\mathcal{A}}$ the inverse of an element is unique. Also for all $\xi, \eta \in \mathcal{A}$, we have $(\xi\eta)^{-1} = \eta^{-1}\xi^{-1}$ and $(\xi^{-1})^{-1} = \xi$.

Proposition 1 ([18]) *Let \mathcal{A} be a Banach algebra with unit $e_{\mathcal{A}}$, the spectral radius of an element $\xi \in \mathcal{A}$ is denoted by $\rho(\xi)$ and defined by $\rho(\xi) = \sup_{r \in \sigma(\xi)} |r| = \lim_{n \rightarrow \infty} \|\xi^n\|^{\frac{1}{n}}$, where $\sigma(\xi)$ is the spectrum of $\xi \in \mathcal{A}$. If $\rho(\xi) < 1$ then $e_{\mathcal{A}} - \xi$ is invertible and $(e_{\mathcal{A}} - \xi)^{-1} = e_{\mathcal{A}} + \sum_{i=1}^{\infty} \xi^i$.*

Remark 1 *The spectral radius $\rho(\xi)$ of $\xi \in \mathcal{A}$ satisfies $\rho(\xi) \leq \|\xi\|$, where \mathcal{A} is a Banach algebra with unity $e_{\mathcal{A}}$.*

Remark 2 *If $\rho(\xi) < 1$, then we get $\|\xi\|^n \rightarrow 0$ as $n \rightarrow \infty$.*

Definition 2 ([7]) *A subset \mathbb{P} of a unital Banach algebra \mathcal{A} is called a cone if*

1. \mathbb{P} is non empty, closed and $\theta_{\mathcal{A}}, e_{\mathcal{A}} \in \mathbb{P}$.
2. If $\xi, \eta \in \mathbb{P}$ and $r, s \geq 0$, then $r\xi + s\eta \in \mathbb{P}$.
3. $\xi, \eta \in \mathbb{P}$ implies $\xi\eta \in \mathbb{P}$.
4. If $\xi, -\xi \in \mathbb{P}$ for some $\xi \in \mathcal{A}$ then $\xi = \theta_{\mathcal{A}}$, where $\theta_{\mathcal{A}}$ is the zero element of \mathcal{A} .

A cone \mathbb{P} is called a solid cone if $\text{int}(\mathbb{P}) \neq \emptyset$. Each cone \mathbb{P} induces a partial ordering \lesssim on \mathcal{A} by $\xi \lesssim \eta$ iff $\eta - \xi \in \mathbb{P}$. We write $\xi \prec \eta$ if $\xi \lesssim \eta$ and $\xi \neq \eta$. When the cone is solid $\xi \ll \eta$ will stand for $\eta - \xi \in \text{int}(\mathbb{P})$. The cone \mathbb{P} is said to be normal if there exists a number $L > 0$ such that $\theta_{\mathcal{A}} \lesssim \xi \lesssim \eta \Rightarrow \|\xi\| \leq L\|\eta\|$. The least positive number L , which satisfies the normality condition is called the normal constant of \mathbb{P} .

Lemma 1 ([17]) *If \mathcal{B} be a real Banach space with a solid cone \mathbb{P} and if $\theta_{\mathcal{B}} \lesssim \xi \ll c$ for all $c \gg \theta_{\mathcal{B}}$, then $\xi = \theta_{\mathcal{B}}$.*

Definition 3 ([19]) *Let \mathbb{P} be a solid cone in a Banach space \mathcal{B} . A sequence $\{\nu_n\} \subset \mathbb{P}$ is called a c -sequence if for each $\theta_{\mathcal{B}} \ll c$ there exists a natural number N such that $\nu_n \ll c$ whenever $n \geq N$.*

Lemma 2 ([19]) *If \mathcal{B} is a real Banach space with a solid cone \mathbb{P} and $\{\nu_n\} \subset \mathbb{P}$ is a sequence with $\|\nu_n\| \rightarrow 0$ as $n \rightarrow \infty$, then $\{\nu_n\}$ is a c -sequence.*

Lemma 3 ([19]) *Let \mathcal{A} be a Banach algebra with unity $e_{\mathcal{A}}$. Let $\xi, \eta \in \mathcal{A}$ such that ξ and η commute, then*

$$(i) \quad \rho(\xi\eta) \leq \rho(\xi)\rho(\eta).$$

$$(ii) \quad \rho(\xi + \eta) \leq \rho(\xi) + \rho(\eta).$$

$$(iii) \quad |\rho(\xi) - \rho(\eta)| \leq \rho(\xi - \eta).$$

Lemma 4 ([8]) *If \mathcal{B} be a real Banach space and \mathbb{P} be a solid cone of \mathcal{B} then for $\xi, \eta, c \in \mathcal{B}$ with $\xi \lesssim \eta \ll c$ implies $\xi \ll c$.*

Lemma 5 ([19]) *Let \mathbb{P} be a solid cone of a Banach algebra \mathcal{A} . Suppose that $k \in \mathbb{P}$ is an arbitrary vector and $\{\nu_n\} \subset \mathbb{P}$ is a c -sequence, then $\{k\nu_n\}$ is also a c -sequence.*

Lemma 6 ([8]) *Let \mathcal{A} be a Banach algebra with unity $e_{\mathcal{A}}$ and $\zeta \in \mathcal{A}$. Let z be a complex constant such that $\rho(\zeta) < |z|$. Then, $\rho((ze_{\mathcal{A}} - \zeta)^{-1}) \leq \frac{1}{|z| - \rho(\zeta)}$.*

Lemma 7 ([8]) *Let \mathcal{B} be a Banach space and \mathbb{P} be a solid cone of \mathcal{B} . Let $\xi, \eta, \zeta \in \mathbb{P}$, $\zeta \lesssim \eta$ and $\xi \lesssim \zeta\xi$ with $\rho(\eta) < 1$. Then $\xi = \theta_{\mathcal{B}}$.*

Lemma 8 ([19]) *Let \mathcal{A} be a Banach algebra with unity $e_{\mathcal{A}}$ and $\{\xi_n\} \subset \mathcal{A}$. Suppose that $\{\xi_n\}$ converges $\xi \in \mathcal{A}$ and that ξ_n and ξ commute for all n . Then we have $\rho(\xi_n) \rightarrow \rho(\xi)$ as $n \rightarrow \infty$.*

