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On A Certain Class of Kolmogorov Systems: Integrability And Non-Existence Of Limit Cycles*

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Abstract

The aim of this paper is to study a class of Kolmogorov polynomial differential systems of the form

$$\begin{aligned} \dot{x} &= x\left(P_m(x,y)+P_n(x,y)+R_{2m-n}(x,y)\right), \\ \dot{y} &= y\left(Q_m(x,y)+Q_n(x,y)+R_{2m-n}(x,y)\right), \end{aligned}$$

where P_i, Q_i and R_i are homogeneous polynomials of degree i. Darboux integrability for these systems for all positive integers m and n is proved. Furthermore, the non-existence of limit cycles is investigated.

1 Introduction and Statement of the Main Results

The well-known Kolmogorov differential systems $\dot{x}_i = x_i f_i(x_1,\dots,x_n)$ for $i = 1,\dots,n$, are usually used to control the interaction of species occupying the same ecological niche. Named after the scientist Kolmogorov who generalized them, they are also called *Lotka–Volterra systems* because they were first studied by them in [14, 19]. They appear in applications where the per unit of change $\dot{x}_i/x_i$ of the dependent variables $x_i = x_i(t)$ are given functions $f_i(x_1,\dots,x_n)$ of these variables at any time.

Studying the integrability and the existence of a limit cycle of real planar differential systems is one of the main problems in the qualitative theory of differential equations, as seen in [1, 4, 7]. Concerning the planar differential systems, a first integral is completely used to determine its phase portrait. However, it is usually very difficult to detect if they are integrable or not.

In this paper, we use Abel differential equation to find new classes of integrable Kolmogorov systems. Then, we study the non-existence of the limit cycles of the resultant integrable systems. Exactly here we consider the planar Kolmogorov differential systems of the form

$$\begin{aligned} \dot{x} &= x\left(P_m(x,y)+P_n(x,y)+R_{2m-n}(x,y)\right), \\ \dot{y} &= y\left(Q_m(x,y)+Q_n(x,y)+R_{2m-n}(x,y)\right), \end{aligned} \tag{1}$$

where P_i, Q_i, and R_i are homogeneous polynomials of degree i. Providing we have $2m-n<0$, we should take $R_{2m-n}\equiv 0$. Many papers have discussed this form where the degree of the system could be quadratic, cubic or quartic [15, 17, 21, 22].

In order to present our results, we need some prerequisites. In polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$, system (1) becomes

$$\begin{aligned} \dot{r} &= f_m(\theta)r^{m+1} + f_n(\theta)r^{n+1} + R_{2m-n}(\theta)r^{2m-n+1}, \\ \dot{\theta} &= g_m(\theta)r^m + g_n(\theta)r^n, \end{aligned} \tag{2}$$

where

$$f_m(\theta) = \cos^2\theta P_m(\cos\theta,\sin\theta) + \sin^2\theta Q_m(\cos\theta,\sin\theta),$$

$$f_n(\theta) = \cos^2\theta P_n(\cos\theta,\sin\theta) + \sin^2\theta Q_n(\cos\theta,\sin\theta),$$

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$$g_m(\theta) = \cos\theta \sin\theta \left(Q_m(\cos\theta, \sin\theta) - P_m(\cos\theta, \sin\theta)\right),$$

$$g_n(\theta) = \cos\theta \sin\theta \left(Q_n(\cos\theta, \sin\theta) - P_n(\cos\theta, \sin\theta)\right).$$

In the region $U = \{(r,\theta) : (g_m(\theta) + g_n(\theta)r^{n-m}) > 0\}$, system (2) will be in the following form

$$\frac{dr}{d\theta} = \frac{r\left(f_m(\theta) + f_n(\theta)r^{n-m} + R_{2m-n}(\theta)r^{m-n}\right)}{g_m(\theta) + g_n(\theta)r^{n-m}}. \tag{3}$$

By the change of variable

$$\rho = \frac{1}{g_m(\theta) + g_n(\theta)r^{n-m}},$$

we can transform (3) to the Abel differential equation of the first kind

$$\frac{d\rho}{d\theta} = A(\theta)\rho^3 + B(\theta)\rho^2 + C(\theta)\rho, \tag{4}$$

with

$$A(\theta) = (n-m)\left(g_n(\theta)g_m(\theta)f_n(\theta) - \frac{g_m^2(\theta)f_n(\theta)}{g_n(\theta)} - g_n(\theta)R_{2m-n}(\theta)\right),$$

$$B(\theta) = \frac{g_n'(\theta)g_m(\theta)}{g_n(\theta)} - g_m'(\theta) + (m-n)\left(\frac{2g_m(\theta)f_n(\theta)}{g_n(\theta)} - f_m(\theta)\right),$$

$$C(\theta) = \frac{(m-n)f_n(\theta)}{g_n(\theta)} - \frac{g_n'(\theta)}{g_n(\theta)}.$$

Now we can state our first result.

