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Systems Having Liouvillian First Integrals And Non-Algebraic Limit Cycles*

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Abstract

We consider the class of polynomial differential equations $\dot{x} = P_m(x, y) + P_{m+n}(x, y)$, $\dot{y} = Q_m(x, y) + Q_{m+n}(x, y)$ for $m,\ n \geq 1$ and where P_i and Q_i are homogeneous polynomials of degree i. Inside this class, we identify a new subclass of Liouvillian integrable systems, under suitable conditions such Liouvillian integrable systems can have at most one limit cycle, and when it exists, is non-algebraic and hyperbolic. Then we study the general systems of the systems studied in [9], which allow us to find the necessary and sufficient conditions for the existence and non-existence of limit cycles.

1. Introduction and Statement of the Main Results

A polynomial differential system on the plane is of the form

$$\begin{aligned} \dot{x} = \frac{dx}{dt} = P(x, y), \\ \dot{y} = \frac{dy}{dt} = Q(x, y), \end{aligned} \tag{1.1}$$

where P and Q are two coprime polynomials of $\mathbb{R}[x, y]$, and the derivatives are performed with respect to the time variable. By definition, the degree of the system (1.1) is the maximum of the degrees of the polynomials P and Q.

System (1.1) is said to be integrable on an open set Ω of $\mathbb{R}^2$ if there exists a non constant continuously differentiable function $H : \Omega \longmapsto \mathbb{R}$ called the first integral of this system on Ω which is constant on the trajectories of the polynomial system (1.1) contained in Ω, i.e., if

$$\frac{dH}{dt}(x, y) = \frac{\partial H}{\partial x}(x, y)\, P(x, y) + \frac{\partial H}{\partial y}(x, y)\, Q(x, y) \equiv 0 \text{ in } \Omega.$$

Moreover, $H = h$ is the general solution of the above equation, where h is an arbitrary constant. It is well known that for the planar differential system, the existence of a first integral determines its phase portrait, see [10].

We recall that in the phase plane, a *limit cycle* of system (1.1) is an isolated periodic solution in the set of all its periodic solutions. If limit cycle contained in the zero set of invariant algebraic curve of the plane, then we say that it is *algebraic*; otherwise, it is called *non-algebraic.* In the qualitative theory of differential systems in the plane, two important problems are to determine the first integrals and the limit cycles.

It is very difficult to detect if a planar differential system is integrable or not and also to know if the limit cycles for this system exist and are algebraic, as well as the determination of their explicit expressions.

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In the beginning, the explicit expressions of limit cycles were algebraic (see, for example, [14, 4, 5, 3] and references therein). It is only after 2006 that it became possible to find explicit expressions of non-algebraic limit cycles [12, 1, 15, 2, 8].

This article deals with two problems for a class of real planar differential systems of the form:

$$\begin{aligned} \dot{x} &= P_m(x,y) + P_{m+n}(x,y), \\ \dot{y} &= Q_m(x,y) + Q_{m+n}(x,y), \end{aligned} \tag{1.2}$$

where $P_i(x,y)$ and $Q_m(x,y)$ are homogeneous polynomials of degree i in the variables x and y with P_m and Q_m satisfying $xQ_m - yP_m \equiv 0$.

In order to present our main results, we take the polar coordinates changes $x = r\cos\theta$, $y = r\sin\theta$, system (1.2) becomes

$$\begin{aligned} \dot{r} &= f_{m+1}(\theta) r^m + f_{m+n+1}(\theta) r^{m+n}, \\ \dot{\theta} &= g_{m+n+1}(\theta) r^{m+n-1}, \end{aligned} \tag{1.3}$$

where

$$\begin{aligned} f_i(\theta) &= \cos\theta P_{i-1}(\cos\theta, \sin\theta) + \sin\theta Q_{i-1}(\cos\theta, \sin\theta), \\ g_i(\theta) &= \cos\theta Q_{i-1}(\cos\theta, \sin\theta) - \sin\theta P_{i-1}(\cos\theta, \sin\theta). \end{aligned}$$

