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Bounds On k-Part Degree Restricted Domination Number Of A Graph*

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Abstract

For a positive integer k, a dominating set D of a graph G is said to be a k-part degree restricted dominating set (k-DRD set) if for all $u \in D$, there exists a set $C_u \subseteq N(u) \cap (V-D)$ such that $|C_u| \leq \left\lceil \frac{d(u)}{k} \right\rceil$ and $\cup_{u \in D} C_u = V - D$. The minimum cardinality of a k-DRD set of a graph G is called the k-part degree restricted domination number of G and is denoted by $\gamma_{\frac{d}{k}}(G)$. In this paper, we provide some bounds on $\gamma_{\frac{d}{k}}$ of join of two graphs, bounds on $\gamma_{\frac{d}{k}}$ in terms of maximum degree, independence number and covering number. Further, we discuss some Nordhaus-Gaddum type results. In addition to this, we prove that for any graph G, $\gamma_{\frac{d}{k}}(G) \leq \gamma_k(G)$, where $\gamma_k(G)$ is the k-domination number of G and we characterize the trees T for which $\gamma_{\frac{d}{k}}(T) = \gamma_k(T)$.

1 Introduction

Let $G = (V, E)$ be a simple, undirected graph. A subset $D \subseteq V$ is called a *dominating set* of G, if every vertex not in D is adjacent to some vertex in D. The *domination number* of G is the minimum cardinality of a dominating set of G and is denoted by $\gamma(G)$. The concept of domination has emerged as one of the most studied areas extensively from a theoretical and algorithmic point of view. The dominating set in a communication network serves as a virtual backbone, and every dominating vertex interacts with all its neighbors. Therefore, a vertex with more neighbors will hold enormous amounts of data, decreasing network performance. Kamath et al. [3] enforced some restrictions on the flow of data from each dominating vertex in order to balance the load, and introduced a new parameter namely, k-part degree restricted domination defined as follows. For a positive integer k, a dominating set D of a graph G is said to be a k-part degree restricted dominating set (k-DRD set) if for all $u \in D$, there exists a set $C_u \subseteq N(u) \cap (V - D)$ such that $|C_u| \leq \left\lceil \frac{d(u)}{k} \right\rceil$ and $\cup_{u \in D} C_u = V - D$. The minimum cardinality of a k-DRD set of a graph G is called the k-part degree restricted domination number of G and is denoted by $\gamma_{\frac{d}{k}}(G)$.

In a graph $G = (V, E)$, the open and closed neighborhood of a vertex $v \in V$ are denoted by $N(v)$and $N[v]$ respectively, where $N(v) = \{u \in V(G) : uv \in E(G)\}$ and $N[v] = N(v) \cup \{v\}$. The degree of a vertex v is $|N(v)|$ and is denoted by $d_G(v)$ or simply $d(v)$. If $d_G(v) = 1$, then v is called a pendant vertex and the support vertex of v is the unique vertex $u \in V(G)$ such that $uv \in E(G)$. A support vertex with exactly one adjacent pendant vertex is called weak support, and a support vertex with at least two adjacent pendant vertices is called a strong support. A rooted tree T is a tree with one vertex $r \in V(T)$ chosen as root. For each vertex $v \in V(T)$, $P(v)$ is the unique $v - r$ path. The parent of $v \in V(T)$ is its neighbor on $P(v)$; its children are its other neighbors. The minimum degree of a graph is $\min\{d_G(v) : v \in V(G)\}$ and is denoted by $\delta(G)$. The maximum degree of a graph G is $\max\{d_G(v) : v \in V(G)\}$ and is denoted by $\Delta(G)$. The

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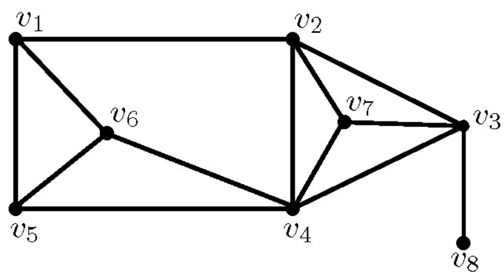

Figure 1: An illustration for the 3-DRD sets in a graph.

complement $\overline{G}$ of a graph G is the graph with vertex set V and two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G. For any real number x, $\lfloor x \rfloor$ is the largest integer not greater than x, called the floor value of x and $\lceil x \rceil$ is the smallest integer not less than x, called the ceiling value of x. A set C of vertices is said to be dominated by a vertex v in a k-DRD set if $C \subseteq C_v$ and a vertex v can dominate at most $\left\lceil \frac{d(v)}{k} \right\rceil$ number of its neighbors. Every graph G has a k-DRD set, where $V(G)$ is a trivial k-DRD set with $C_v = \emptyset$ for every $v \in V(G)$. For every k-DRD set D of a graph G, we can partition the set $V - D$ with the collection of sets $\{C_u : u \in D\}$ and in this paper, we assume that $C_u \cap C_v = \emptyset$ for every $u, v \in D$. For notations and graph theory terminologies not defined here, we refer [4, 2]

Example 1 *In Figure 1 a 3-part degree restricted domination is illustrated. Since $k = 3$, vertices of degree one, two and three can dominate at most one of its neighbors and vertices of degree four, five and six can dominate at most two of its neighbors. Here, $D = \{v_1, v_3, v_4\}$ is a 3-DRD set with $C_{v_1} = \{v_6\}$, $C_{v_3} = \{v_2, v_8\}$, $C_{v_4} = \{v_7, v_5\}$ and*

$$\cup_{u \in D} C_u = C_{v_1} \cup C_{v_3} \cup C_{v_4} = \{v_2, v_5, v_6, v_7, v_8\} = V - D.$$

(We can also consider $C_{v_3} = \{v_7, v_8\}$, $C_{v_4} = \{v_2, v_6\}$, $C_{v_1} = \{v_5\}$ or $C_{v_3} = \{v_7, v_8\}$, $C_{v_4} = \{v_2, v_5\}$, $C_{v_1} = \{v_6\}$.) Also, $\{v_2, v_3, v_5\}$ is a 3-DRD set with $C_{v_2} = \{v_1, v_4\}$ and $C_{v_3} = \{v_7, v_8\}$, $C_{v_5} = \{v_6\}$, $\{v_2, v_4, v_8\}$ is a 3-DRD set with $C_{v_2} = \{v_7, v_1\}$, $C_{v_4} = \{v_5, v_6\}$ and $C_{v_8} = \{v_3\}$. The 3-part degree restricted domination number of graph in Figure 1 is 3, that is $\gamma_{\frac{d}{3}} = 3$.

In this paper, we provide some bounds on $\gamma_{\frac{d}{k}}$ in terms of maximum degree, independence number and covering number. Further, we discuss some Nordhaus-Gaddum type results and we give bounds on $\gamma_{\frac{d}{k}}$ of join of two graphs. Let $\gamma_k(G)$ be the k-domination number of a graph G. We prove that for any graph G, $\gamma_{\frac{d}{k}}(G) \leq \gamma_k(G)$ and we characterize the trees T for which $\gamma_{\frac{d}{k}}(T) = \gamma_k(T)$.

2 Bounds on k-Part Degree Restricted Domination Number of a Graph

In this section, we will discuss some bounds on k-part degree restricted domination number.

