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A Note On A Conjecture Of R. Brück*

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Abstract

In connection to a conjecture of R. Brück we improve a result of Z. X. Chen and K. H. Shon [4] concerning value sharing by an entire function with its derivative.

1 Introduction, Definitions and Results

Let f be an entire function and $M(r,f) = \max_{|z|=r} |f(z)|$ be the maximum modulus function of f. The order $\sigma(f)$ and the lower order $\lambda(f)$ of f are defined respectively by

$$\sigma(f) = \limsup_{r\to\infty} \frac{\log\log M(r,f)}{\log r}$$

and

$$\lambda(f) = \liminf_{r\to\infty} \frac{\log\log M(r,f)}{\log r}.$$

The first iterated order or hyper order $\sigma_2(f)$ and the first iterated lower order or hyper lower order $\lambda_2(f)$ are defined respectively by

$$\sigma_2(f) = \limsup_{r\to\infty} \frac{\log\log\log M(r,f)}{\log r}$$

and

$$\lambda_2(f) = \liminf_{r\to\infty} \frac{\log\log\log M(r,f)}{\log r}.$$

If the Taylor expansion of f is $f(z) = \sum_{n=0}^{\infty} a_n z^n$, then the power series $\sum_{n=0}^{\infty} |a_n| r^n$ converges for every $r > 0$ and so for any given $r > 0$, we have $\lim_{r\to\infty} |a_n| r^n = 0$. Hence the maximum term $\mu(r,f) = \max_{n\geq 0} |a_n| r^n$ is well defined.

Also we define $\nu(r,f)$, the central index of f, as the greatest exponent m such that $\mu(r,f) = |a_m| r^m$ (see [7, p.50]).

It is well known that

$$\sigma(f) = \limsup_{r\to\infty} \frac{\log \nu(r,f)}{\log r}$$

(see [7, p.51]). Similarly it can be verified that

$$\lambda(f) = \liminf_{r\to\infty} \frac{\log \nu(r,f)}{\log r}.$$

By Lemma 2 in [3] we see that

$$\sigma_2(f) = \limsup_{r\to\infty} \frac{\log\log \nu(r,f)}{\log r}$$

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and in a similar fashion we can prove that

$$\lambda_2(f) = \liminf_{r\to\infty} \frac{\log\log \nu(r,f)}{\log r}.$$

Let $u(z)$ be a nonconstant subharmonic function in the open complex plane. We put $B(r,u) = \sup_{|z|=r} u(z)$. The order $\sigma(u)$ and the lower order $\lambda(u)$ of u are defined by

$$\sigma(u) = \limsup_{r\to\infty} \frac{\log B(r,u)}{\log r}$$

and

$$\lambda(u) = \liminf_{r\to\infty} \frac{\log B(r,u)}{\log r}$$

(see [1]).

Let $E \subset [1,\infty)$ and χ_E be the characteristic function of E. The upper and the lower logarithmic densities of E are respectively defined by

$$\overline{\text{logdens}}(E) = \limsup_{r\to\infty} \frac{\int_1^r \frac{\chi_E(t)}{t}dt}{\log r}$$

and

$$\underline{\text{logdens}}(E) = \liminf_{r\to\infty} \frac{\int_1^r \frac{\chi_E(t)}{t}dt}{\log r}.$$

The quantity $\lim_{r\to\infty} \int_1^r \frac{\chi_E(t)}{t}dt$ is called the logarithmic measure of E. It is easy to note that if $\overline{\text{logdens}}(E) > 0$, then E has infinite logarithmic measure.

Let f and g be two entire functions and a be also an entire function, which, in particular, may be a constant. We say that f and g share the function a CM (counting multiplicities) if $f - a$ and $g - a$ have the same set of zeros with counting multiplicities.

L. A Rubel and C. C. Yang [8] were the first to consider the uniqueness problem of an entire function sharing two values with its derivative. Afterwards in 1996 R. Brück [2] considered the problem of a single value sharing by an entire function with its derivative and proposed the following conjecture.

Brück's Conjecture: Let f be a nonconstant entire function with $\sigma_2(f) < \infty$ and $\sigma_2(f)$ is not a positive integer. If f and $f^{(1)}$ share a finite value a CM, then $f^{(1)} - a = c(f - a)$, where c is a nonzero constant.

If $a = 0$, then the conjecture was resolved by Brück himself [2], but the case $a \neq 0$ is not yet fully resolved.

For entire functions of finite order, G. G. Gundersen and L. Z. Yang [6] resolved the conjecture and proved the following result.

Theorem 1 ([6]) *Let f be a nonconstant entire function of finite order. If f and $f^{(1)}$ share one finite value a CM then $f^{(1)} - a = c(f - a)$ for some nonzero constant c.*

Generalizing Theorem 1 to higher order derivatives, L. Z. Yang [10] proved the following result.