We note that if $g_{m+n+1}(\theta)$ vanishes for some $\theta = \theta^*$ then it has $\{\theta = \theta^*\}$ as an invariant straight line. From the uniqueness of solutions, we get that system (1.3) has no limit cycles. Since our goal is to study the limit cycles, we limit the study to region $W = \{\theta : g_{m+n+1}(\theta) \neq 0\}$. In this case, we remark that for any equilibrium point (x_0, y_0) of the systems, we have $x_0 Q(x_0, y_0) - y_0 P(x_0, y_0) = 0$, but $\dot{x}$ and $\dot{y}$ are related to $\dot{\theta}$ by $\dot{\theta} = \frac{xQ(x,y) - yP(x,y)}{x^2+y^2}$, we deduce then that at each point $(x_0, y_0) \neq (0,0)$ we have $\dot{\theta} = 0$. As $\dot{\theta} = \dfrac{d\theta}{dt}$ is positive or negative for all t, this means that $(0,0)$ is the unique equilibrium point of system (1.2) and the orbits $(r(t), \theta(t))$ of system (1.3) have same or opposite orientation with respect to $(x(t), y(t))$ of system (1.2).

Our results are the following

Theorem 1. *For system (1.2) the following statements hold.*

(1) *The system (1.2) has the Liouvillian first integral*

$$H(x,y) = \left(x^2+y^2\right)^{\frac{n}{2}} \exp\left(-\int_0^{\arctan\frac{y}{x}} F(s)\,ds\right) - \int_0^{\arctan\frac{y}{x}} G(s) \exp\left(-\int_0^s F(w)\,dw\right) ds,$$

where

$$F(s) = n \times \tfrac{f_{m+n+1}(s)}{g_{m+n+1}(s)} \text{ and } G(s) = n \times \tfrac{f_{m+1}(s)}{g_{m+n+1}(s)} .$$

(2) *The system (1.2) can have at most one limit cycle. When it exists, it is hyperbolic, and given in polar coordinates by the equation*

$$r(\theta, r_0) = \exp\left(\tfrac{1}{n}\int_0^\theta F(s)\,ds\right)\left(r_0^n + \int_0^\theta G(s)\exp\left(-\int_0^s F(w)\,dw\right) ds\right)^{\frac{1}{n}},$$

where

$$r_0 = \sqrt[n]{\tfrac{\exp\left(\int_0^{2\pi} F(s)ds\right)}{1-\exp\left(\int_0^{2\pi} F(s)ds\right)} \int_0^{2\pi} G(s)\exp\left(-\int_0^s F(w)\,dw\right) ds}.$$

Moreover, there exist such systems which have one non-algebraic limit cycle.

We will apply our theorem to the subsystem of system (1.2)

$$\begin{aligned} \dot{x} &= x + (\alpha y - \beta x)\, U, \\ \dot{y} &= y - (\beta y + \alpha x)\, U, \end{aligned} \tag{1.4}$$

where U is a homogenous polynomial of degree n in the variables x and y.

The rest of this paper is organized as follows. Section 2 is dedicated to prove theorem 1. In Section 3, we present the necessary and sufficient conditions for systems (1.4) to have non-algebraic and hyperbolic limit cycles.