Lemma 9 ([8]) *Let \mathcal{A} be a Banach algebra with unity $e_{\mathcal{A}}$ and \mathbb{P} be a solid cone of \mathcal{A} . Let $\zeta \in \mathcal{A}$ and $\nu_n = \zeta^n$ for all $n \in \mathbb{N}$. If $\rho(\zeta) < 1$, then $\{\nu_n\}$ is a c -sequence.*

Lemma 10 ([10]) *For a cone \mathbb{P} in the Banach space $(\mathcal{B}, \|\cdot\|)$, the followings are equivalent:*

1. \mathbb{P} is normal.
2. for arbitrary sequence $\{\xi_n\}, \{\eta_n\}, \{\zeta_n\}$ in \mathcal{B} , with $\xi_n \lesssim \eta_n \lesssim \zeta_n \forall n \in \mathbb{N}$, if $\lim_{n \rightarrow \infty} \xi_n = \lim_{n \rightarrow \infty} \zeta_n = \xi$, then $\lim_{n \rightarrow \infty} \eta_n = \xi$.
3. there exists a norm $\|\cdot\|_1$ on \mathcal{B} equivalent to $\|\cdot\|$ such that the cone is monotone with respect to $\|\cdot\|_1$.

Definition 4 ([2]) *Let X be a nonempty set and b be a real number satisfying $b \geq 1$. A function $D_b : X \times X \rightarrow [0, \infty)$ is said to be a b -metric on X if for all $\xi, \eta, \zeta \in X$, the following conditions hold:*

1. $D_b(\xi, \eta) = 0$ iff $\xi = \eta$.
2. $D_b(\xi, \eta) = D_b(\eta, \xi)$.
3. $D_b(\xi, \zeta) \leq b[D_b(\xi, \eta) + D_b(\eta, \zeta)]$.

The space (X, D_b) is called a b -metric space.

Definition 5 ([11]) *Let X be a nonempty set and $\theta : X \times X \rightarrow [1, \infty)$. A function $D_{eb} : X^2 \rightarrow [0, \infty)$ is called an extended b -metric if for all $\xi, \eta, \zeta \in X$ it satisfies:*

1. $D_{eb}(\xi, \eta) = 0$ if and only if $\xi = \eta$.
2. $D_{eb}(\xi, \eta) = D_{eb}(\eta, \xi)$.
3. $D_{eb}(\xi, \zeta) \leq \theta(\xi, \zeta)[D_{eb}(\xi, \eta) + D_{eb}(\eta, \zeta)]$.

The pair (X, D_{eb}) is called an extended b -metric space.

Definition 6 ([14]) Let X be a nonempty set and \mathcal{A} be a real unital Banach algebra with a solid cone \mathbb{P} . The mapping $D_c : X \times X \rightarrow \mathcal{A}$ is said to be a cone metric over Banach algebra on X if it satisfies the following conditions:

1. $D_c(\xi, \eta) \succeq \theta_{\mathcal{A}}$ for all $\xi, \eta \in X$ and $D_c(\xi, \eta) = \theta_{\mathcal{A}}$ iff $\xi = \eta$.
2. $D_c(\xi, \eta) = D_c(\eta, \xi)$ for all $\xi, \eta \in X$.
3. $D_c(\xi, \zeta) \preceq D_c(\xi, \eta) + D_c(\eta, \zeta)$ for all $\xi, \eta, \zeta \in X$.

The space (X, D_c) is called a cone metric space over the Banach algebra \mathcal{A} .

Definition 7 ([8]) Let X be a nonempty set, \mathcal{A} be a real unital Banach algebra with a solid cone \mathbb{P} and $s \geq 1$ be a constant. The mapping $D_{cb} : X \times X \rightarrow \mathcal{A}$ is called a cone b -metric over Banach algebra on X if the following conditions hold:

1. $D_{cb}(\xi, \eta) \succeq \theta_{\mathcal{A}}$ for all $\xi, \eta \in X$ and $D_{cb}(\xi, \eta) = \theta_{\mathcal{A}}$ iff $\xi = \eta$.
2. $D_{cb}(\xi, \eta) = D_{cb}(\eta, \xi)$ for all $\xi, \eta \in X$.
3. $D_{cb}(\xi, \zeta) \preceq s[D_{cb}(\xi, \eta) + D_{cb}(\eta, \zeta)]$ for all $\xi, \eta, \zeta \in X$.

The space (X, D_{cb}) is called a cone b -metric space over the Banach algebra \mathcal{A} .

2 Main Results

In this section we introduce generalized cone b -metric space over Banach algebra without the assumption of normality of cones.

In the following definitions and theorems we always suppose that \mathcal{A} is a real Banach algebra with a unity, \mathbb{P} is a solid cone in \mathcal{A} , not necessarily normal and $\preceq, <$ and \ll are partial orderings with respect to \mathbb{P} .

Definition 8 Let X be a nonempty set, $\theta : X^2 \rightarrow [1, \infty)$ be a mapping. A mapping $\mathcal{D}_{ecb} : X \times X \rightarrow \mathcal{A}$ is said to be an extended cone b -metric over Banach algebra if it satisfies the following conditions:

- ($\mathcal{D}_{ecb}1$) $\mathcal{D}_{ecb}(\xi, \eta) \succeq \theta_{\mathcal{A}}$ and $\mathcal{D}_{ecb}(\xi, \eta) = \theta_{\mathcal{A}}$ iff $\xi = \eta$;
- ($\mathcal{D}_{ecb}2$) $\mathcal{D}_{ecb}(\xi, \eta) = \mathcal{D}_{ecb}(\eta, \xi)$;
- ($\mathcal{D}_{ecb}3$) $\mathcal{D}_{ecb}(\xi, \eta) \preceq \theta(\xi, \eta)[\mathcal{D}_{ecb}(\xi, \zeta) + \mathcal{D}_{ecb}(\zeta, \eta)]$.

The triplet $(X, \mathcal{D}_{ecb}, \mathcal{A})$ is called an extended cone b -metric space over Banach algebra.