Theorem 2 ([10]) *Let f be a nonconstant entire function of finite order. If f and $f^{(k)}$ share one finite value a CM, then $f^{(k)} - a = c(f - a)$ for some nonzero constant c.*

In 2004, J. P. Wang [9] improved Theorem 2 in the following manner.

Theorem 3 ([9]) *Let f be a nonconstant entire function of finite order and a be a nonconstant polynomial. If f and $f^{(k)}$ share a CM, then $f^{(k)} - a = c(f - a)$ for some nonzero constant c.*

In the same year Z. X. Chen and K. H. Shon [4] extended Theorem 1 to a class of entire functions of unrestricted order and proved the following theorem.

Theorem 4 ([4]) *Let f be a nonconstant entire function with $\sigma_2(f) < \frac{1}{2}$. If f and $f^{(1)}$ share a finite value a CM, then $f^{(1)} - a = c(f - a)$, where c is a nonzero constant.*

Noting that Brück conjecture remains open for the case $\sigma_2(f) \geq \frac{1}{2}$, the purpose of the paper is to improve both Theorem 3 and Theorem 4 and prove the following result. Also our proof is simpler than Z. X. Chen and K. H. Shon [4].

Theorem 5 *Let f be a nonconstant entire function with $\lambda_2(f) < \frac{1}{2}$ and $\sigma_2(f) < \infty$. Suppose that $a = a(z)$ is a polynomial. If f and $f^{(k)}$ share a CM, then $f^{(k)} - a = c(f - a)$, where c is a nonzero constant.*

2 Lemmas

In this section we present some necessary lemmas.

Lemma 6 ([7, p.9]) *Let $P(z) = b_n z^n + b_{n-1} z^{n-1} + \cdots + b_0 (b_n \neq 0)$ be a polynomial of degree n. Then for every $\epsilon (> 0)$ there exists $R (> 0)$ such that for all $|z| = r > R$ we get*

$$(1 - \epsilon)|b_n| r^n \leq |P(z)| \leq (1 + \epsilon)|b_n| r^n.$$

Lemma 7 ([7, p.51]) *Let f be a transcendental entire function. Then there exists a set $E \subset (1, \infty)$ with finite logarithmic measure such that for $|z| = r \notin [0, 1] \cup E$ and $|f(z)| = M(r, f)$ we get*

$$\frac{f^{(k)}(z)}{f(z)} = (1 + o(1)) \left(\frac{\nu(r, f)}{z} \right)^k. \tag{1}$$

Lemma 8 ([7, p.5]) *Let $g : (0, +\infty) \to \mathbb{R}$ and $h : (0, +\infty) \to \mathbb{R}$ be monotone increasing functions such that $g(r) \leq h(r)$ outside of an exceptional set E of finite logarithmic measure. Then for any $\alpha > 1$, there exists $R > 0$ such that $g(r) \leq h(r^\alpha)$ holds for $r > R$.*

Lemma 9 ([1]) *Let $u(z)$ be a nonconstant subharmonic function in the open complex plane $\mathbb{C}$ of lower order $\lambda, 0 \leq \lambda < 1$. If $\lambda < \alpha < 1$, then*

$$\overline{logdens}\{r : A(r) > (\cos \alpha\pi) B(r)\} \geq 1 - \frac{\lambda}{\alpha},$$

where $A(r) = \inf_{|z|=r} u(z)$ and $B(r) = \sup_{|z|=r} u(z)$.

Remark 1 *Since for an entire function Q, $\log |Q(z)|$ is a subharmonic function in $\mathbb{C}$ ([5, p.394]), we can apply Lemma 9 to the function $u(z) = \log |Q(z)|$.*

3 Proof of Theorem 5

Proof. Since $f^{(k)} - a$ and $f - a$ share 0 CM, there exists an entire function Q such that

$$\frac{f^{(k)} - a}{f - a} = e^Q. \tag{2}$$

If Q is a constant, then we are done. So we suppose that Q is nonconstant and consider the following cases.
Case 1. Let $\sigma(f) < \infty$. Then from (2) we see that Q is a polynomial. Further $\sigma(f) \geq 1$, because if $\sigma(f) < 1$, then (2) implies that Q is a constant. Therefore f is transcendental.

Now for any z with $|f(z)| = M(r, f)$, noting that f is transcendental, we get by Lemma 6

$$|\frac{a(z)}{f(z)}| \leq \frac{M(r, a)}{M(r, f)} \leq \frac{2|\beta| r^{\deg a}}{M(r, f)} \to 0 \tag{3}$$

as $r \to \infty$, where β is the leading coefficient of $a = a(z)$.