Theorem 1 *If D is a $\gamma_{\frac{d}{k}}$-set of a graph G such that $C_u \neq \emptyset$ for every $u \in D$ and $C_u \cap C_v = \emptyset$ for every $u, v \in D$, then $V - D$ is a k-DRD set of G and $\gamma_{\frac{d}{k}}(G) \leq \frac{n}{2}$.*

Proof. Let $D = \{v_1, v_2, \ldots, v_m\}$ be a $\gamma_{\frac{d}{k}}$-set of G satisfying the conditions in the hypothesis. For each $v_i \in D$, choose a vertex say $a_i \in C_{v_i}$ and $A = \{a_1, a_2, \ldots, a_m\}$. Clearly, $A \subseteq V - D$. For every $a_i \in A$ define, $C_{a_i} = \{v_i\}$ and for every $a_j \in V - (D \cup A)$, $C_{a_j} = \emptyset$. Then, for each $a_i \in A$, $C_{a_i} \subseteq N(a_i) \cap D$, $|C_{a_i}| = 1$ and

$$\cup_{a_j \in V-D} C_{a_j} = (\cup_{a_j \in A} C_{a_j}) \cup (\cup_{a_j \in (V-(D \cup A)} C_{a_j}) = D = V - (V - D).$$

Hence, $V-D$ is a k-DRD set and $|D| \leq |V-D|$, which implies $\gamma_{\frac{d}{k}}(G) \leq \frac{n}{2}$. ■

Theorem 2 *Let G be a graph such that every vertex of G is either a pendant vertex or adjacent to at least one pendant vertex. If $A = \{u \in V : d(u) > 1\}$ and k_u is the number of pendant vertices in $N(u)$, for each $u \in A$, then*

$$\gamma_{\frac{d}{k}}(G) = |A| + \sum_{u \in A / k_u \geq \lceil \frac{d(u)}{k} \rceil} k_u - \left\lceil \frac{d(u)}{k} \right\rceil,$$

where the summation is taken over all the vertices $u \in A$ such that $k_u \geq \left\lceil \frac{d(u)}{k} \right\rceil$.

Proof. For each $u \in A$, we define $C_u = N(u) - A$ if $k_u \leq \left\lceil \frac{d(u)}{k} \right\rceil$ and $C_u \subseteq N(u) - A$ of cardinality $\left\lceil \frac{d(u)}{k} \right\rceil$ if $k_u > \left\lceil \frac{d(u)}{k} \right\rceil$. Then $D = \cup_{u \in A}(N(u) - C_u) \cup A$ is a k-DRD set of G. Since $N(v) \subseteq D$ for every $v \in V - A$, $C_v = \emptyset$ for every $v \in (V - A) \cap D$. Also, the vertices in A dominate its maximum possible vertices in $V - A$. Hence, we get D as a minimum k-DRD set of G. ■

Corollary 3 *For the corona of any connected graph G with K_1, $\gamma(GoK_1) = \gamma_{\frac{d}{k}}(GoK_1) = |V(G)|$.*

Theorem 4 *Let G be a connected graph of order n. Then,*

$$\left\lceil \frac{n}{\lceil \frac{\Delta(G)}{k} \rceil + 1} \right\rceil \leq \gamma_{\frac{d}{k}}(G) \leq n - \left\lceil \frac{\Delta(G)}{k} \right\rceil.$$

Proof. Let G be a graph of order n and D be a $\gamma_{\frac{d}{k}}$-set of G. Since for every $u \in D$ order of C_u cannot exceed $\lceil \frac{\Delta(G)}{k} \rceil$, we have

$$\left\lceil \frac{n}{\lceil \frac{\Delta(G)}{k} \rceil + 1} \right\rceil \leq \gamma_{\frac{d}{k}}(G).$$

Let $v \in V$ such that $d(v) = \Delta(G)$ and $N(v) = \{u_1, u_2, \ldots, u_{\Delta(G)}\}$. Choosing arbitrarily $\lceil \frac{\Delta(G)}{k} \rceil$ number of vertices from $N(v)$, we define $C_v = \{u_1, u_2, \ldots, u_{\lceil \frac{\Delta(G)}{k} \rceil}\}$ and $C_w = \emptyset$, for every $w \in V - (C_v \cup \{v\})$. Then, $V - C_v$ is a *k-DRD* set of G and

$$\gamma_{\frac{d}{k}}(G) \leq |V - C_v| = n - \lceil \frac{\Delta(G)}{k} \rceil.$$

■

Remark 1 *The upper and lower bounds cited in Theorem 4 are attained by the graphs $K_{1,m}$ and K_n, respectively.*

Proposition 5 *For any connected graph G of order $n \geq 6$ and $k > 1$, $\gamma_{\frac{d}{k}}(G) = n - \left\lceil \frac{\Delta(G)}{k} \right\rceil$ if and only if $G = K_{1,n-1}$.*

Proof. Let $\gamma_{\frac{d}{k}}(G) = n - \lceil \frac{\Delta(G)}{k} \rceil$ and $v \in V(G)$ such that $d(v) = \Delta(G)$. We claim that $d(v) = n - 1$. Suppose $d(v) \neq n - 1$. Then, there exists an edge uw such that at least one of u, w is not a neighbour of v. Note that, if $\lceil \frac{d(v)}{k} \rceil > d(v) - 1$, then $d(v) \leq 2$. If $d(v) = \Delta(G) = 1$, then $G = K_1$. If $d(v) = \Delta(G) = 2$, then $G = P_n$ or $G = C_n$. Note that, for $k > 1$,

$$\gamma_{\frac{d}{k}}(C_n) = \gamma_{\frac{d}{k}}(P_n) = \lceil \frac{n}{2} \rceil.$$

Since $n \geq 6$ and $d(v) = \Delta(G)$, $\lceil \frac{d(v)}{k} \rceil \leq d(v) - 1$. Then, we can find a subset S of $N(v) - \{u, w\}$ of cardinality $\lceil \frac{\Delta(G)}{k} \rceil$ and $D = V - (S \cup \{u\})$ is a k-DRD set of G with $C_v = S$, $C_w = \{u\}$, $C_x = \emptyset$ for all $x \in D - \{v, w\}$. Then

$$|D| = |V - (S \cup \{u\})| = n - \lceil \frac{\Delta(G)}{k} \rceil - 1 < n - \lceil \frac{\Delta(G)}{k} \rceil,$$

which is a contradiction. Next, we claim that $G - v = \overline{K}_{n-1}$. Suppose $G - v \neq \overline{K}_{n-1}$. Then, $G - v$ has at least one edge, say uw. If $\lceil \frac{d(v)}{k} \rceil > d(v) - 2$, then $d(v) \leq 4$. Since $n \geq 6$ and $d(v) = n - 1$, $\lceil \frac{d(v)}{k} \rceil \leq d(v) - 2$. Then, we can find a subset S of $N(v) - \{u, w\}$ of cardinality $\lceil \frac{\Delta(G)}{k} \rceil$ and $V - (S \cup \{u\})$ is a k-DRD set of G. Also,

$$|V - (S \cup \{u\})| = n - \lceil \frac{\Delta(G)}{k} \rceil - 1 < n - \lceil \frac{\Delta(G)}{k} \rceil,$$

which is a contradiction. Hence $G = K_{1,n-1}$. ■

Theorem 6 *Let G be a connected graph of order $n \geq 4$. Then, $\gamma_{\frac{d}{k}}(G) = n - 1$ if and only if $G = K_{1,n-1}$ and $k \geq n - 1$.*

Proof. If $G = K_{1,n-1}$ and $k \geq n-1$, then $\gamma_{\frac{d}{k}}(G) = n-1$. On the other hand, assume that G is a connected graph of order $n \geq 4$ and $\gamma_{\frac{d}{k}}(G) = n - 1$. Clearly, P_4 is not a subgraph of G. If P_4 is a subgraph of G, then $\gamma_{\frac{d}{k}}(G) \leq n - 2$, a contradiction.

Claim 1: $\Delta(G) = n - 1$.

Since $n \geq 4$ and G is connected, $\Delta(G) \geq 2$. If $\Delta(G) < n - 1$ and u is a vertex of maximum degree in G, then there exists a vertex, which is not adjacent to u but adjacent to some vertices in $N(u)$, which implies P_4 is a subgraph of G, a contradiction.