Remark 3 An extended cone b -metric space over Banach algebra generalizes several known metric structures, such as:

- (i) If $\theta(\xi, \eta) = 1$ for all $\xi, \eta \in X$ then an extended cone b -metric space over Banach algebra reduces to a cone metric space over Banach algebra.
- (ii) A cone b -metric space over Banach algebra is an extended cone b -metric space over Banach algebra for $\theta(\xi, \eta) = s > 1$ for all $\xi, \eta \in X$.
- (iii) If $\theta(\xi, \eta) = 1$ for all $\xi, \eta \in X$ and $\mathcal{A} = \mathbb{R}$ with the cone $\mathbb{P} = [0, \infty)$ then an extended cone b -metric space over Banach algebra reduces to an usual metric space.
- (iv) A b -metric space is an extended cone b -metric space over Banach algebra for $\theta(\xi, \eta) = s > 1$ for all $\xi, \eta \in X$ and $\mathcal{A} = \mathbb{R}$ with the cone $\mathbb{P} = [0, \infty)$.

(v) If $\mathcal{A} = \mathbb{R}$ with the cone $\mathbb{P} = [0, \infty)$ then an extended cone b -metric space over Banach algebra reduces to an extended b -metric space.

Example 1 Let $\mathcal{A} = C[0, 1]$ be the usual unital Banach algebra with the sup norm. Let $\mathbb{P} = \{f \in C[0, 1] : f(t) \geq 0 \text{ for all } t \in [0, 1]\}$ and $X = \mathbb{R}$. Define a mapping $\mathcal{D}_{ecb} : X^2 \rightarrow \mathcal{A}$ by $\mathcal{D}_{ecb}(\xi, \eta)(t) = (1 + |\xi| + |\eta|)|\xi - \eta| \exp(t)$ for any $\xi, \eta \in X$ and for all $t \in [0, 1]$. Conditions $(\mathcal{D}_{ecb}1)$ and $(\mathcal{D}_{ecb}2)$ are clearly satisfied by \mathcal{D}_{ecb} . Now we check the condition $(\mathcal{D}_{ecb}3)$. For this we take $u, v, w \in X$ as arbitrary. Then we see that

$$\begin{aligned} \mathcal{D}_{ecb}(u, w)(t) &= (1 + |u| + |w|)|u - w| \exp(t) \\ &\leq (1 + |u| + |w|)(|u - v| + |v - w|) \exp(t) \\ &\leq (1 + |u| + |w|)(\mathcal{D}_{ecb}(u, v)(t) + \mathcal{D}_{ecb}(v, w)(t)) \\ &\leq (1 + |u| + |w|)(\mathcal{D}_{ecb}(u, v) + \mathcal{D}_{ecb}(v, w))(t) \text{ for all } t \in [0, 1]. \end{aligned}$$

Therefore $\mathcal{D}_{ecb}(u, w) \lesssim (1 + |u| + |w|)(\mathcal{D}_{ecb}(u, v) + \mathcal{D}_{ecb}(v, w))$. Since u, v, w are chosen arbitrarily, we have \mathcal{D}_{ecb} is an extended cone b -metric space over Banach algebra with $\theta(\xi, \eta) = (1 + |\xi| + |\eta|)$ for all $\xi, \eta \in X$.

Example 2 Let $\mathcal{A} = C[0, 1]$ be the usual unital Banach algebra with the sup norm. Let $\mathbb{P} = \{f \in C[0, 1] : f(t) \geq 0 \text{ for all } t \in [0, 1]\}$ and $X = \mathbb{R}$. Define a mapping $\mathcal{D}_{ecb} : X^2 \rightarrow \mathcal{A}$ by $\mathcal{D}_{ecb}(\xi, \eta)(t) = (1 + |\xi - \eta|)|\xi - \eta| \exp(t)$ for any $\xi, \eta \in X$ and for all $t \in [0, 1]$. Then clearly \mathcal{D}_{ecb} satisfies conditions $(\mathcal{D}_{ecb}1)$ and $(\mathcal{D}_{ecb}2)$. To check the condition $(\mathcal{D}_{ecb}3)$ let us choose $u, v, w \in X$ as arbitrary. Then we have

$$\begin{aligned} \mathcal{D}_{ecb}(u, w)(t) &= (1 + |u - w|)|u - w| \exp(t) \\ &\leq (1 + |u - w|)(|u - v| + |v - w|) \exp(t) \\ &\leq (1 + |u - w|)(\mathcal{D}_{ecb}(u, v)(t) + \mathcal{D}_{ecb}(v, w)(t)) \\ &\leq (1 + |u - w|)(\mathcal{D}_{ecb}(u, v) + \mathcal{D}_{ecb}(v, w))(t) \text{ for all } t \in [0, 1]. \end{aligned}$$

Therefore $\mathcal{D}_{ecb}(u, w) \lesssim (1 + |u - w|)(\mathcal{D}_{ecb}(u, v) + \mathcal{D}_{ecb}(v, w))$. Since u, v, w are chosen arbitrarily it is seen that \mathcal{D}_{ecb} satisfies condition $(\mathcal{D}_{ecb}3)$ also. Hence \mathcal{D}_{ecb} is an extended cone b -metric space over Banach algebra with $\theta(\xi, \eta) = (1 + |\xi - \eta|)$ for all $\xi, \eta \in X$ but it is not a cone metric space over Banach algebra since

$$\mathcal{D}_{ecb}(1, 1.1)(t) + \mathcal{D}_{ecb}(1.1, 2)(t) = 1.82 \exp(t) < 2 \exp(t) = \mathcal{D}_{ecb}(1, 2)(t) \text{ for all } t \in [0, 1].$$

Lemma 11 Let \mathcal{A} be a real Banach algebra and $l(\neq \theta_{\mathcal{A}}) \in \mathcal{A}$. If $\lim_{n \rightarrow \infty} \frac{\|l^{n+1}\|}{\|l^n\|}$ exists then this limit is equal to $\rho(l)$.

Proof. The proof can be done easily. So we omit the proof. ■

Now we prove some fixed point theorems in the setting of an extended cone b -metric space over Banach algebra. Let us define a subset of \mathcal{A} as follows:

$$\mathbb{P}^* := \left\{ l(\neq \theta_{\mathcal{A}}) \in \mathbb{P} : \lim_{n \rightarrow \infty} \frac{\|l^{n+1}\|}{\|l^n\|} \text{ exists} \right\}.$$

Definition 9 Let $(X, \mathcal{D}_{ecb}, \mathcal{A})$ be an extended cone b -metric space over Banach algebra and $\{\xi_n\}$ be a sequence in X . Then

(i) $\{\xi_n\}$ is called \mathcal{D}_{ecb} -convergent sequence if there exists some $\xi \in X$ such that for every $l \in \mathcal{A}$ with $l \gg \theta_{\mathcal{A}}$ there exists some $N \in \mathbb{N}$ such that $\mathcal{D}_{ecb}(\xi_n, \xi) \ll l$ for all $n \geq N$.