From (2) we get

$$e^{Q} = \frac{\frac{f^{(k)}}{f} - \frac{a}{f}}{1 - \frac{a}{f}}. \tag{4}$$

Now by Lemma 7 there exists $E \subset (1, \infty)$ with finite logarithmic measure such that for all large $|z| = r \notin [0,1] \cup E$ and $|f(z)| = M(r,f)$ we get in view of (3),(4) and (1)

$$e^{Q(z)} = (1 + o(1)) \left(\frac{\nu(r,f)}{z} \right)^{k}. \tag{5}$$

Now from (5) we get for all large $|z| = r \notin [0,1] \cup E$ with $|f(z)| = M(r,f)$

$$\begin{aligned} |Q(z)| &= |\log e^{Q(z)}| \\ &= |\log \left(\frac{\nu(r,f)}{z} \right)^{k}| + o(1) \\ &= |k \log \nu(r,f) - k \log z| + o(1) \\ &\le k \log \nu(r,f) + k \log r + 6k\pi \\ &< 2k(\sigma(f) + 1) \log r + 6k\pi. \end{aligned} \tag{6}$$

Also by Lemma 6 we obtain for all large $|z| = r$

$$\frac{1}{2}|\delta| r^{\deg Q} \le |Q(z)|, \tag{7}$$

where δ is the leading coefficient of Q.

Now (6) and (7) together imply $\deg Q = 0$, which is a contradiction.

Case 2. Let $\sigma(f) = \infty$. We note from (2) that $\lambda(Q) \le \lambda_2(f) < \frac{1}{2}$. We now consider the following subcases.

Subcase 2.1. Let Q be a polynomial. Then from (5) we get for all large $|z| = r \notin [0,1] \cup E$ with $|f(z)| = M(r,f)$

$$|Q(z)| \le k \log \nu(r,f) + k \log r + 6k\pi. \tag{8}$$

From (7) and (8) we obtain for all large $|z| = r \notin [0,1] \cup E$ with $|f(z)| = M(r,f)$

$$\frac{1}{2}|\delta| r^{\deg Q} \le k \log \nu(r,f) + k \log r + 6k\pi.$$

So for all large $|z| = r \notin [0,1] \cup E$ we get

$$\frac{1}{2}|\delta| r^{\deg Q} \le k \log \nu(r,f) + k \log r + 6k\pi.$$

Hence by Lemma 8 for given α, $1 < \alpha < \frac{3}{2}$, we get for all large values of r

$$\frac{1}{2}|\delta| r^{\deg Q} \le k \log \nu(r^{\alpha}, f) + k\alpha \log r + 6k\pi$$

and so

$$r^{\deg Q} \left(\frac{1}{2}|\delta| - \frac{k\alpha \log r}{r^{\deg Q}} \right) \le k \log \nu(r^{\alpha}, f) + 6k\pi.$$

This implies $\deg Q \le \alpha \lambda_2(f) < \frac{\alpha}{2} < 1$, which is a contradiction.

Subcase 2.2. Let Q be a transcendental entire function. We see by Note 1 that $u(z) = \log |Q(z)|$ is a subharmonic function and also $\lambda(u) = \lambda(Q) < \frac{1}{2}$. Suppose that $H = \{r : A(r) > (\cos \alpha\pi) B(r)\}$, where $A(r) = \inf_{|z|=r} \log |Q(z)|$, $B(r) = \sup_{|z|=r} \log |Q(z)|$ and $\lambda(Q) < \alpha < \frac{1}{2}$.

Then by Lemma 9 H has infinite logarithmic measure. Also by Lemma 7 for $|z| = r \in H \backslash \{[0,1] \cup E\}$ with $|f(z)| = M(r,f)$ we get (1).

Now by (3), (4) and (1) for all large $|z| = r \in H \backslash \{[0,1] \cup E\}$ with $|f(z)| = M(r,f)$ we get (5), where Q is transcendental entire, and so

$$\begin{aligned} |Q(z)| &= |\log e^{Q(z)}| \\ &= \left|\log \left(\frac{\nu(r,f)}{z}\right)^k\right| + o(1) \\ &= |k \log \nu(r,f) - k \log z| + o(1) \\ &\leq k \log \nu(r,f) + k \log r + 6k\pi \\ &< 2kr^{\sigma_2(f)+1}. \end{aligned} \tag{9}$$

So by (9) and by Lemma 9 there exists a constant $d, 0 < d \leq 1$, such that $(M(r,Q))^d \leq 2kr^{\sigma_2(f)+1}$ for all large values of $|z| = r \in H \backslash \{[0,1] \cup E\}$ and $|f(z)| = M(r,f)$. This is impossible because Q is transcendental and so $\lim_{r\to\infty} \frac{(M(r,Q))^d}{r^{\sigma_2(f)+1}} = \infty$. This proves the theorem. ■

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