2. Proof of Theorem 1

(1) In the region $W = \{\theta : g_{m+n+1}(\theta) \neq 0\}$, system (1.3) becomes

$$\frac{dr}{d\theta} = \frac{f_{m+1}(\theta)}{g_{m+n+1}(\theta)} r^{1-n} + \frac{f_{m+n+1}(\theta)}{g_{m+n+1}(\theta)} r, \tag{2.1}$$

which is a Bernoulli equation. By introducing the standard change of variables $\rho = r^n$, we can transform (2.1) into the linear differential equation

$$\frac{d\rho}{d\theta} = F(\theta)\, \rho + G(\theta), \tag{2.2}$$

with

$$F(\theta) = n \times \frac{f_{m+n+1}(\theta)}{g_{m+n+1}(\theta)} \quad \text{and} \quad G(\theta) = n \times \frac{f_{m+1}(\theta)}{g_{m+n+1}(\theta)}.$$

The general solution of equation (2.2) is

$$\rho(\theta) = \exp\left(\int_0^\theta F(s)\, ds\right) \left(k + \int_0^\theta G(s) \exp\left(-\int_0^s F(w)\, dw\right) ds\right),$$

with $k \in \mathbb{R}$, which implies that the general solution of the equation (2.1) is

$$r(\theta) = \exp\left(\frac{1}{n}\int_0^\theta F(s)\, ds\right) \left(k + \int_0^\theta G(s) \exp\left(-\int_0^s F(w)\, dw\right) ds\right)^{\frac{1}{n}},$$

with $k \in \mathbb{R}$. From this solution, we can obtain a first integral in the variables (x, y) of the form

$$H(x, y) = \left(x^2 + y^2\right)^{\frac{n}{2}} \exp\left(-\int_0^{\arctan \frac{y}{x}} F(s)\, ds\right) - \int_0^{\arctan \frac{y}{x}} G(s) \exp\left(-\int_0^s F(w)\, dw\right) ds.$$

Since this first integral is a function that can be expressed by quadratures of elementary functions, it is a Liouvillian function; consequently, system (1.2) is Liouvillian integrable. Hence, statement (1) is proved.
(2) Notice that system (1.2) has a periodic orbit if and only if equation (2.1) has a strictly positive 2π-periodic solution.

The general solution of equation (2.1), with initial condition $r(0) = r_0$, is

$$r(\theta, r_0) = \exp\left(\frac{1}{n}\int_0^\theta F(s)\, ds\right) \left(r_0^n + \int_0^\theta G(s) \exp\left(-\int_0^s F(w)\, dw\right) ds\right)^{\frac{1}{n}}.$$

The condition that the solution starting at $r = r_0$ is periodic reads as

$$r_0 = \sqrt[n]{\frac{\exp\left(\int_0^{2\pi} F(s) ds\right)}{1 - \exp\left(\int_0^{2\pi} F(s) ds\right)} \int_0^{2\pi} G(s) \exp\left(-\int_0^s F(w)\, dw\right) ds}.$$

Periodicity of $r(\theta, r_0)$

We have

$$r(\theta+2\pi) = \exp\left(\frac{1}{n}\int_0^{\theta+2\pi} F(s)\,ds\right)\left(\begin{array}{c}\frac{\exp\left(\int_0^{2\pi} F(s)ds\right)}{1-\exp\left(\int_0^{2\pi} F(s)ds\right)}\int_0^{2\pi} G(s)\exp\left(-\int_0^s F(w)\,dw\right)ds \\ \int_0^{\theta+2\pi} G(s)\exp\left(-\int_0^s F(w)\,dw\right)ds\end{array}\right)^{\frac{1}{n}},$$

it follows

$$r(\theta+2\pi) = \exp\left(\frac{1}{n}\left(\int_0^{2\pi} F(s)\,ds + \int_{2\pi}^{\theta+2\pi} F(s)\,ds\right)\right)\left(\begin{array}{c}\frac{\exp\left(\int_0^{2\pi} F(s)ds\right)\int_0^{2\pi} G(s)\exp\left(-\int_0^s F(w)dw\right)ds}{1-\exp\left(\int_0^{2\pi} F(s)ds\right)} \\ \int_0^{2\pi} G(s)\exp\left(-\int_0^s F(w)\,dw\right)ds \\ +\int_{2\pi}^{\theta+2\pi} G(s)\exp\left(-\int_0^s F(w)\,dw\right)ds\end{array}\right)^{\frac{1}{n}},$$

i.e.