Claim 2: K_3 is not a subgraph of G.

Assume that K_3 is a subgraph of G. Since $n \geq 4$, there exists a vertex $v \in V$ such that $v \notin V(K_3)$ and adjacent to some vertices in $V(K_3)$. Then, P_4 is a subgraph of G, a contradiction.

From **Claim 1** and **Claim 2** it follows that $G = K_{1,n-1}$. Suppose $k < n - 1 = \Delta(G)$. Then, $\lceil \frac{\Delta(G)}{k} \rceil \geq 2$ and hence $\gamma_{\frac{d}{k}}(G) \leq n - 2$, a contradiction to the assumption $\gamma_{\frac{d}{k}}(G) = n - 1$. ■

Lemma 7 *If tree T has no strong support and degree of each vertex is odd, then T is an infinite tree.*

Proof. Let T be a finite rooted tree, $v \in V(T)$ be a vertex in the last level say m and u be the parent vertex of v. Since degree of each vertex is odd, $d(u) \geq 3$. Also note that u has no strong support, which implies there exists a vertex at a distance two from u and lies in $(m+1)^{th}$ level, a contradiction. Hence, T is an infinite tree. ■

Lemma 8 *For any tree T and a pendant vertex v of T, $\gamma_{\frac{d}{k}}(T - v) \leq \gamma_{\frac{d}{k}}(T)$.*

Lemma 9 *For any finite tree T, $\gamma_{\frac{d}{2}}(T) \leq \lceil \frac{n}{2} \rceil$.*

Proof. We prove the result by induction on n. Clearly, the result holds for $n = 1, 2, 3, 4$. Assume that the result holds for all the trees of order less than n. Let T be a tree of order n.

Case 1: n is odd.

For each edge $e \in E(T)$, $T - e$ has two components say, T_1 and T_2 such that the order of T_1 is even and the order of T_2 is odd. Then, by the induction assumption,

$$\gamma_{\frac{d}{2}}(T) \leq \gamma_{\frac{d}{2}}(T_1) + \gamma_{\frac{d}{2}}(T_2) \leq \lceil \frac{|V(T_1)|}{2} \rceil + \lceil \frac{|V(T_2)|}{2} \rceil \leq \lceil \frac{n}{2} \rceil.$$

Case 2: n is even.

If T has an edge $e \in E(T)$ such that $T - e$ has two components of even order, then the result holds. Suppose for every edge $e \in E(T)$, $T - e$ has two components of odd order. Then, degree of each vertex in T is odd. By Lemma 7, there exists a vertex say, w such that at least two pendant vertices say, w_1, w_2 are adjacent to w. Let D be a minimum 2-DRD set of $T - w_2$. Then, any one of the vertex in $\{w, w_1\}$ should be in D. Assume that $w \in D$. Since $d_T(w)$ is odd, $\lceil \frac{d_T(w)-1}{2} \rceil + 1 = \lceil \frac{d_T(w)}{2} \rceil$. Now w dominates w_1 in T and D is a 2-DRD set of T. Hence,

$$\gamma_{\frac{d}{2}}(T) \leq |D| = \gamma_{\frac{d}{2}}(T - w_2) \leq \lceil \frac{n-1}{2} \rceil \leq \lceil \frac{n}{2} \rceil.$$

■

Theorem 10 *For any connected graph G, $\gamma_{\frac{d}{2}}(G) \leq \lceil \frac{n}{2} \rceil$.*

Proof. Let T be a spanning tree of G. Then, by Lemma 9 $\gamma_{\frac{d}{2}}(T) \leq \lceil \frac{n}{2} \rceil$. Note that $d_T(w) \leq d_G(w)$ for every $w \in V$ and hence $\gamma_{\frac{d}{2}}(G) \leq \gamma_{\frac{d}{2}}(T) \leq \lceil \frac{n}{2} \rceil$. ■

Remark 2 *The bound stated in Theorem 10 does not hold for some graphs, if $k > 2$. For $n \geq 6$,*

$$\gamma_{\frac{d}{3}}(K_{1,m}) = n - \lceil \frac{n-1}{3} \rceil > \lceil \frac{n}{2} \rceil.$$

2.1 Nordhaus-Gaddum Type Results

Proposition 11 *For any graph G, $\gamma_{\frac{d}{2}}(G) + \gamma_{\frac{d}{2}}(\overline{G}) \leq n + \frac{m}{2}$, where m is the number of odd order components of $G \cup \overline{G}$.*

Corollary 12 *Let G be a graph such that the components of G and $\overline{G}$ are of even order. Then, $\gamma_{\frac{d}{2}}(G) + \gamma_{\frac{d}{2}}(\overline{G}) = n$ if and only if $\gamma_{\frac{d}{2}}(G) = \gamma_{\frac{d}{2}}(\overline{G}) = \frac{n}{2}$.*

Theorem 13 *For any nontrivial tree other than star,*

1. $\gamma_{\frac{d}{2}}(T) + \gamma_{\frac{d}{2}}(\overline{T}) \leq n$.
2. $\gamma_{\frac{d}{2}}(T) + \gamma_{\frac{d}{2}}(\overline{T}) = n$ *if and only if* $T = P_4$ *or* $T = P_5$.

Proof. Let T be a tree such that $T \neq K_{1,n-1}$. Then, T has a vertex which is not adjacent to a vertex of maximum degree and there are at least 2 pendant vertices having no common neighbors. Then, $\overline{T}$ is connected and has at least two vertices of degree $n-2$. By Proposition 11, $\gamma_{\frac{d}{2}}(T) + \gamma_{\frac{d}{2}}(\overline{T}) \leq n + 1$. If $\gamma_{\frac{d}{2}}(T) + \gamma_{\frac{d}{2}}(\overline{T}) = n + 1$, then n must be odd. Suppose n is even. Then,

$$\gamma_{\frac{d}{2}}(T) + \gamma_{\frac{d}{2}}(\overline{T}) \leq \lceil \frac{n}{2} \rceil + \lceil \frac{n}{2} \rceil = n.$$

Further, as $\overline{T}$ has at least two vertices of degree $n-2$ and has no common neighbors in T, we get $\gamma_{\frac{d}{2}}(\overline{T}) = 2$. By Lemma 9, $\gamma_{\frac{d}{2}}(T) \leq \lceil \frac{n}{2} \rceil$. Then,

$$n + 1 = \gamma_{\frac{d}{2}}(T) + \gamma_{\frac{d}{2}}(\overline{T}) \leq \frac{n+1}{2} + 2,$$

which implies $n \leq 3$. Hence, T must be a star, a contradiction. Therefore, $\gamma_{\frac{d}{2}}(T) + \gamma_{\frac{d}{2}}(\overline{T}) \leq n$. Suppose that $\gamma_{\frac{d}{2}}(T) + \gamma_{\frac{d}{2}}(\overline{T}) = n$. By Theorem 10, $\gamma_{\frac{d}{2}}(T) \leq \lceil \frac{n}{2} \rceil$ and $\gamma_{\frac{d}{2}}(\overline{T}) \leq \lceil \frac{n}{2} \rceil$, which implies $\gamma_{\frac{d}{2}}(T) = \lceil \frac{n}{2} \rceil$ and $\gamma_{\frac{d}{2}}(\overline{T}) = \lfloor \frac{n}{2} \rfloor$ or $\gamma_{\frac{d}{2}}(T) = \lfloor \frac{n}{2} \rfloor$ and $\gamma_{\frac{d}{2}}(\overline{T}) = \lceil \frac{n}{2} \rceil$. Since $\gamma_{\frac{d}{2}}(\overline{T}) = 2$, we get $n \leq 5$. If $n = 4$, then tree with 4 vertices other than $K_{1,3}$ is P_4. If $n = 5$, then

$$n = \gamma_{\frac{d}{2}}(T) + \gamma_{\frac{d}{2}}(\overline{T}) = \gamma_{\frac{d}{2}}(T) + 2 = 5.$$