(ii) $\{\xi_n\}$ is said to be \mathcal{D}_{ecb} -Cauchy sequence if for every $l \in \mathcal{A}$ with $l \gg \theta_{\mathcal{A}}$ there exists $N \in \mathbb{N}$ such that $\mathcal{D}_{ecb}(\xi_n, \xi_{n+p}) \ll l$ for all $n \geq N$ and for every $p = 1, 2, \dots$.

(iii) $(X, \mathcal{D}_{ecb}, \mathcal{A})$ is called \mathcal{D}_{ecb} -complete if every \mathcal{D}_{ecb} -Cauchy sequence in X is \mathcal{D}_{ecb} -convergent.

Definition 10 Let $(X, \mathcal{D}_{ecb}, \mathcal{A})$ be an extended cone b-metric space over Banach algebra. A mapping $G : (X, \mathcal{D}_{ecb}) \rightarrow (X, \mathcal{D}_{ecb})$ is said to be \mathcal{D}_{ecb} -orbitally continuous at a point $u \in X$ if for some $\xi \in X$, $\lim_{i \rightarrow \infty} G^{n_i} \xi = u$ implies $\lim_{i \rightarrow \infty} G^{n_i+1} \xi = Gu$. If G is \mathcal{D}_{ecb} -orbitally continuous at each point of X , then we say that G is \mathcal{D}_{ecb} -orbitally continuous in X .

Theorem 1 Let $(X, \mathcal{D}_{ecb}, \mathcal{A})$ be a \mathcal{D}_{ecb} -complete extended cone b-metric space over Banach algebra. Let \mathbb{P} be a solid cone in \mathcal{A} not necessarily normal in \mathcal{A} . Suppose that $T : X \rightarrow X$ be a mapping such that for all $\xi \in X$

$$\mathcal{D}_{ecb}(T\xi, T^2\xi) \lesssim l \mathcal{D}_{ecb}(\xi, T\xi), \quad (1)$$

where $l \in \mathbb{P}^*$, $\rho(l) < 1$ with $\lim_{n, m \rightarrow \infty} \theta(\xi_n, \xi_m) < \frac{1}{\rho(l)}$ and $\{\xi_n\} = \{T^n \xi_0\}$ is the Picard iterating sequence generated by $\xi_0 \in X$. Then T has a fixed point in X provided that T is \mathcal{D}_{ecb} -orbitally continuous in X .

Proof. From the contractive condition (1) we have

$$\begin{aligned} \mathcal{D}_{ecb}(\xi_n, \xi_{n+1}) &= \mathcal{D}_{ecb}(T\xi_{n-1}, T^2\xi_{n-1}) \\ &\lesssim l \mathcal{D}_{ecb}(\xi_{n-1}, T\xi_{n-1}) \\ &= l \mathcal{D}_{ecb}(\xi_{n-1}, \xi_n) \\ &\vdots \\ &\lesssim l^n \mathcal{D}_{ecb}(\xi_0, \xi_1). \end{aligned} \quad (2)$$

Now, for all $n \in \mathbb{N}$ and for any $p = 1, 2, \dots$,

$$\begin{aligned} \mathcal{D}_{ecb}(\xi_n, \xi_{n+p}) &\lesssim \theta(\xi_n, \xi_{n+p}) [\mathcal{D}_{ecb}(\xi_n, \xi_{n+1}) + \mathcal{D}_{ecb}(\xi_{n+1}, \xi_{n+p})] \\ &\lesssim \theta(\xi_n, \xi_{n+p}) \mathcal{D}_{ecb}(\xi_n, \xi_{n+1}) \\ &\quad + \theta(\xi_n, \xi_{n+p}) \theta(\xi_{n+1}, \xi_{n+p}) [\mathcal{D}_{ecb}(\xi_{n+1}, \xi_{n+2}) + \mathcal{D}_{ecb}(\xi_{n+2}, \xi_{n+p})] \\ &\lesssim \theta(\xi_n, \xi_{n+p}) \mathcal{D}_{ecb}(\xi_n, \xi_{n+1}) + \theta(\xi_n, \xi_{n+p}) \theta(\xi_{n+1}, \xi_{n+p}) \mathcal{D}_{ecb}(\xi_{n+1}, \xi_{n+2}) \\ &\quad + \dots + \theta(\xi_n, \xi_{n+p}) \theta(\xi_{n+1}, \xi_{n+p}) \dots \theta(\xi_{n+p-2}, \xi_{n+p}) [\mathcal{D}_{ecb}(\xi_{n+p-2}, \xi_{n+p-1}) \\ &\quad \quad \quad + \mathcal{D}_{ecb}(\xi_{n+p-1}, \xi_{n+p})] \\ &\lesssim \theta(\xi_n, \xi_{n+p}) \mathcal{D}_{ecb}(\xi_n, \xi_{n+1}) + \theta(\xi_n, \xi_{n+p}) \theta(\xi_{n+1}, \xi_{n+p}) \mathcal{D}_{ecb}(\xi_{n+1}, \xi_{n+2}) \\ &\quad + \dots + \theta(\xi_n, \xi_{n+p}) \theta(\xi_{n+1}, \xi_{n+p}) \dots \theta(\xi_{n+p-2}, \xi_{n+p}) \mathcal{D}_{ecb}(\xi_{n+p-2}, \xi_{n+p-1}) \\ &\quad \quad \quad + \theta(\xi_n, \xi_{n+p}) \theta(\xi_{n+1}, \xi_{n+p}) \dots \theta(\xi_{n+p-1}, \xi_{n+p}) \mathcal{D}_{ecb}(\xi_{n+p-1}, \xi_{n+p})] \\ &\lesssim [\theta(\xi_n, \xi_{n+p}) l^n + \theta(\xi_n, \xi_{n+p}) \theta(\xi_{n+1}, \xi_{n+p}) l^{n+1} + \dots \\ &\quad \quad \quad + \theta(\xi_n, \xi_{n+p}) \theta(\xi_{n+1}, \xi_{n+p}) \dots \theta(\xi_{n+p-2}, \xi_{n+p}) l^{n+p-2} \\ &\quad \quad \quad + \theta(\xi_n, \xi_{n+p}) \theta(\xi_{n+1}, \xi_{n+p}) \dots \theta(\xi_{n+p-1}, \xi_{n+p}) l^{n+p-1}] \mathcal{D}_{ecb}(\xi_0, \xi_1) \\ &\lesssim \left[\sum_{r=n}^{n+p-1} l^r \prod_{s=1}^r \theta(\xi_s, \xi_{n+p}) \right] \mathcal{D}_{ecb}(\xi_0, \xi_1). \end{aligned} \quad (3)$$