$$r(\theta+2\pi) = \exp\left(\frac{1}{n}\int_{2\pi}^{\theta+2\pi} F(s)\,ds\right)\exp\left(\frac{1}{n}\int_0^{2\pi} F(s)\,ds\right)\left(\begin{array}{c}\frac{\int_0^{2\pi} G(s)\exp\left(-\int_0^s F(w)dw\right)ds}{1-\exp\left(\int_0^{2\pi} F(s)ds\right)} \\ +\int_{2\pi}^{\theta+2\pi} G(s)\exp\left(-\int_0^s F(w)\,dw\right)ds\end{array}\right)^{\frac{1}{n}},$$

by the change of variable $u = s - 2\pi$, we obtain

$$\int_{2\pi}^{\theta+2\pi} F(s)\,ds = \int_0^{\theta} F(s)\,ds,$$

and

$$\int_{2\pi}^{\theta+2\pi} G(s)\exp\left(-\int_0^s F(w)\,dw\right)ds = \exp\left(-\int_0^{2\pi} F(s)\,ds\right)\int_0^{\theta} G(s)\exp\left(-\int_0^s F(w)\,dw\right)ds,$$

then

$$r(\theta+2\pi) = r(\theta).$$

Therefore $r(\theta, r_0)$ is 2π-periodic.

In order to prove the hyperbolicity of the limit cycle, we introduce the Poincaré return map

$$\gamma \mapsto \Pi(\gamma) = r(2\pi, \gamma) = \exp\left(\frac{1}{n}\int_0^{2\pi} F(s)\,ds\right)\left(\gamma^n + \int_0^{2\pi} G(s)\exp\left(-\int_0^s F(w)\,dw\right)ds\right)^{\frac{1}{n}}$$

and show that the function of Poincaré first return verify $\left.\frac{d\Pi}{d\gamma}(\gamma)\right|_{\gamma=r_0} \neq 1$, see [11]. We have

$$\frac{d\Pi}{d\gamma}(\gamma) = \left(\exp\left(\frac{1}{n}\int_0^{2\pi} F(s)\,ds\right)\right)\gamma^{n-1}\left(\gamma^n + \int_0^{2\pi} G(s)\exp\left(-\int_0^s F(w)\,dw\right)ds\right)^{\frac{1}{n}-1},$$

which implies that

$$\left.\frac{d\Pi}{d\gamma}(\gamma)\right|_{\gamma=r_0} = \left(\exp\left(\frac{1}{n}\int_0^{2\pi} F(s)\,ds\right)\right)r_0^{n-1}\left(r_0^n + \int_0^{2\pi} G(s)\exp\left(-\int_0^s F(w)\,dw\right)ds\right)^{\frac{1}{n}-1}.$$

After the substitution of the value of r_0 into the previous relationship , we obtain

$$\left.\frac{d\Pi}{d\gamma}(\gamma)\right|_{\gamma=r_0} = \exp\left(\int_0^{2\pi} F(s)\,ds\right) \neq 1.$$

Therefore the solution of the differential equation (2.1) is a hyperbolic limit cycle; consequently, it is a hyperbolic limit cycle for the system (1.2).

We consider the system (1.2) with $m = 1$ and $n = 2$ having the form

$$\begin{aligned} \dot{x} &= x + (y - x)\left(x^2 - xy + y^2\right), \\ \dot{y} &= y - (y + x)\left(x^2 - xy + y^2\right), \end{aligned} \tag{2.3}$$

It is easy to check that system (2.3) is a subclass of (1.2) because of $P_1(x, y) = x$ and $Q_1(x, y) = y$ i.e., satisfying $xQ_1 - yP_1 \equiv 0$. These systems have a non-algebraic and hyperbolic limit cycle, see [8]. Hence, statement (2) is proved.