Then, tree with 5 vertices having 2-domination number 3 is P_5. Conversely, $\gamma_{\frac{d}{2}}(P_4) = \gamma_{\frac{d}{2}}(\overline{P_4}) = 2$ and $\gamma_{\frac{d}{2}}(P_5) = 3$, $\gamma_{\frac{d}{2}}(\overline{P_5}) = 2$. Hence, result holds. ■

2.2 Bounds on $\gamma_{\frac{d}{k}}$ of Join of Two Graphs

In this section, we discuss bounds on k-part degree restricted domination number of join of two graphs. For any graph G_1, G_2 we know that $\gamma(G_1 + G_2) \leq 2$; but

$$\gamma_{\frac{d}{k}}(G_1 + G_2) > \max\{\gamma_{\frac{d}{k}}(G_1), \gamma_{\frac{d}{k}}(G_2)\},$$

for some graphs. Throughout this section, it is assumed that $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are two connected graphs of order n_1 and n_2, respectively, unless otherwise specified. "The join $G = G_1 + G_2$ of two graphs G_1 and G_2 with disjoint vertex sets V_1 and V_2 and edge sets E_1 and E_2, is the graph with vertex set $V = V_1 \cup V_2$ and edge set $E = E_1 \cup E_2 \cup \{uv : u \in V(G_1) \text{ and } v \in V(G_2)\}$" [2]

Proposition 14 *For any two graphs $G_1 \neq K_1$ and G_2,*

1. $2 \leq \gamma_{\frac{d}{k}}(G_1 + G_2) \leq \gamma_{\frac{d}{k}}(G_1) + \gamma_{\frac{d}{k}}(G_2)$.
2. *If $k \geq \Delta(G_1 + G_2)$ and $n_1 \leq n_2$, then $n_1 \leq \gamma_{\frac{d}{k}} G_1 + G_2) \leq n_2$.*
3. $\gamma_{\frac{d}{k}}(K_{n,n}) = \begin{cases} 2\left\lfloor \frac{n}{m} \right\rfloor & \text{if } n \equiv 0 (mod\ m), \\ 2\left\lfloor \frac{n}{m} \right\rfloor + 1 & \text{if } n \equiv 1 (mod\ m), \\ 2\left\lfloor \frac{n}{m} \right\rfloor + 2 & \text{otherwise.} \end{cases}$ *where* $m = \left\lceil \frac{n}{k} \right\rceil + 1$.

Proposition 15 *For $k > 1$, $\gamma_{\frac{d}{k}}(G_1 + G_2) = 1$ if and only if $G_1 = G_2 = K_1$.*

Proof. If $\gamma_{\frac{d}{k}}(G_1 + G_2) = 1$, then $D = \{u\}$ is a $\gamma_{\frac{d}{k}}$-set of $G_1 + G_2$ for some $u \in V(G_1 + G_2)$. Let $n_1 + n_2 = n$. Then, clearly $n \geq 2$,

$$|C_u| = n - 1 \leq \lceil \frac{d(u)}{k} \rceil \leq \lceil \frac{n-1}{k} \rceil \leq \lceil \frac{n-1}{2} \rceil,$$

which implies $n \leq 2$. Hence, $n = 2$ and $G_1 = G_2 = K_1$. Converse is obvious. ∎

Proposition 16 *For $n_1, n_2 \geq k$, $\gamma_{\frac{d}{k}}(G_1 + G_2) \leq 2k$.*

Proof. In the graph $G_1 + G_2$, at most $l = \left\lceil \frac{n_2}{\lceil \frac{n_2}{k} \rceil} \right\rceil$ vertices from V_1 will be sufficient to dominate V_2; and the remaining $n_1 - l$ vertices of V_1 will require at most $\left\lceil \frac{n_1 - l}{\lceil \frac{n_1}{k} \rceil} \right\rceil$ vertices from V_2. Then,

$$\gamma_{\frac{d}{k}}(G_1 + G_2) \leq \left\lceil \frac{n_1 - l}{\lceil \frac{n_1}{k} \rceil} \right\rceil + l \leq 2k.$$

∎

Proposition 17 1. *If $\gamma_{\frac{d}{k}}(G_1) \geq k$ and $n_2 \equiv 0 (mod\ k)$, then $\gamma_{\frac{d}{k}}(G_1 + G_2) \leq \gamma_{\frac{d}{k}}(G_1)$.*

2. *For, $2 \leq k < n_2 \leq n_1$, $\gamma_{\frac{d}{k}}(G_1 + G_2) < \gamma_{\frac{d}{k}}(G_1) + k$.*

Proof. Let D be $\gamma_{\frac{d}{k}}$-set of G_1. Since $n_2 \equiv 0 (mod\ k)$, we get

$$\left\lceil \frac{d(u) + n_2}{k} \right\rceil = \left\lceil \frac{d(u)}{k} \right\rceil + \left\lceil \frac{n_2}{k} \right\rceil,$$

for any $u \in D$. Hence, each vertex in D can dominate $\frac{n_2}{k}$ vertices from V_2 and $\gamma_{\frac{d}{k}}(G_1)(\frac{n_2}{k}) \geq k(\frac{n_2}{k}) \geq n_2$. Therefor, D is a k-DRD set of $G_1 + G_2$. Similarly if $k \nmid n_2$, then each vertex in D can dominate at least $\lceil \frac{n_2}{k} \rceil - 1$ vertices from V_2. Hence, D can dominate $\gamma_{\frac{d}{k}}(G_1)(\lceil \frac{n_2}{k} \rceil - 1)$ vertices from V_2. Since $\lceil \frac{n_2}{k} \rceil \geq 2$

and $n_1 \geq n_2$, we get $n_2 - \gamma_{\frac{d}{k}}(\lceil\frac{n_2}{k}\rceil - 1) \leq n_1 - \gamma_{\frac{d}{k}}$. So we can find a subset D' of $V_1 - D$ of order $\left\lceil\frac{n_2-\gamma_{\frac{d}{k}}(\lceil\frac{n_2}{k}\rceil-1)}{\lceil\frac{n_2}{k}\rceil}\right\rceil$, which dominate all the remaining vertices in V_2 which are not dominated by D in the graph $G_1 + G_2$. Therefore,

$$\gamma_{\frac{d}{k}}(G_1 + G_2) \leq \gamma_{\frac{d}{k}}(G_1) + \left\lceil\frac{n_2 - \gamma_{\frac{d}{k}}(\lceil\frac{n_2}{k}\rceil - 1)}{\lceil\frac{n_2}{k}\rceil}\right\rceil < \gamma_{\frac{d}{k}}(G_1) + k.$$

■

Remark 3 *The following examples illustrate that the bounds in Proposition 16 and Proposition 17 are sharp.*

1. *Let G_1 and G_2 be two graphs each having perfect matching and $k > \Delta(G_1+G_2)$. Then $\gamma_{\frac{d}{k}}(G_1+G_2) > \max\{\gamma_{\frac{d}{k}}(G_1), \gamma_{\frac{d}{k}}(G_2)\}$.*

2. *For $k = 3$, $\gamma_{\frac{d}{k}}(K_{12,12}) = \gamma_{\frac{d}{k}}(\overline{K}_{12} + \overline{K}_{12}) = 6 = 2k$ (from Proposition 14 (3)). In this example bound in Proposition 16 is attained.*

3. *For $G_1 = C_5$ and $G_2 = C_6$, $\gamma_{\frac{d}{3}}(C_5 + C_6) = 3 = \gamma_{\frac{d}{3}}(C_5)$, which shows that the first equality given in the Proposition 17 can be attained.*

4. *Let G_1 be a connected graph of order 11 satisfying the following conditions:*

   (a) $u, v \in V(G_1)$ such that $d(u) = 9$ and $d(v) = 7$.
   (b) $d(w) \leq 7$ for all $w \in V(G_1) - \{u, v\}$.
   (c) $N[u] \cup N[v] = V(G_1)$.