Again we have,

$$\left\| \sum_{r=n}^{n+p-1} l^r \prod_{s=1}^r \theta(\xi_s, \xi_{n+p}) \right\| \leq \sum_{r=n}^{n+p-1} \prod_{s=1}^r \theta(\xi_s, \xi_{n+p}) \|l^r\|.$$

For $r \in \mathbb{N}$, let us define $u_r^{(n+p)} = \prod_{s=1}^r \theta(\xi_s, \xi_{n+p}) \|l^r\|$. Then for any $p = 1, 2, \dots$,

$$\lim_{n \rightarrow \infty} \frac{u_{n+1}^{(n+p)}}{u_n^{(n+p)}} = \lim_{n \rightarrow \infty} \theta(\xi_{n+1}, \xi_{n+p}) \frac{\|l^{n+1}\|}{\|l^n\|} = \lim_{n \rightarrow \infty} \theta(\xi_{n+1}, \xi_{n+p}) \lim_{n \rightarrow \infty} \frac{\|l^{n+1}\|}{\|l^n\|}.$$

Since $l \in \mathbb{P}^*$ we have $\lim_{n \rightarrow \infty} \frac{\|l^{n+1}\|}{\|l^n\|} = \rho(l)$ and therefore for any $p = 1, 2, \dots$,

$$\lim_{n \rightarrow \infty} \frac{u_{n+1}^{(n+p)}}{u_n^{(n+p)}} = \lim_{n \rightarrow \infty} \theta(\xi_{n+1}, \xi_{n+p})\rho(l) < 1.$$

So by Ratio test we have for any $p = 1, 2, \dots$,

$$\lim_{n \rightarrow \infty} \left\| \sum_{r=n}^{n+p-1} l^r \prod_{s=1}^r \theta(\xi_s, \xi_{n+p}) \right\| \leq \lim_{n \rightarrow \infty} \sum_{r=n}^{n+p-1} \prod_{s=1}^r \theta(\xi_s, \xi_{n+p}) \|l^r\| = 0.$$

Therefore from (3) we conclude that $\{\xi_n\}$ is \mathcal{D}_{ecb} -Cauchy sequence in X . Since X is \mathcal{D}_{ecb} -complete we have $\{\xi_n\}$ is \mathcal{D}_{ecb} -convergent sequence in X and let it be convergent to $\xi \in X$. Now if T is \mathcal{D}_{ecb} -orbitally continuous in X then $\xi_0^n = \xi_n \rightarrow \xi$ implies $T\xi_0^n = T\xi_n = \xi_{n+1} \rightarrow T\xi$. Hence $T\xi = \xi$ and ξ is a fixed point of T in X . ■

Now here we give some supporting examples in respect of Theorem 1.

Example 3 (Banach algebra with non-normal cone)

Let us consider the Banach algebra $\mathcal{A} = C_{\mathbb{R}}^1[0, 1]$ with the norm $\|x\|_{\mathcal{A}} = \|x\|_{\infty} + \|x'\|_{\infty}$ and usual pointwise multiplication. Obviously \mathcal{A} is a Banach algebra with unity $e_{\mathcal{A}} = 1$. Let us take $\mathbb{P} = \{x \in \mathcal{A} : x(t) \geq 0 \text{ for all } t \in [0, 1]\}$. Then it can be verified that \mathbb{P} is a non-normal cone.

Now let $X = \{1, 2, 3\}$. Define $\theta : X^2 \rightarrow [1, \infty)$ and $\mathcal{D}_{ecb} : X \times X \rightarrow \mathcal{A}$ as:
 $\theta(\xi, \eta) = 1 + \xi + \eta$ for all $\xi, \eta \in X$ and $\mathcal{D}_{ecb}(1, 1)(t) = \mathcal{D}_{ecb}(2, 2)(t) = \mathcal{D}_{ecb}(3, 3)(t) = 0$, $\mathcal{D}_{ecb}(1, 2)(t) = \mathcal{D}_{ecb}(2, 1)(t) = 10e^t$, $\mathcal{D}_{ecb}(2, 3)(t) = \mathcal{D}_{ecb}(3, 2)(t) = 40e^t$, $\mathcal{D}_{ecb}(1, 3)(t) = \mathcal{D}_{ecb}(3, 1)(t) = 80e^t$ for all $t \in [0, 1]$. Then $(X, \mathcal{D}_{ecb}, \mathcal{A})$ is a \mathcal{D}_{ecb} -complete extended cone b-metric space over Banach algebra but not a usual cone metric space over Banach algebra.

Let $T : X \rightarrow X$ be defined by $T1 = T2 = 1$ and $T3 = 2$. Then $\mathcal{D}_{ecb}(T\xi, T^2\xi) \lesssim l \mathcal{D}_{ecb}(\xi, T\xi)$ for all $\xi \in X$, with $l(t) = \frac{1}{4} + \frac{1}{48}t$ for all $t \in [0, 1]$. We show that $l \in \mathbb{P}^*$. Here it is seen that $l^n(t) = (\frac{1}{4} + \frac{1}{48}t)^n$ for all $t \in [0, 1]$ and for all $n \in \mathbb{N}$. Therefore $(l^n)'(t) = \frac{n}{48} (\frac{1}{4} + \frac{1}{48}t)^{n-1}$ for all $t \in [0, 1]$ and for all $n \in \mathbb{N}$. Thus $\|l^n\|_{\mathcal{A}} = (\frac{1}{4} + \frac{1}{48})^{n-1} [(\frac{1}{4} + \frac{1}{48}) + \frac{n}{48}]$ for all $n \geq 1$. Hence we get

$$\begin{aligned} \frac{\|l^{n+1}\|_{\mathcal{A}}}{\|l^n\|_{\mathcal{A}}} &= \frac{(\frac{1}{4} + \frac{1}{48})^n [(\frac{1}{4} + \frac{1}{48}) + \frac{n+1}{48}]}{(\frac{1}{4} + \frac{1}{48})^{n-1} [(\frac{1}{4} + \frac{1}{48}) + \frac{n}{48}]} \\ &= \left(\frac{1}{4} + \frac{1}{48}\right) \frac{1 + \frac{14}{n}}{1 + \frac{13}{n}} \text{ for all } n \geq 1. \end{aligned} \tag{4}$$

So from equation (4) it follows that $l \in \mathbb{P}^*$. Also one can check that T satisfies all the conditions of Theorem 1 and $1 \in X$ is the unique fixed point of T .