Theorem 1 *For system (2), the following statements hold.*

(a) If $B(\theta)A(\theta)C(\theta)$ is not identically zero and the following relation holds

$$\left(\frac{A(\theta)}{B(\theta)}\right)' = aB(\theta) - \frac{A(\theta)C(\theta)}{B(\theta)}, \qquad a \in \mathbb{R},$$

then system (2) is Darboux integrable with the first integral in the form

$$H(\rho,\theta) = \begin{cases} \dfrac{\rho\exp(-\int C(\theta))\exp\left(-\frac{1}{\sqrt{4a-1}}\arctan\left(\frac{(1+2\rho\frac{A(\theta)}{B(\theta)})}{\sqrt{4a-1}}\right)\right)}{\sqrt{\rho^2\frac{A^2(\theta)}{B^2(\theta)}+\rho\frac{A(\theta)}{B(\theta)}+\mu}} & \textit{if } a > \frac{1}{4}, \\ \dfrac{\rho\exp(-\int C(\theta))\exp\left(\frac{1}{1+2\rho\frac{A(\theta)}{B(\theta)}}\right)}{1+2\rho\frac{A(\theta)}{B(\theta)}} & \textit{if } a = \frac{1}{4}, \\ \dfrac{\rho\exp(-\int C(\theta))\left|\sqrt{1-4a}+1+2\rho\frac{A(\theta)}{B(\theta)}\right|^{\frac{1}{2}(-1+\frac{1}{\sqrt{1-4a}})}}{\left|\sqrt{1-4a}-1-2\rho\frac{A(\theta)}{B(\theta)}\right|^{\frac{1}{2}(1+\frac{1}{\sqrt{1-4a}})}} & \textit{if } a < \frac{1}{4}. \end{cases}$$

(b) If $C(\theta)$ is not identically zero and $A(\theta)B(\theta) \equiv 0$, then system (2) is Darboux integrable with the first integral

$$H(\rho,\theta) = \begin{cases} \frac{1}{\rho}\left(\exp\left(\int C(\theta)d\theta\right) + \rho\int B(\theta)\exp\left(\int C(s)ds\right)d\theta\right) & \textit{if } A(\theta) \equiv 0, \\ \frac{1}{\rho^2}\left(\exp\left(2\int C(\theta)d\theta\right) + 2\rho^2\int A(\theta)\exp\left(2\int C(s)ds\right)d\theta\right) & \textit{if } B(\theta) \equiv 0, \\ \rho\exp\left(-\int C(\theta)d\theta\right) & \textit{if } A(\theta) \equiv B(\theta) \equiv 0. \end{cases}$$

Starting with D. Hilbert, mathematicians have been interested in planar differential systems. More particularly, in their inverse integrating factors and limit cycles, see for example [2, 3, 5, 6, 11]. Our second main result on the inverse integrating factors and the limit cycles of the Kolmogorov system (1) is the following.

Theorem 2 *Considering system (2), the following statements hold:*

(a) If $B(\theta)A(\theta)C(\theta)$ is not identically zero and the relation

$$\left(\frac{A(\theta)}{B(\theta)}\right)' = aB(\theta) - \frac{A(\theta)C(\theta)}{B(\theta)}, \qquad a \in \mathbb{R},$$

holds, then the function

$$V(\rho,\theta) = \rho\left(\rho^2\frac{A^2(\theta)}{B^2(\theta)} + \rho\frac{A(\theta)}{B(\theta)} + a\right),$$

is an inverse integrating factor of system (2) in the region (where $g_n(\theta)$ has one sign), which makes $V(\rho,\theta)$ well defined. In this case system (2) can have at most two limit cycles which must contain the origin inside.

(b) If $B(\theta)C(\theta)$ is not identically zero and $A(\theta) \equiv 0$, then

$$V(\rho,\theta) = \rho + \rho^2 \exp\left(-\int C(\theta)d\theta\right)\int \exp\left(\int C(\theta)d\theta\right)B(\theta)d\theta,$$

is an inverse integrating factor of system (2) in the region where $V(\rho,\theta)$ is well defined. Furthermore, system (2) can have at most one limit cycle which must contain the origin inside.

(c) If $A(\theta)C(\theta)$ is not identically zero and $B(\theta) \equiv 0$, then

$$V(\rho,\theta) = \frac{\rho}{2} + \rho^3 \exp\left(-2\int C(\theta)d\theta\right)\int \exp\left(2\int C(\theta)d\theta\right)A(\theta)d\theta,$$

is an inverse integrating factor of system (2) in the region where $V(\rho,\theta)$ is well defined. Furthermore, system (2) can have at most two limit cycles which must contain the origin inside.

The next result proved in [9], is used to prove Theorem 2.