   Then, $\gamma_{\frac{d}{2}}(G_1) = 2$ but $\gamma_{\frac{d}{2}}(G_1 + P_{11}) = 3 < 2 + 2 = \gamma_{\frac{d}{2}}(G_1) + k$, which satisfies the second inequality in the Proposition 17.

2.3 Bounds on $\gamma_{\frac{d}{k}}$ in Terms of Independence and Covering

In this section, we find some bounds on k-part degree restricted domination number $\gamma_{\frac{d}{k}}$ in terms of vertex cover α_0, edge cover α_1, matching number β_1 and vertex independence number β_0. (See [2].) Though we know that $\gamma(G) \leq \beta_1(G)$ and $\gamma(G) \leq \alpha_0(G)$ for any graph G. But $\gamma_{\frac{d}{k}}(G)$, $\beta_1(G)$ and $\gamma_{\frac{d}{k}}(G)$, $\alpha_0(G)$ are incomparable. If $C_u \neq \emptyset$ for every $u \in D$ or $|V - D| = \sum_{u\in D}\lceil\frac{d(u)}{k}\rceil$, then $\gamma_{\frac{d}{k}}(G) \leq \beta_1(G)$, where D is a $\gamma_{\frac{d}{k}}$-set of given graph G.

For any given subset $D \subseteq V$ to determine whether it is a k-DRD set or not, first we have to construct C_u, for every $u \in D$. Here, we give a general construction of C_u for every $u \in D$ and we use this construction throughout this paper.

Let $D = \{v_1, v_2, \ldots, v_m\}$ and choose a vertex v_1 from D. If

$$\big|N(v_1) \cap (V - D)\big| \leq \lceil\frac{d(v_1)}{k}\rceil,$$

then $C_{v_1} = N(v_1) \cap (V - D)$. Otherwise, choose $\lceil\frac{d(v_1)}{k}\rceil$ number of vertices from the set $N(v_1) \cap (V - D)$ and name that set as C_{v_1}. For all i, $2 \leq i \leq m$, if

$$\big|N(v_i) \cap (V - (D \cup (\cup_{j=1}^{i-1} C_{v_j})))\big| \leq \lceil\frac{d(v_i)}{k}\rceil,$$

then $C_{v_i} = N(v_i) \cap (V - (D \cup (\cup_{j=1}^{i-1} C_{v_j})))$. Otherwise, choose $\lceil\frac{d(v_i)}{k}\rceil$ number of vertices from the set $N(v_i) \cap (V - (D \cup (\cup_{j=1}^{i-1} C_{v_j})))$ and name it as C_{v_i}.

Theorem 18 *For any graph G and $k \geq \Delta(G)$,*

1. $\gamma_{\frac{d}{k}}(G) \geq \frac{n}{2}$.
2. $\gamma_{\frac{d}{k}}(G) = n - \beta_1(G)$.
3. $\gamma_{\frac{d}{k}}(G) = \frac{n}{2}$ *if and only if G has a perfect matching.*
4. $\gamma(G) + \gamma_{\frac{d}{k}}(G) = n$ *if and only if* $\gamma(G) = \beta_1(G)$.

Proof.

1. Since $k \geq \Delta(G)$, each vertex can dominate at most one vertex other than itself. If every vertex dominate exactly two vertices including itself, then $\gamma_{\frac{d}{k}}(G) = \frac{n}{2}$. Otherwise $\gamma_{\frac{d}{k}}(G) > \frac{n}{2}$.
2. Let M be a maximum matching of G and U be the set of vertices saturated by M. Since $k \geq \Delta(G)$, each vertex in U can dominate at most one saturated vertex other than itself. Hence, all the neighbors of unsaturated vertices are dominated. Since M is a maximum matching set, only $|M|$ number of vertices can dominate two vertices including itself. Hence,
$$\gamma_{\frac{d}{k}}(G) = n - 2\beta_1(G) + \beta_1(G) = n - \beta_1(G).$$
3. We know that $\beta_1(G) = \frac{n}{2}$ if and only if G has a perfect matching and from statement 2, statement 3 is trivial.
4. From statement 2, we have
$$\gamma(G) + \gamma_{\frac{d}{k}}(G) = n \Leftrightarrow \gamma(G) + n - \beta_1(G) = n \Leftrightarrow \gamma(G) = \beta_1(G).$$

■

Proposition 19 *For any graph G,*

1. $\gamma_{\frac{d}{k}}(G) + \beta_1(G) \leq n$.
2. *If G has a perfect matching, then* $\gamma_{\frac{d}{k}}(G) \leq \frac{n}{2}$.
3. $\gamma_{\frac{d}{k}}(G) + \gamma(G) \leq n$.
4. *If G is Hamiltonian, then* $\gamma_{\frac{d}{k}}(G) \leq \lceil \frac{n}{2} \rceil$.

Proof.

1. We know that for any positive integer k, $\gamma_{\frac{d}{k}}(G) \leq \gamma_{\frac{d}{k+1}}(G)$. Therefore, for any $k \leq \Delta(G)$, $\gamma_{\frac{d}{k}}(G) \leq \gamma_{\frac{d}{\Delta(G)}}(G) = n - \beta_1(G)$.
2. The second statement follows trivially from the first statement.
3. Since $\gamma(G) \leq \beta_1(G)$ and from the first inequality, we get $\gamma_{\frac{d}{k}}(G) + \gamma(G) \leq n$.
4. If G is Hamiltonian, then $\beta_1(G) = \lfloor \frac{n}{2} \rfloor$ and from the first inequality $\gamma_{\frac{d}{k}}(G) \leq \lceil \frac{n}{2} \rceil$.

■

Lemma 20 *For any $k \geq 1$, if $\gamma(G) + \gamma_{\frac{d}{k}}(G) = n$, then $\gamma(G) = \beta_1(G)$ and $\gamma_{\frac{d}{k}}(G) \geq \frac{n}{2}$.*

Remark 4 *For $k \leq 2$, $\gamma(G) + \gamma_{\frac{d}{k}}(G) = n$ if and only if the components of G are cycle C_4 or the corona HoK_1 for any connected graph H. Suppose $D \subseteq V$ is both $\gamma(G)$-set and k-DRD set of a graph G. Then, $\gamma(G) + \gamma_{\frac{d}{k}}(G) = n$, if and only if the components of G are cycle C_4 or the corona HoK_1 for any connected graph H.*

Proposition 21 *Let G be a graph having an r-factor. If $\lceil \frac{\delta(G)}{k} \rceil \geq r$, then $\gamma_{\frac{d}{k}}(G) \leq \frac{n}{2}$.*

Proof. Let $G_1, G_2, ..., G_m$ be the components of an r-regular spanning subgraph of G. Since $\lceil \frac{\delta(G)}{k} \rceil \geq r$, union of dominating (1-DRD) set of each G_i's, $1 \leq i \leq m$, will be a k-DRD set of G. Hence,

$$\gamma_{\frac{d}{k}}(G) \leq \sum_{i=1}^{m} \gamma(G_i) \leq \sum_{i=1}^{m} \frac{|V(G_i)|}{2} = \frac{n}{2}.$$

■

Theorem 22 *For any graph G with $\delta(G) \geq k$, $\gamma(G) \leq \gamma_{\frac{d}{k}}(G) \leq \alpha_0(G)$.*