Example 4 (Banach algebra with normal cone)

Let us consider the Banach algebra $\mathcal{A} = l^1 = \{u = (u_n)_{n \in \mathbb{N}} : \sum_{n=1}^{\infty} |u_n| < \infty\}$ with the norm $\|u\|_{\mathcal{A}} = \sum_{n=1}^{\infty} |u_n|$ and the multiplication defined by

$$uv = (u_n)_{n \in \mathbb{N}}(v_n)_{n \in \mathbb{N}} = \left(\sum_{i+j=n+1} u_i v_j \right)_{n \in \mathbb{N}}.$$

Then \mathcal{A} is a Banach algebra with unity $e_{\mathcal{A}} = (1, 0, 0, \dots)$. Also put $\mathbb{P} = \{u = (u_n)_{n \in \mathbb{N}} : u_n \geq 0 \text{ for all } n \geq 1\}$.

Now let us take $X = l^p = \{\xi = (\xi_n)_{n \in \mathbb{N}} : \sum_{n=1}^{\infty} |\xi_n|^p < \infty\}$, $0.4 < p < 1$. Let $\sigma : X^2 \rightarrow \mathbb{R}^+$ be defined by $\sigma(\xi, \eta) = (\sum_{n=1}^{\infty} |\xi_n - \eta_n|^p)^{\frac{1}{p}}$ for all $\xi = (\xi_n)_{n \in \mathbb{N}}$, $\eta = (\eta_n)_{n \in \mathbb{N}}$ in X . Let $\mathcal{D}_{ecb} : X^2 \rightarrow \mathcal{A}$ be given by $\mathcal{D}_{ecb}(\xi, \eta) = \left(\frac{\sigma(\xi, \eta)}{2^n}\right)_{n \in \mathbb{N}}$ for all $\xi, \eta \in X$. Then $(X, \mathcal{D}_{ecb}, \mathcal{A})$ is a \mathcal{D}_{ecb} -complete extended cone b-metric space

over Banach algebra with $\theta(\xi, \eta) = 2^{\frac{1}{p}-1}$ for all $\xi, \eta \in X$ but not a usual cone metric space over Banach algebra (See [8]).

Define $T : X \rightarrow X$ by $T(\xi_n)_{n \geq 1} = \left(\frac{\xi_n}{4} + \frac{1}{3^n}\right)_{n \geq 1}$ for all $(\xi_n)_{n \geq 1} \in X$. Then $\mathcal{D}_{ecb}(T\xi, T^2\xi) \lesssim l \mathcal{D}_{ecb}(\xi, T\xi)$ for all $\xi \in X$, with $l = (\frac{1}{3}, 0, 0, \dots)$. Then one can verify that T satisfies all the conditions of Theorem 1 and $(\frac{4}{3^{n+1}})_{n \geq 1}$ is the unique fixed point of T in X .

Our contractive condition generalizes several contractive operators with different Lipschitz constants on an extended cone b -metric space over Banach algebra. Moreover it can be easily verified that each of these contractive mappings are \mathcal{D}_{ecb} -orbitally continuous and thus Theorem 1 holds good for such contractive mappings also.

Corollary 1 Consider the following contractive mappings over $(X, \mathcal{D}_{ecb}, \mathcal{A})$.

- (i) If a mapping $T : X \rightarrow X$ satisfies $\mathcal{D}_{ecb}(T\xi, T\eta) \lesssim k\mathcal{D}_{ecb}(\xi, \eta)$ for all $\xi, \eta \in X$, where $k \in \mathbb{P}$ with $\rho(k) < 1$, i.e., T is a Banach contraction then it satisfies the contractive condition (1) with $l = k$.
- (ii) A mapping $T : X \rightarrow X$, which satisfies $\mathcal{D}_{ecb}(T\xi, T\eta) \lesssim k[\mathcal{D}_{ecb}(\xi, T\xi) + \mathcal{D}_{ecb}(\eta, T\eta)]$ for all $\xi, \eta \in X$ and for $k \in \mathbb{P}$ with $\rho(k) < \frac{1}{2}$, i.e., T is a Kannan type contractive map, also satisfies the contractive condition (1) with $l = (e_{\mathcal{A}} - k)^{-1}k$.
- (iii) If $T : X \rightarrow X$ satisfies $\mathcal{D}_{ecb}(T\xi, T\eta) \lesssim k[\mathcal{D}_{ecb}(\xi, T\eta) + \mathcal{D}_{ecb}(\eta, T\xi)]$ for all $\xi, \eta \in X$, where $k \in \mathbb{P}$ with $\rho(k) < \frac{1}{2}$, i.e., T is a Chatterjea type contractive map then it satisfies the contractive condition (1) with $l = (e_{\mathcal{A}} - k)^{-1}k$.
- (iv) A mapping $T : X \rightarrow X$ satisfying

$$\mathcal{D}_{ecb}(T\xi, T\eta) \lesssim k_1\mathcal{D}_{ecb}(\xi, \eta) + k_2\mathcal{D}_{ecb}(\xi, T\xi) + k_3\mathcal{D}_{ecb}(\eta, T\eta), \text{ for all } \xi, \eta \in X$$

and for $k_1, k_2, k_3 \in \mathbb{P}$, which commute with each other, with $\rho(k_1) + \rho(k_2) + \rho(k_3) < 1$, i.e., T is a Reich type contractive map, also satisfies the contractive condition (1) with $l = (e_{\mathcal{A}} - k_3)^{-1}(k_1 + k_2)$.