3. Application of Theorem 1

In this section, we apply Theorem 1 to systems (1.4) for studying their integrability and the existence of non-algebraic limit cycles.

Corollary 1. *For system (1.4) the following statements hold.*

(1) *The system (1.4) has the Liouvillian first integral*

$$H(x, y) = \left(x^2 + y^2\right)^{\frac{n}{2}} \exp\left(\frac{-n\beta}{\alpha} \arctan \frac{y}{x}\right) + \frac{n}{\alpha} \int_0^{\arctan \frac{y}{x}} \frac{\exp\left(-\frac{n\beta}{\alpha} s\right)}{U(s)} ds.$$

(2) *If $\beta U > 0$ and $\alpha \neq 0$, the system (1.4) has exactly one non-algebraic, stable and hyperbolic limit cycle explicitly given in polar coordinates by*

$$r(\theta, r_0) = \exp\left(\frac{\beta}{\alpha}\theta\right)\left(r_0^n + \int_0^{\theta} \frac{-n \exp\left(-\frac{n\beta}{\alpha} s\right)}{\alpha U(s)} ds\right)^{\frac{1}{n}},$$

where

$$r_0 = \sqrt[n]{\frac{\exp\left(2\pi n \frac{\beta}{\alpha}\right)}{\exp\left(2\pi n \frac{\beta}{\alpha}\right) - 1} \int_0^{2\pi} \frac{n \exp\left(-\frac{n\beta}{\alpha} s\right)}{\alpha U(s)} ds}.$$

(3) *If $\beta U \leq 0$ or $\alpha = 0$, the system (1.4) has no periodic orbits.*

Proof of Corollary 1. Taking polar coordinates (r, θ), the systems (1.4) can be written as

$$\begin{aligned} \dot{r} &= r - \beta U r^{n+1}, \\ \dot{\theta} &= -\alpha U r^n. \end{aligned} \tag{3.1}$$

(1) Using the statement (1) of Theorem 1, we find that the systems (1.4) admits

$$H(x, y) = \left(x^2 + y^2\right)^{\frac{n}{2}} \exp\left(-\int_0^{\arctan \frac{y}{x}} F(s)\, ds\right) - \int_0^{\arctan \frac{y}{x}} G(s) \exp\left(-\int_0^{s} F(w)\, dw\right) ds$$

as a Liouvillian first integral, where

$$\begin{aligned} F(\theta) &= \tfrac{n\beta}{\alpha}, \\ G(\theta) &= \tfrac{-n}{\alpha U(\theta)}. \end{aligned}$$

After calculation, we get

$$H(x,y)=\left(x^2+y^2\right)^{\frac{n}{2}}\exp\left(\frac{-n\beta}{\alpha}\arctan\frac{y}{x}\right)+\frac{n}{\alpha}\int_0^{\arctan\frac{y}{x}}\frac{\exp\left(-\frac{n\beta}{\alpha}s\right)}{U(s)}ds.$$

Hence, statement (1) of the corollary is proved.
(2) Using the statement (2) of theorem 1, we find that the limit cycle, if it exists, is of the form:

$$r(\theta)=\exp\left(\frac{\beta}{\alpha}\theta\right)\left(r_0^n+\int_0^{\theta}\frac{-n}{\alpha U(s)}\exp\left(-\frac{n\beta}{\alpha}s\right)ds\right)^{\frac{1}{n}},$$

where

$$r_0=\sqrt[n]{\frac{\exp\left(2\pi n\frac{\beta}{\alpha}\right)}{\exp\left(2\pi n\frac{\beta}{\alpha}\right)-1}\int_0^{2\pi}\frac{n\exp\left(-\frac{n\beta}{\alpha}s\right)}{\alpha U(s)}ds}.$$