Proof. Let $D = \{v_1, v_2, \ldots, v_m\}$ be a minimum vertex cover set of G and for each $v_i \in D$, $C_{v_i} \subseteq V - D$ as constructed in the beginning of the Section 2.3. If $\cup_{v_i \in D} C_{v_i} = V - D$, then D is a k-DRD set of G and result holds. Suppose that $\cup_{v_i \in D} C_{v_i} \neq V - D$. Then, we can find a vertex $w^* \in (V - D) - (\cup_{i=1}^{m} C_{v_i})$. Since D is a vertex cover and $\delta(G) \geq k$, w^* is adjacent to at least k vertices in D. For every $v \in N(w^*) \cap D = B$, $|C_v| = \lceil \frac{d(v)}{k} \rceil$. Also, for every $u \in N(\cup_{x \in B} C_x) \cap D = B_1$, $|C_u| = \lceil \frac{d(u)}{k} \rceil$. Otherwise, we can find a path $P = w^*, v_1, v_2, v_3$ such that $v_1, v_3 \in D$, $|C_{v_3}| < \lceil \frac{d(v_3)}{k} \rceil$ and $v_2 \in C_{v_1}$. We redefine, $C_{v_3} = C_{v_3} \cup \{v_2\}, C_{v_1} = (C_{v_1} - \{v_2\}) \cup \{w^*\}$. Then, w^* is dominated by v_1 and D is a k-DRD set. If for every $u \in N(\cup_{x \in B_1} C_x) \cap D = B_2$, $|C_u| = \lceil \frac{d(u)}{k} \rceil$, then continuing the above process, we get the set $C \subseteq D$ with following properties:

($\mathcal{P}_{11}$) $|C_w| = \lceil \frac{d(w)}{k} \rceil$ for all $w \in C$.

($\mathcal{P}_{12}$) $C_{w_i} \cap C_{w_j} = \emptyset$ for all $w_i, w_j \in C$.

($\mathcal{P}_{13}$) $N(C_w) \cap D \subseteq C$ for all $w \in C$.

Since D is a vertex cover and $\delta(G) \geq k$, by the above properties we have $k \sum_{w \in C} |C_w| \leq \sum_{w \in C} d(w)$. If $k \sum_{w \in C} |C_w| = \sum_{w \in C} d(w)$, then the vertices in C are adjacent to only the vertices in $\cup_{w \in C} C_w$. But vertices in C are adjacent to w^* and $w^* \notin \cup_{w \in C} C_w$. Therefore, $k \sum_{w \in C} |C_w| < \sum_{w \in C} d(w)$, which implies $\sum_{w \in C} |C_w| < \sum_{w \in C} \lceil \frac{d(w)}{k} \rceil$, a contradiction to Property $\mathcal{P}_{11}$. Hence, w^* should be dominated by some vertices in D, D is a k-DRD set and $\gamma_{\frac{d}{k}}(G) \leq |D| = \alpha_0(G)$. ■

Lemma 23 *For any caterpillar T and $k > 2$, $\gamma_{\frac{d}{k}}(T) \geq \alpha_0(T) \geq \beta_1(T)$.*

Proof. Let $A = \{u \in V(T) : d(u) > 2\}$ and S be a minimum vertex cover set of T such that $A \subseteq S$. Then, vertices in A is adjacent to at least one pendant vertex. Since $k > 2$ and as per the definition of k-DRD set, vertex $v \in A$ can dominate at most $d(v) - 2$ number of its neighbors. Also note that vertices in $S - A$ can dominate at most one vertex other than itself. Hence, $|S'| \geq |S| = \alpha_0(T) \geq \beta_1(T)$ for any $\gamma_{\frac{d}{k}}$-set S' of T and $\gamma_{\frac{d}{k}}(T) \geq \alpha_0(T) \geq \beta_1(T)$. ■

Theorem 24 *For any graph G with $\delta(G) > 0$, $\gamma_{\frac{d}{k}}(G) \leq \alpha_1(G)$.*

Proof. Since $\delta(G) > 0$, each vertex can dominate at least one vertex other than itself. By taking one end vertex of each edge in a minimum edge cover, we can construct a k-DRD set of graph G. Hence, $\gamma_{\frac{d}{k}}(G) \leq \alpha_1(G)$. ■

Theorem 25 *For any graph G with $\delta(G) > k$, $\gamma_{\frac{d}{k}}(G) \leq \beta_1(G)$.*

Proof. Let M be a maximum matching set of G and $D = \{v_1, v_2, \dots, v_p\}$ be a dominating set (1-DRD set) of G obtained from the maximum matching M such that $|D| = |M|$. Suppose M is a perfect matching. Then, clearly D is a k-DRD set of G and result holds. Assume M is not a perfect matching and construct C_{v_i} for every $v_i \in D$ as provided in the beginning of Section 2.3 along with one additional condition. That is, for all i, $1 \le i \le p$, if

$$\left|N(v_i) \cap (V - (D \cup (\cup_{j=1}^{i-1} C_{v_j})))\right| > \lceil \frac{d(v_i)}{k} \rceil,$$

then choose $\lceil \frac{d(v_i)}{k} \rceil$ number of vertices along with a vertex u_i such that $v_i u_i \in M$ from the set $N(v_i) \cap (V - (D \cup (\cup_{j=1}^{i-1} C_{v_j})))$ and name it as C_{v_i}. If $\cup_{v_j \in D} C_{v_j} = V - D$, then D is a k-DRD set. Further, if a vertex $v \in V - (\cup_{v_j \in D} C_{v_j} \cup D)$ is adjacent to some vertex $u \in C_w$ with $|C_w| = 1$, then add u to D, w to $V - D$ and construct the set C_u for $u \in D$ as defined above. Since $\delta(G) > k$, vertex u can dominate both w, v. Let $A = V - (\cup_{v_j \in D} C_{v_j} \cup D)$. If $A = \emptyset$, then clearly D is a k-DRD set and result holds. Assume that $A \neq \emptyset$ and $w^* \in A$. Since M is a maximum matching and by the above constructions, w^* is not adjacent to any vertices in $V - D$. Hence, w^* is adjacent to at least k vertices in D.

Then, as in the proof of the Theorem 22 either w^* is dominated by some vertices in D or we get a set $C \subseteq D$ satisfying following properties $(\mathcal{P})$:

($\mathcal{P}_{21}$) $|C_w| = \lceil \frac{d(w)}{k} \rceil$ for all $w \in C$.

($\mathcal{P}_{22}$) $C_{w_i} \cap C_{w_j} = \emptyset$ for all $w_i, w_j \in C$.

($\mathcal{P}_{23}$) $N(C_w) \cap D \subseteq C$ for all $w \in C$.

($\mathcal{P}_{24}$) $|C_w| > 1$ for all $w \in C$.

($\mathcal{P}_{25}$) The vertices in $\cup_{w \in C} C_w$ has its all neighbor in C.