- (v) A mapping $T : X \rightarrow X$ satisfying

$$\mathcal{D}_{ecb}(T\xi, T\eta) \lesssim k_1\mathcal{D}_{ecb}(\xi, \eta) + k_2 \frac{\mathcal{D}_{ecb}(\xi, T\xi) + \mathcal{D}_{ecb}(\eta, T\eta)}{2} + k_3 \frac{\mathcal{D}_{ecb}(\xi, T\eta) + \mathcal{D}_{ecb}(\eta, T\xi)}{2}$$

for all $\xi, \eta \in X$ and for $k_1, k_2, k_3 \in \mathbb{P}$, which commute with each other, with $\rho(k_1) + \rho(k_2) + \rho(k_3) < 1$, i.e., T is a Hardy-Rogers type contractive map, also satisfies the contractive condition (1) with $l = (e_{\mathcal{A}} - \frac{k_2}{2} - \frac{k_3}{2})^{-1}(k_1 + \frac{k_2}{2} + \frac{k_3}{2})$.

- (vi) If a mapping $T : X \rightarrow X$ satisfies $\mathcal{D}_{ecb}(T\xi, T\eta) \lesssim k\mathcal{D}_{ecb}(\xi, \eta) + L\mathcal{D}_{ecb}(\eta, T\xi)$ for all $\xi, \eta \in X$, where $k \in \mathbb{P}$ with $\rho(k) < 1$ and $L \lesssim \theta_{\mathcal{A}}$, i.e., T is a weak contraction then it satisfies the contractive condition (1) with $l = k$.

3 An Application of Fixed Point Theorem to Dynamical Programming

Bellman [3] was the first who introduced the existence and successive approximations of solutions for several functional equations arising in dynamical programming, one of them is as follows:

$$f(t) = \sup_{t' \in \Theta} \{p(t, t') + G(t, t', f(q(t, t')))\} \quad (5)$$

where X and Y are Banach spaces over the field \mathbb{R} . $\Delta \subset X$ denotes the state space (i.e., the set of initial state, action and transition of the process) and $\Theta \subset Y$ denotes the decision space (i.e., the set of all possible actions of the process) and t, t' denotes the state and decision vectors respectively.

Here $f(t)$ denotes the optimal return function with initial state t and $p : \Delta \times \Theta \rightarrow \mathbb{R}$, $q : \Delta \times \Theta \rightarrow \Delta$, $G : \Delta \times \Theta \times \mathbb{R} \rightarrow \mathbb{R}$. There after several authors have studied the properties of solutions for various functional equations arising in dynamical programming and have tried to solve them by using various fixed point theorems, for these we may refer to [4], [13].

Let $B_d(\Delta)$ be the set of all real-valued bounded functions on Δ . We define a norm on $B_d(\Delta)$ by $\|\beta\| = \sup_{t \in \Delta} |\beta(t)|$ for all $\beta \in B_d(\Delta)$. Then $(B_d(\Delta), \|\cdot\|)$ forms a Banach space.

Let $\mathcal{A} = \mathbb{R}^2$ with $\|(a_1, a_2)\|_{\mathcal{A}} = |a_1| + |a_2|$ for all $(a_1, a_2) \in \mathbb{R}^2$ and the multiplication is defined by $ab = (a_1, a_2)(b_1, b_2) = (a_1b_1, a_1b_2 + b_1a_2)$ for all $a = (a_1, a_2), b = (b_1, b_2) \in \mathbb{R}^2$. Then \mathcal{A} will be a unital Banach algebra with unity $(1, 0)$. Let us take $\mathbb{P} = \{(a_1, a_2) \in \mathbb{R}^2 : a_1, a_2 \geq 0\}$. Then \mathbb{P} is a normal cone on \mathcal{A} .

Let us define a metric $\mathcal{D}_{ecb} : B_d(\Delta) \times B_d(\Delta) \rightarrow \mathcal{A}$ by $\mathcal{D}_{ecb}(\xi, \eta) = (\|\xi - \eta\|^2, \|\xi - \eta\|^2) = (\{\sup_{t \in \Delta} |\xi(t) - \eta(t)|\}^2, \{\sup_{t \in \Delta} |\xi(t) - \eta(t)|\}^2)$ for all $\xi, \eta \in B_d(\Delta)$. Then $(B_d(\Delta), \mathcal{D}_{ecb})$ is a \mathcal{D}_{ecb} -complete extended cone b -metric space over Banach algebra with $\theta(\xi, \eta) = 2$ for all $\xi, \eta \in B_d(\Delta)$.

Now we are going to discuss the existence of the solution of the functional equation (5) by applying our established fixed point theorem. For this first we define a mapping $\Phi : B_d(\Delta) \rightarrow B_d(\Delta)$ by

$$(\Phi f)(t) = \sup_{t' \in \Theta} \{p(t, t') + G(t, t', f(q(t, t')))\}. \quad (6)$$

for all $f \in B_d(\Delta)$. Clearly if the functions p and G are bounded then Φ becomes well-defined.

Theorem 2 Let $\Phi : B_d(\Delta) \rightarrow B_d(\Delta)$ be a mapping defined by (6) and suppose the following conditions hold.

(D1) $p : \Delta \times \Theta \rightarrow \mathbb{R}$ and $G : \Delta \times \Theta \times \mathbb{R} \rightarrow \mathbb{R}$ are continuous and bounded.

(D2) For all $\beta \in B_d(\Delta)$, G satisfies

$$|G(t, t', \beta(t'')) - G(t, t', \Phi(\beta)(t''))| \leq \sqrt{k} \|\beta - \Phi(\beta)\| \quad (7)$$

for all $(t, t', \beta(t'')) \in \Delta \times \Theta \times \mathbb{R}$, where $k \in (0, \frac{1}{2})$. Then the functional equation (7) has a bounded solution.