Notice that

$$A=\frac{\exp\left(2\pi n\frac{\beta}{\alpha}\right)}{\exp\left(2\pi n\frac{\beta}{\alpha}\right)-1},$$

$$B=\frac{n\exp\left(-\frac{n\beta}{\alpha}s\right)}{\alpha U(s)}.$$

i.e.

$$r_0=\sqrt[n]{A\int_0^{2\pi}Bds}.$$

As $\beta U>0$ and $\alpha\neq 0$, then $(A>0$ and $B>0)$ or $(A<0$ and $B<0)$, which implies that $r_0>0$.
By the proof of Theorem 1, $r(\theta,r_0)$ is $2\pi-$periodic. To demonstrate that the solution $r(\theta,r_0)$ is periodic, it suffices to show that it is strictly positive.

Strict Positivity of $r(\theta,r_0)$ for $\theta\in[0,2\pi[$.

To study the strict positivity of $r(\theta,r_0)$, we distinguish two cases $\alpha U<0$ and $\alpha U>0$.
When $\alpha U<0$, It's clear that $r(\theta,r_0)$ is strictly positive.
When $\alpha U>0$, we have $\alpha\beta$ is strictly positive, which implies that $A>1$ and therefore

$$\begin{aligned}
r(\theta,r_0) &= \exp\left(\frac{\beta}{\alpha}\theta\right)\left(A\int_0^{2\pi}\frac{n\exp\left(-\frac{n\beta}{\alpha}s\right)}{\alpha U(s)}ds-\int_0^{\theta}\frac{n\exp\left(-\frac{n\beta}{\alpha}s\right)}{\alpha U(s)}ds\right)^{\frac{1}{n}}\\
&\geq \exp\left(\frac{\beta}{\alpha}\theta\right)\left(\int_0^{2\pi}\frac{n\exp\left(-\frac{n\beta}{\alpha}s\right)}{\alpha U(s)}ds-\int_0^{\theta}\frac{n\exp\left(-\frac{n\beta}{\alpha}s\right)}{\alpha U(s)}ds\right)^{\frac{1}{n}} \text{ because of } A>1\\
&\geq \exp\left(\frac{\beta}{\alpha}\theta\right)\left(\int_{\theta}^{2\pi}\frac{n\exp\left(-\frac{n\beta}{\alpha}s\right)}{\alpha U(s)}ds\right)^{\frac{1}{n}}>0 \text{ because of } \alpha U>0.
\end{aligned}$$

Stability of $r(\theta,r_0)$

We have

$$F(s)=\frac{n\beta}{\alpha}\text{ and }\dot{\theta}=-\alpha U r^{n+1}.$$

It has been shown in the proof of statement 1 of Theorem1 that the limit cycle in the case of its existence is hyperbolic. To study stability, two cases arise.

When $\alpha U < 0$, it's easy to see that $F < 0$ and $\dot{\theta} > 0$ which implies that

$$\left.\frac{d\Pi}{d\gamma}(\gamma)\right|_{\gamma=r_0} = \exp\left(\int_0^{2\pi} F(s)\,ds\right) < 1.$$

Therefore the solution of the differential equation (3.1) is a stable and hyperbolic limit cycle; consequently, it is a stable and hyperbolic limit cycle for the system (1.4).

When $\alpha U > 0$, we have $F > 0$ and $\dot{\theta} < 0$ which implies that

$$\left.\frac{d\Pi}{d\gamma}(\gamma)\right|_{\gamma=r_0} = \exp\left(\int_0^{2\pi} F(s)\,ds\right) > 1.$$

Therefore the solution of the differential equation (3.1) is an unstable and hyperbolic limit cycle. Consequently, it is a stable and hyperbolic limit cycle for the system (1.4).