This leads to a contradiction. Hence, $A = \emptyset$, D is a k-DRD set and $\gamma_{\frac{d}{k}}(G) \le |D| = \beta_1(G)$. ∎

Corollary 26 *For any graph G of even order n with $\delta(G) > k$, $\gamma_{\frac{d}{k}}(G) + \beta_1(G) \le n$. If $\gamma_{\frac{d}{k}}(G) + \beta_1(G) = n$, then G has a perfect matching.*

Theorem 27 *For any tree T, $\gamma_{\frac{d}{k}}(T) \le \beta_0(T)$.*

Proof. Let T be a rooted tree with m levels. Now, label all the vertices in m^{th} level as "0". Label all the vertices in $(m-1)^{th}$ level having child in m^{th} level labeled "0" as "1" and label all the remaining vertices in $(m-1)^{th}$ level as "0". Similarly, label all the vertices in $(m-2)^{th}$ level having child in $(m-1)^{th}$ level labeled "0" as "1" and label all the remaining vertices in $(m-2)^{th}$ level as "0". Continue the process for all the m levels. Let D be the set of all the vertices labeled "0". Then, D is an independent vertex set. Also, note that all the vertices labeled "1" will be dominated by its child vertices labeled "0". Hence, D is a k-DRD set and $\gamma_{\frac{d}{k}}(T) \le |D| \le \beta_0(T)$. ∎

Remark 5 *For any graph other than tree, the vertex independence number β_0 and $\gamma_{\frac{d}{k}}$ are incomparable. For example the graph G of order $n > 6$ formed by joining two complete graphs by an edge, we get $\beta_0(G) < \gamma_{\frac{d}{2}}(G) \le \gamma_{\frac{d}{k}}(G)$. For complete graph K_n, $n > 2$, $1 = \beta_0(K_n) < \gamma_{\frac{d}{2}}(K_n) = 2 \le \gamma_{\frac{d}{k}}(K_n)$.*

3 Bounds on k-Part Degree Restricted Domination Number in Terms of k-Domination Number

In this section, we find some bounds on k-part degree restricted domination number in terms of k-domination number.

Definition 1 *[1] For a positive integer k, a dominating set D of a graph G is called a k-dominating set, if every vertex of $V-D$ is adjacent to at least k vertices in D. The k-domination number of G is the minimum cardinality of a k-dominating set of G and is denoted by $\gamma_k(G)$.*

Theorem 28 *In any graph G, every k-dominating set is a k-DRD set.*

Proof. Without loss of generality, we assume G to be connected (otherwise, we can apply the following argument for each of the components of G). Let D be a k-dominating set of G. Then, each vertex in $V-D$ is adjacent to at least k vertices in D. Construct C_u for every $u \in D$ and the proof follows by the similar argument used in the proof of Theorem 22. ■

Corollary 29 *For any graph G, $\gamma_{\frac{d}{k}}(G) \leq \gamma_k(G)$.*

Corollary 30 *For any graph G with $\delta(G) \geq k$, $\gamma(G) \leq \gamma_{\frac{d}{k}}(G) \leq \gamma_k(G) \leq \alpha_0(G)$.*

Remark 6 *For $k = 2$, the bound stated in Corollary 29 can be attained by the graph P_n , C_n and K_n, $n > 2$. Also for any graph G and $k > \Delta(G)$, $\gamma_{\frac{d}{k}}(G) < \gamma_k(G)$ and $\gamma_{\frac{d}{k}}(G) + \beta_1(G) = \gamma_k(G)$.*

Proposition 31 *For any graph G with $\delta(G) \geq k$,*

$$\frac{\gamma_{\frac{d}{k}}(G) + \gamma_k(G)}{2} \leq n - \beta_0(G).$$

Lemma 32 *For any graph G, $\gamma_{\frac{d}{k}}(G) = \gamma_k(G)$ if and only if G has a $\gamma_{\frac{d}{k}}$-set which is a k-dominating set.*

Lemma 33 *For any tree $T \neq K_2$ and $k > 1$, $\gamma_{\frac{d}{k}}(T) = \gamma_k(T)$ if and only if there exists a set $D \subseteq V(T)$ satisfying the following properties ($\mathcal{P}$):*

($\mathcal{P}_1$) All the pendant vertices are in D.

($\mathcal{P}_2$) $d(u) = k$ for all $u \in V-D$.

($\mathcal{P}_3$) If $uv \in E(T)$, then either $u \in D$ and $v \in V-D$ or $u \in V-D$ and $v \in D$.

Proof. Assume that T is a rooted tree such that $\gamma_{\frac{d}{k}}(T) = \gamma_k(T)$. Then, there exists a $\gamma_{\frac{d}{k}}$-set D which is a k-dominating set. Since D is a k-dominating set, property $\mathcal{P}_1$ is trivially holds.

We claim that $|C_u| \leq 1$ for all $u \in D$ and if $|C_u| = 1$, then C_u contains the parent vertex of u. Let $u \in D$ be a vertex in the i^{th} level of rooted tree T such that $|C_u| = 2$ and $|C_v| \leq 1$ for all the vertices v in the succeeding level. (If $|C_u| > 2$, then we can apply the same following argument for each of the child neighbor of u in C_u.) Then, at least one vertex in $C_u \subseteq V-D$ say u_1 should be a child of u and $d(u_1) > 1$. (Since $u_1 \in V-D$ and D is k-dominating set.) Since D is a k-dominating set, at least one child of u_1 say u_2 should be in D. If $|C_{u_2}| = 0$ (or $d(u_2) = 1$), then u_2 can dominate u_1 and $|C_u| = 1$. If not, then u_2 has at least one child say $u_3 \in C_{u_2}$ in $V-D$. Since D is a k-dominating set, at least one child of u_3 say u_4 should be in D. If $|C_{u_4}| = 0$ (or $d(u_4) = 1$), then u_4 can dominate u_3, u_2 can dominate u_1 and $|C_u| = 1$. If not, then continuing this process we get a path $P = u, u_1, u_2 .... u_l$ such that $u_i \in D$ if i is even, $u_i \in V-D$ if i is odd and $d(u_l) = 1$. Then, by similar rearrangements we can modify C_u such that $|C_u| = 1$ and C_u contains the parent vertex of u. Now, D is a minimum k-DRD set, which is a k-dominating set such that $|C_u| \leq 1$ and if $|C_u| = 1$, then C_u contains the parent vertex of u.

We prove that $d(u) = k$ for all $u \in V-D$. Since D is a k-dominating set, $d(u) \geq k$ for all $u \in V-D$. Let $d(u) = k+1$ and N be the set of k neighbor of u in D. By above claim there exists two vertices $v, w \in N$ such that $C_v = \{u\}$ and $C_w = \emptyset$. Since $d(u) = k+1$ and $\lceil \frac{d(u)}{k} \rceil \geq 2$, u can dominate two of its neighbors. Hence, $D - \{v, w\} \cup \{u\}$ is a k-DRD set of tree T with $C_u = \{v, w\}$, a contradiction to the fact that D is a minimum k-DRD set. Hence, property $\mathcal{P}_2$ holds.

If $uv \in E(T)$, then by $\mathcal{P}_2$ both u, v are not in $V - D$. Assume that $u, v \in D$ such that u lies in l^{th} level and v lies in the $l + 1^{th}$ level. Then, $C_v = \emptyset$ and $|C_u| = 1$. Let $C_u = \{u_1\} \subseteq V - D$. Since $d(u_1) = k$ and D is a k-dominating set, all the neighbors of u_1 is in D. If u_1 has at least one child neighbor say $u_2 \neq u$ in D, then u_2 can dominate u_1 and v can dominate u, a contradiction to the fact that D is minimum k-DRD set. Assume that u_1 has no child other than u in D. Then, $d(u_1) = k = 2$ and parent vertex of u_1 say u_3 is in D. If $|C_{u_3}| = 0$, then it is a contradiction to the fact that D is $\gamma_{\frac{d}{k}}$-set. If $C_{u_3} = \{u_4\}$, then parent vertex of u_4 is in D, continuing like this we get a path $P = v, u, u_1, u_3, u_4 \ldots, u_r$ from v to root vertex u_r such that $u_i \in D$ if i is odd, $u_i \in V - D$ if i is even for $i > 1$. Suppose $u_r \in D$. Then, $C_{u_r} = \emptyset$ and v can dominate u (by some rearrangement in $V - D$), contradiction. If $u_r \in V - D$, then at least two child vertices of u_r should be in D. Then, v can dominate u (by some rearrangement in $V - D$), a contradiction. Hence, property $\mathcal{P}_3$ holds.