Theorem 3 *Let (P,Q) be a C^1 vector field de ned in an open subset U of $\mathbb{R}^2$, and let $V(x,y)$ be a C^1 incerse integrating factor of this vector eld in an open subset $U \subset \mathbb{R}^2$. If γ is a limit cycle of (P,Q) contained in U, it must be contained in $\{(x,y) \in U : V(x,y) = 0\}$.*

2 Proof of Theorems 1 and 2.

Here the details will be omitted, since the proof of Theorem 1 is similar to that given in [7].

Proof of statement (a) **of Theorem 2.** We know that the inverse integrating factor is not unique. Therefore, if $V(\rho,\theta)$is an inverse integrating factor, then the product $V(\rho,\theta)H(\rho,\theta)$ is also an inverse integrating factor. Hence, we have

$$\frac{\partial H(\rho,\theta)}{\partial\rho} = \frac{a\rho\exp\left(-\int C(\theta)\right)\exp\left(-\frac{1}{\sqrt{4a-1}}\arctan\left(\frac{(1+2\rho\frac{A(\theta)}{B(\theta)})}{\sqrt{4a-1}}\right)\right)}{\left(\rho^2\frac{A^2(\theta)}{B^2(\theta)} + \rho\frac{A(\theta)}{B(\theta)} + a\right)^{\frac{3}{2}}},$$

and

$$Q(\rho,\theta)=1,\quad \frac{1}{V(\rho,\theta)}=\frac{\partial H(\rho,\theta)}{\partial\rho},$$

we get the following inverse integrating factor

$$V(\rho,\theta)H(\rho,\theta)=\frac{1}{a}\rho\left(\rho^2\frac{A^2(\theta)}{B^2(\theta)}+\rho\frac{A(\theta)}{B(\theta)}+a\right).$$

This expression for the inverse integrating factor gives the two possible formulas of limit cycles

$$\begin{array}{ll}\rho_1(\theta)=\frac{\left(-1\pm\sqrt{1-4a}\right)B(\theta)}{2A(\theta)}, & \text{if } a<\frac{1}{4},\\ \rho_2(\theta)=-\frac{B(\theta)}{2A(\theta)}, & \text{if } a=\frac{1}{4}.\end{array}$$

These limit cycles cannot be contained in one of the open quadrants. Besides, they cannot be cutting any of the axes since are trajectories for system (2), these limit cycles have at most a unique point on every ray $\theta=\theta^*$ for all $\theta^*\in[0,2\pi)$. ■

Proof of statement (b) **of Theorem 2.** Since $B(\theta)C(\theta)$ is not identically zero and $A(\theta)\equiv 0$, the parial derivative of $H(\rho,\theta)$ with respect to ρ is

$$\frac{\partial H(\rho,\theta)}{\partial\rho}=-\exp\left(\int C(\theta)d\theta\right)\rho^{-2},$$

so we obtain

$$V(\rho,\theta)=-\exp\left(-\int C(\theta)d\theta\right)\rho^2,$$

and hence

$$V(\rho,\theta)H(\rho,\theta)=-\left(\rho+\rho^2\exp\left(-\int C(\theta)d\theta\right)\int\exp\left(\int C(\theta)d\theta\right)B(\theta)d\theta\right).$$

Therefore, the possible formula of a limit cycle is given by

$$\rho(\theta)=-\frac{\exp\left(\int C(\theta)d\theta\right)}{\int\exp\left(\int C(\theta)d\theta\right)B(\theta)d\theta},$$

and it must contain the origin inside for the same reason mentioned previously in statement (a). This completes the proof. ■

Proof of statement (c) **of Theorem 2.** Because $A(\theta)C(\theta)$ is not identically zero and $B(\theta)\equiv 0$, the parial derivative of $H(\rho,\theta)$ with respect to ρ is

$$\frac{\partial H(\rho,\theta)}{\partial\theta}=-\frac{1}{2}\exp\left(2\int C(\theta)d\theta\right)\rho^{-3},$$

we obtain

$$V(\rho,\theta)=-2\exp\left(-2\int C(\theta)d\theta\right)\rho^3,$$

and hence

$$V(\rho,\theta)H(\rho,\theta)=-4\left(\frac{\rho}{2}+\rho^3\exp\left(-2\int C(\theta)d\theta\right)\int\exp\left(2\int C(\theta)d\theta\right)A(\theta)d\theta\right).$$

From the expression of the inverse integrating factor, the unique possible limit cycle must be given by

$$\rho(\theta)=\pm\frac{\exp\left(\int C(\theta)d\theta\right)}{\sqrt{-2\int\exp\left(2\int C(\theta)d\theta\right)A(\theta)d\theta}}.$$

This limit cycle has to contain the origin inside as mentioned previously in statement (a). Therefore statement (c) is proved. ∎

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