Clearly in the (x, y) plane, the curve $(r\cos\theta, r\sin\theta)$ with

$$r^n(\theta, r_0) = \exp\left(\tfrac{n\beta}{\alpha}\theta\right)\left(r_0^n + \int_0^\theta \frac{-n\exp\left(-\frac{n\beta}{\alpha}s\right)}{\alpha U(s)}ds\right)$$

is not algebraic, due to the expression $\exp\left(\frac{n\beta}{\alpha}\theta\right) r_0^n$. So the limit cycle it also non-algebraic. Since the Poincaré return map possesses only one fixed point r_0, the system (1.4) admits exactly one limit cycle. This completes the proof of statement (2) of corollary.

(3) It is easy to check that if $\beta U < 0$, then A and B have different signs, and if $\alpha\beta U = 0$, then A or B is not defined. This implies that if $\beta U \leq 0$ or $\alpha = 0$, then r_0 is not defined, or it is negative. So the systems (1.4) do not have periodic orbits. Hence, statement (3) of the corollary is proved. ■

4. Application of Corollary 1

In this section, we apply the Corollary 1 to system (1.4) for $U = \left(ax^2 - bxy + ay^2\right)^n$. The system becomes

$$\begin{aligned} \dot{x} &= x + (\alpha y - \beta x)\left(ax^2 - bxy + ay^2\right)^n, \\ \dot{y} &= y - (\beta y + \alpha x)\left(ax^2 - bxy + ay^2\right)^n. \end{aligned} \tag{4.1}$$

In [9], Bokoucha determined sufficient conditions for the existence of a limit cycle for systems (4.1). In the following proposition, we complete what has been done, where we establish sufficient and necessary conditions for its existence.

Proposition 1. *For system (4.1), these following assertions are true.*

(1) *The system (4.1) has the Liouvillian first integral*

$$H(x, y) = \left(x^2 + y^2\right)^n \exp\left(\tfrac{-2n\beta}{\alpha}\arctan\tfrac{y}{x}\right) + \tfrac{2n}{\alpha}\int_0^{\arctan\frac{y}{x}} \frac{\exp\left(-\frac{2n\beta}{\alpha}s\right)}{\left(a - \frac{b}{2}\sin 2s\right)^n}ds.$$

(2) *The system (4.1) has exactly one non-algebraic, stable and hyperbolic limit cycle if and only if one of the following statements holds*

i) n *is even,* $\alpha \neq 0$, $\beta > 0$ *and* $|b| < 2|a|$.

ii) n *is odd,* $\alpha \neq 0$, $\beta > 0$ *and* $|b| < 2a$.

iii) n is odd, $\alpha \neq 0$, $\beta < 0$ and $2a < -|b|$.

Moreover, this limit cycle is given in polar coordinates by

$$r(\theta, r_0) = \exp\left(\frac{\beta}{\alpha}\theta\right)\left(r_0^n + \int_0^\theta \frac{-2n\exp\left(-\frac{2n\beta}{\alpha}s\right)}{\alpha\left(a-\frac{b}{2}\sin 2s\right)^n}ds\right)^{\frac{1}{2n}},$$

where

$$r_0 = \sqrt[2n]{\frac{\exp\left(4\pi n\frac{\beta}{\alpha}\right)}{\exp\left(4\pi n\frac{\beta}{\alpha}\right)-1}\int_0^{2\pi} \frac{2n\exp\left(-\frac{2n\beta}{\alpha}s\right)}{\alpha\left(a-\frac{b}{2}\sin 2s\right)^n}ds}.$$

(3) *The system (4.1) has no periodic orbits if and only if one of the following statements holds*

i) $\alpha\beta = 0$ or $|b| \geq 2|a|$.

ii) n is even, $\alpha \neq 0$, $\beta < 0$ and $|b| < 2|a|$.

iii) n is odd, $\alpha \neq 0$, $\beta < 0$ and $|b| < 2a$.

iv) n is odd, $\alpha \neq 0$, $\beta > 0$ and $2a < -|b|$.

Remark 1. *Bokoucha in [9], studied only the case where $\alpha > 0$, $\beta > 0$ and $|b| < 2a$.*

The following lemma gives necessary and sufficient conditions on the sign of $U = (ax^2 - bxy + ay^2)^n$.