Conversely, assume that T is a rooted tree having m levels and $D \subseteq V$ satisfying all the above properties. Then, by property $\mathcal{P}_2$ and $\mathcal{P}_3$, D is a k-dominating set and hence k-DRD set of T. Also, Property $\mathcal{P}_1$ and $\mathcal{P}_3$ implies that, vertices in $(m - i)^{th}$ level lie in $V - D$ if i is odd and vertices in $(m - i)^{th}$ level lie in D if i is even for all i, $1 \leq i < m$. We prove that D is a minimum k-DRD set of T, then converse part holds. Let D^* be a minimum k-DRD set of tree T such that $\cup_{u \in D^*} C'_u = V - D^*$. We construct a minimum k-DRD set D' of T from D^* such that $V - D \subseteq D'$. Now all the vertices in m^{th} level are in D and all the vertices in $(m - 1)^{th}$ level are in $V - D$. If there is a vertex $v \in V - D^*$ lies in $(m - 1)^{th}$ level, then pendant neighbour (Since $v \in V - D$, $d(v) > 1$) of v say u should be in D^* with $C'_u = \emptyset$ (or $C'_u = \{v\}$). Define, $D_1 = D^* \cup \{v\} - \{u\}$ and $C_v = \{u\}$. If there is a vertex $v \in D_1$ lies in $(m - 1)^{th}$ level, such that $C_v = \{v'\}$ and v' is the parent vertex of v, then also pendant neighbour (since $v \in V - D$, $d(v) > 1$) of v say w should be in D_1 with $C'_w = \emptyset$. Define, $D_2 = D_1 \cup \{v'\} - \{w\}$, $C_{v'} = \emptyset$ and $C_v = \{w\}$. Then, D_2 is a minimum k-DRD set of T such that all the vertices in $(m - 1)^{th}$ level is in D_2 and dominates only its pendant neighbour. Since vertices lie in $(m - 3)^{th}$ level are in $V - D$, the $(m - 3)^{th}$ level vertices are not pendant vertices. If there is a vertex $w \in V - D_2$ that lies in $(m - 3)^{th}$ level, then child neighbour of w say w' is in D_2 with $C'_{w'} = \emptyset$ (or $C'_{w'} = \{w\}$). (Since w' has all neighbors except w in $m - 1^{th}$ level and all the vertices in $m - 1^{th}$ level are in D_2 and only dominating its child vertices.) Define, $D_3 = D_2 \cup \{w\} - \{w'\}$, $C_w = \{w'\}$. If there is a vertex $u \in D_3$ that lies in $(m - 3)^{th}$ level, such that $C_u = \{u'\}$ and u' is the parent vertex of u, then child of u say w^* should be in D_3 with $C'_{w^*} = \emptyset$. (Since $u \in V - D$, $d(u) = k$ and $|C_u|$ can not exceed 1. Also w^* has all neighbors except u in $m - 1^{th}$ level and all the vertices in $m - 1^{th}$ level are in D_3 and only dominating its child vertices.) Proceeding in this manner, we get a minimum k-DRD set $D_r = D'$ such that all the vertices in $(m - i)^{th}$ level lie in D' if i is odd and $V - D \subseteq D'$. Then, $V - D' \subseteq D$ and $D' = (V - D) \cup (D \cap D')$. Since all the neighbors of D lie in $V - D$, $C'_w = \emptyset$ for all $w \in D \cap D' \subseteq D'$. Since $d(u) = k$ for all $u \in (V - D)$, $|C'_u| = 1$ for all $u \in (V - D) \subseteq D'$. Hence, vertices in $V - D'$ should be dominated by vertices in $V - D$ in D', which implies $|V - D'| \leq |V - D|$. Since D' is a $\gamma_{\frac{d}{k}}$-set of T, we get $|D'| = |D|$. Since D' is a minimum k-DRD set and a k-dominating set of T, D is a minimum k-dominating set of T and $\gamma_{\frac{d}{k}}(T) = \gamma_k(T)$. ■

Let ψ be the collection of all trees T such that for any $u \in V(T)$ either all the pendant vertices are at odd distance from u or all the pendant vertices are at even distance from u. If a vertex u is at odd distance from a pendant vertex, then $d(u) = k$.

Theorem 34 *For any tree $T \neq K_2$, $\gamma_{\frac{d}{k}}(T) = \gamma_k(T)$ if and only if $T \in \psi$.*

Proof. Let T be a rooted tree and $\gamma_{\frac{d}{k}}(T) = \gamma_k(T)$. Then, by Lemma 33 there exists a subset $D \subseteq V(T)$ satisfying properties $\mathcal{P}$. Suppose there exists a vertex $v \in V(T)$ such that v is at odd distance from a pendant vertex v_1 and v is at even distance from a pendant vertex v_2. Then, the first condition $\mathcal{P}_1$ of Lemma 33 implies that, $v_1, v_2 \in D$. By the third condition $\mathcal{P}_3$ parent vertex of v_1 say v_3 lies in $V - D$ and parent vertex of v_3 lies in D. Since v is at odd distance from v_1, we have $v \in V - D$. Now, $v_2 \in D$ and v is at even distance from v_2. Then, by the similar argument as above $v \in D$, a contradiction. Note that all the vertices at odd distance from a pendant vertex lie in $V - D$. Then, by Property $\mathcal{P}_2$ degree of all the vertices are at odd distance from a pendant vertex is k. Hence, $T \in \psi$. Conversely, assume that $T \in \psi$. Let $D \subseteq V(T)$ be the collection of all the pendant vertices in $V(T)$ and all the vertices at even distance from pendant vertices.

Since $T \in \psi$, we have $V - D$ is the collection of all the vertices at odd distance from pendant vertices and $d(u) = k$ for all $u \in V - D$. Then, $D \subseteq V$ satisfying the first and second condition in Lemma 33. Let $uv \in E(T)$. If any one of u, v is a pendant vertex, then third condition in Lemma 33 holds. Suppose both u, v are not pendant vertices. If $u \in D$, then u is at even distance from a pendant vertex say v_1 and v is at odd distance from the pendant vertex v_1. Hence, $v \in V - D$. If $u \in V - D$, then u is at odd distance from a pendant vertex say v_2 and v is at even distance from the pendant vertex v_2. Hence, $v \in D$. Therefore, we can find a set $D \subseteq V(T)$ satisfying all the three condition stated in Lemma 33 and hence $\gamma_{\frac{d}{k}}(T) = \gamma_k(T)$. ■

Corollary 35 *For any caterpillar T with diametral path $P = v_1, v_2, \ldots, v_m$, where v_1, v_m are pendant vertices, $\gamma_{\frac{d}{k}}(T) = \gamma_k(T)$ if and only if T satisfies following properties.*

1. *m is odd.*
2. *$d(v_{2l+1}) = 2$ for all $1 \le l \le \frac{m-3}{2}$.*
3. *$d(v_{2l}) = k$ for all $2 \le l \le \frac{m-1}{2}$.*

Corollary 36 *For any caterpillar T of order n, $\gamma_{\frac{d}{2}}(T) = \gamma_2(T)$ if and only if $T = P_n$, n is odd.*

4 Conclusion

In this paper, we provided some bounds on $\gamma_{\frac{d}{k}}$ of join of two graphs and bounds on $\gamma_{\frac{d}{k}}$ in terms of maximum degree, independence number and covering number. We are working on some bounds for $\gamma_{\frac{d}{k}}$ of the graph obtained from other graph operators like Cartesian product. We also discussed bounds on $\gamma_{\frac{d}{k}}$ in terms of k-domination number γ_k. We proved that $\gamma_{\frac{d}{k}}(G) \le \gamma_k(G)$ for any graph G, and we characterized the trees T, for which $\gamma_{\frac{d}{k}}(T) = \gamma_k(T)$.

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