Proof. Let $\lambda > 0$ be chosen arbitrarily. Then there exists $t'_1, t'_2 \in \Theta$ such that for all $t \in \Delta$ and $\beta \in B_d(\Delta)$,

$$p(t, t'_1) + G(t, t'_1, \beta(q(t, t'_1))) > \Phi(\beta)(t) - \lambda, \quad (8)$$

$$p(t, t'_2) + G(t, t'_2, \Phi(\beta)(q(t, t'_2))) < \Phi(\beta)(t) + \lambda, \quad (9)$$

$$\Phi(\beta)(t) \geq p(t, t'_2) + G(t, t'_2, \beta(q(t, t'_2))) \quad (10)$$

and

$$\Phi^2(\beta)(t) \geq p(t, t'_1) + G(t, t'_1, \Phi(\beta)(q(t, t'_1))). \quad (11)$$

From inequality (8) and (11) we have for all $t \in \Delta$,

$$\begin{aligned} \Phi(\beta)(t) - \Phi^2(\beta)(t) &< G(t, t'_1, \beta(q(t, t'_1))) - G(t, t'_1, \Phi(\beta)(q(t, t'_1))) + \lambda \\ &\leq |G(t, t'_1, \beta(q(t, t'_1))) - G(t, t'_1, \Phi(\beta)(q(t, t'_1)))| + \lambda \\ &\leq \sqrt{k} \|\beta - \Phi(\beta)\| + \lambda. \end{aligned} \quad (12)$$

In a similar way, from equation (9) and (10) we can show that

$$\Phi^2(\beta)(t) - \Phi(\beta)(t) < \sqrt{k} \|\beta - \Phi(\beta)\| + \lambda \text{ for all } t \in \Delta. \quad (13)$$

Since $\lambda > 0$ is arbitrary it follows from equations (12) and (13) that $|\Phi(\beta)(t) - \Phi^2(\beta)(t)| \leq \sqrt{k}\|\beta - \Phi(\beta)\|$ for all $t \in \Delta$ and for all $\beta \in B_d(\Delta)$. Therefore for all $\beta \in B_d(\Delta)$, we have $\|\Phi(\beta) - \Phi^2(\beta)\| \leq \sqrt{k}\|\beta - \Phi(\beta)\|$. Hence

$$\begin{aligned} \mathcal{D}_{ecb}(\Phi(\beta), \Phi^2(\beta)) &= (\|\Phi(\beta) - \Phi^2(\beta)\|^2, \|\Phi(\beta) - \Phi^2(\beta)\|^2) \\ &\prec_{\lambda} (k, 0)(\|\beta - \Phi(\beta)\|^2, \|\beta - \Phi(\beta)\|^2) \\ &= (k, 0)\mathcal{D}_{ecb}(\beta, \Phi(\beta)) \end{aligned}$$

i.e. $\mathcal{D}_{ecb}(\Phi(\beta), \Phi^2(\beta)) \prec_{\lambda} (k, 0)\mathcal{D}_{ecb}(\beta, \Phi(\beta))$ for all $\beta \in B_d(\Delta)$. Thus by Theorem 1, Φ has a fixed point in $B_d(\Delta)$ and consequently the functional equation (5) has a bounded solution. ■

Acknowledgment. The authors are grateful to the honorable reviewer for his valuable suggestions and comments for the improvement of this manuscript. Also first and second authors acknowledge financial support awarded by the Council of Scientific and Industrial Research, New Delhi, India, through research fellowship for carrying out research work leading to the preparation of this manuscript.

References

- [1] A. A. Firozjah, H. Rahimi, M. D. la Sen, G. Soleimani Rad, Fixed point results under generalized c -distance in cone b -metric spaces over Banach algebras, *Axioms*, 9(2020), 31.
- [2] I. A. Bakhtin, The contraction mapping principle in quasi-metric spaces, *Funct. Anal. Unianowsk Gos. Ped. Inst.*, 30(1989), 26–37.
- [3] R. Bellman, *Dynamic Programming*, Princeton University Press, Princeton, NJ, USA, 1957.
- [4] R. Bellman, *Methods of Nonlinear Analysis*, vol. 2, Academic Press, New York, NY, USA, 1973.
- [5] S. Czerwik, Contraction mappings in b -metric spaces, *Acta Math. Inform. Univ. Ostrav.*, 1(1993), 5–11.
- [6] D. Dey and M. Saha, Partial cone metric space and some fixed point theorems, *Turkic World Math. Soc. J. App. Eng. Math.*, 3(2013), 1–9.
- [7] J. Fernandez, N. Malviya, S. Radenović and K. Saxena, F -cone metric spaces over Banach algebra, *Fixed Point Theory Appl.*, 7(2017), 18 pp.
- [8] H. Huang and S. Radenović, Some fixed point results of generalized Lipschitz mappings on cone b -metric spaces over Banach algebras, *J. Comput. Anal. Appl.*, 20(2016), 566–583.
- [9] L. G. Huang and X. Zhang, Cone metric spaces and fixed point theorem for contractive mappings, *J. Math. Anal. Appl.*, 332(2007), 1468–1476.
- [10] Z. Kadelburg and S. Radenović, A note on various types of cones and fixed point results in cone metric spaces, *Asian J. Math. Appl. 2013*, Article ID ama0104 (2013).
- [11] T. Kamran, M. Samreen and Q. U. Ain, A generalization of b -metric space and some fixed point theorems, *Mathematics*, 5,19; doi:10.3390/math5020019, (2017).
- [12] A. K. Laha and M. Saha, Fixed point on $\alpha - \psi$ multivalued contractive mappings in cone metric space, *Acta Comment. Univ. Tartu. Math.*, 20(2016), 35–43.
- [13] Z. Liu, J. Ume and S. Kang, On properties of solutions for two functional equations arising in dynamic programming, *Fixed Point Theory Appl.*, 2010, Art. ID 905858, 19 pp.
- [14] H. Liu and S. Xu, Cone metric spaces with Banach algebras and fixed point theorems of generalized Lipschitz mappings, *Fixed Point Theory Appl.*, 2013, 2013:320, 10 pp.

- [15] K. Roy and M. Saha, Common fixed point theorems for generalized contractive mappings in an F -cone metric space over a Banach algebra, *Malaya J. Mat.*, 7(2019), 751–758.
- [16] K. Roy and M. Saha, Generalized contractions and fixed point theorems over bipolar cone $_{tvs}$ b -metric spaces with an application to homotopy theory, *Mat. Vesnik*, 72(2020), 281–296.
- [17] S. Radenović and B. E. Rhoades, Fixed point theorem for two non-self mappings in cone metric spaces, *Comput. Math. Appl.*, 57(2009), 1701–1707.
- [18] W. Rudin, *Functional Analysis*, 2nd Edn. McGraw-Hill, NewYork (1991).
- [19] S. Xu and S. Radenović, Fixed point theorems of generalized Lipschitz mappings on cone metric spaces over Banach algebras without assumption of normality, *Fixed Point Theory Appl.*, 2014, 2014:102, 12 pp.