Lemma 2. *Consider $U = (ax^2 - bxy + ay^2)^n$.*

1) *$U > 0$ if and only if one of the following statements holds*

i) n is even and $|b| < 2|a|$.

ii) n is odd and $|b| < 2a$.

2) *$U < 0$ if and only if n is odd and $2a < -|b|$.*

3) *$U = 0$ if and only if $|b| \geq 2|a|$.*

Proof of proposition 1. The proof of proposition is an immediate consequence of Corollary 1 and Lemma 2. ■

5. Examples

In this section, we present some examples to illustrate the applicability of the our main results. In addition, plots of phase portraits on the Poincaré disc for each example are performed.

Example 1. *In the system (4.1), we take $\alpha = \beta = a = -b = 1$ and $n = 1$, we obtain*

$$\begin{aligned}\dot{x} &= x + (y - x)\left(x^2 + xy + y^2\right),\\ \dot{y} &= y - (y + x)\left(x^2 + xy + y^2\right).\end{aligned} \tag{5.1}$$

which has a non-algebraic, stable and hyperbolic limit cycle whose expression in polar coordinates is

$$r(\theta, r_0) = \exp(\theta)\sqrt{r_0 + \int_0^\theta \frac{-2\exp(-2s)}{\left(1-\frac{1}{2}\sin 2s\right)}ds},$$

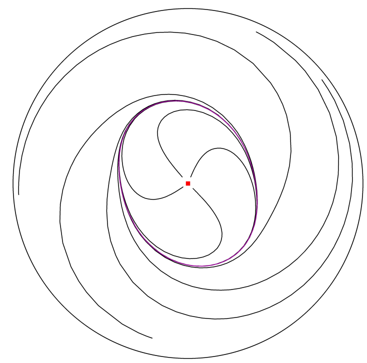

Figure 5.1: The phase portrait on the Poincaré disc of the system (5.1), showing a limit cycle.

where

$$r_0 = \sqrt{\frac{\exp(4\pi)}{\exp(4\pi)-1} \int_0^{2\pi} \frac{2\exp(-2s)}{\left(1-\frac{1}{2}\sin 2s\right)} ds} \simeq 1.1912.$$

Example 2. *In the system (4.1), we take $\alpha = \beta = 1$, $a = b = -2$ and $n = 2$, we obtain*

$$\begin{aligned} \dot{x} &= x + (y-x)\left(-2x^2 + 2xy - 2y^2\right)^2, \\ \dot{y} &= y - (y+x)\left(-2x^2 + 2xy - 2y^2\right)^2. \end{aligned} \quad (5.2)$$

which has a non-algebraic, stable and hyperbolic limit cycle whose expression in polar coordinates is

$$r(\theta, r_0) = \exp(\theta)\left(r_0^2 + \int_0^{\theta} \frac{-4\exp(-4s)}{(-2+\sin 2s)^2} ds\right)^{\frac{1}{4}},$$

where

$$r_0 = \sqrt[4]{\frac{\exp(8\pi)}{\exp(8\pi)-1} \int_0^{2\pi} \frac{4\exp(-4s)}{(-2+\sin 2s)^2} ds} \simeq 0.8163.$$

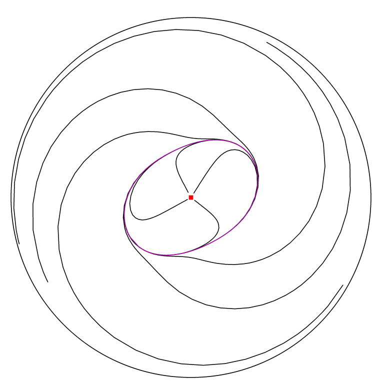

Figure 5.2: The phase portrait on the Poincaré disc of the system (5.2), showing a limit cycle

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