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Investigation Of Generalized Fibonacci Hybrid Numbers And Their Properties*

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Abstract

In [16], M. Özdemir defined a new non-commutative number system called hybrid numbers. In this paper, we consider the generalized Fibonacci hybrid numbers and investigate some basic properties of these hybrid numbers by using the Binet's formula. We also get some generalized identities for (p,q)-Fibonacci hybrid numbers and (p,q)-Lucas hybrid numbers.

1 Introduction

The most famous generalization of the set of complex numbers is the set of quaternions. In 1843, William Hamilton described the set of quaternions

$$\mathbb{H} = \{a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} :\ \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1\}$$

and James Cockle defined coquaternions (split quaternions)

$$\overline{\mathbb{H}} = \{a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} :\ \mathbf{i}^2 = -1,\ \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = 1\}$$

in 1849 (see [5]). Quaternions and coquaternions are used to define 3D Euclidean and Lorentzian rotations, respectively. A set of split quaternions is non-commutative and contains zero divisors, nilpotent elements, and nontrivial idempotents (see [15, 17]). Previous studies have examined the geometric and physical applications of split quaternions, which are required in solving split quaternionic equations [7].

In particular, Fibonacci and Lucas quaternions cover a wide range of interests in modern mathematics as they appear in the comprehensive works of [11, 12]. Furthermore, quaternions with third-order sequences are studied in [2, 3, 4]. For example, the Fibonacci quaternion denoted by $Q_{F,n}$, is the n-th term of the sequence where each term is the sum of the two previous terms beginning with the initial values $Q_{F,0} = \mathbf{i}+\mathbf{j}+2\mathbf{k}$ and $Q_{F,1} = 1+\mathbf{i}+2\mathbf{j}+3\mathbf{k}$. The well-known Fibonacci quaternion $Q_{F,n}$ is defined as

$$Q_{F,n} = F_n + \mathbf{i}F_{n+1} + \mathbf{j}F_{n+2} + \mathbf{k}F_{n+3} \tag{1}$$

and the Lucas quaternion is defined as $Q_{L,n} = L_n + \mathbf{i}L_{n+1} + \mathbf{j}L_{n+2} + \mathbf{k}L_{n+3}$ for $n \geq 0$, where F_n and L_n are n-th Fibonacci and Lucas number, respectively.

Ipek [13] studied the (p,q)-Fibonacci quaternions $Q_{\mathcal{F},n}$ which is defined as

$$Q_{\mathcal{F},n} = pQ_{\mathcal{F},n-1} + qQ_{\mathcal{F},n-2},\ n \geq 2 \tag{2}$$

with initial conditions $Q_{\mathcal{F},0} = \mathbf{i} + p\mathbf{j} + (p^2+q)\mathbf{k}$, $Q_{\mathcal{F},1} = 1 + p\mathbf{i} + (p^2+q)\mathbf{j} + (p^3+2pq)\mathbf{k}$ and $p^2+4q > 0$. If $p = q = 1$, we get the classical Fibonacci quaternion $Q_{F,n}$ [8]. If $p = 2q = 2$, we get the Pell quaternion $Q_{P,n} = P_n + \mathbf{i}P_{n+1} + \mathbf{j}P_{n+2} + \mathbf{k}P_{n+3}$ (see [6]), where P_n is the n-th Pell number.

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The well-known Binet's formulas for (p,q)-Fibonacci quaternion and (p,q)-Lucas quaternion, see [13], are given by

$$Q_{\mathcal{F},n}=\frac{\underline{\alpha}\alpha^n-\underline{\beta}\beta^n}{\alpha-\beta} \text{ and } Q_{\mathcal{L},n}=\underline{\alpha}\alpha^n+\underline{\beta}\beta^n, \tag{3}$$

where α,β are roots of the characteristic equation $t^2-pt-q=0$, and $\underline{\alpha}=1+\alpha\mathbf{i}+\alpha^2\mathbf{j}+\alpha^3\mathbf{k}$ and $\underline{\beta}=1+\beta\mathbf{i}+\beta^2\mathbf{j}+\beta^3\mathbf{k}$. We note that $\alpha+\beta=p$, $\alpha\beta=-q$ and $\alpha-\beta=\sqrt{p^2+4q}$.

The generalized Fibonacci quaternion $Q_{w,n}$ is defined recently by Halici and Karataş in [10] as $Q_{w,0}=a+b\mathbf{i}+(pb+qa)\mathbf{j}+((p^2+q)b+pqa)\mathbf{k}$, $Q_{w,1}=b+(pb+qa)\mathbf{i}+((p^2+q)b+pqa)\mathbf{j}+((p^3+2pq)b+q(p^2+q)a)\mathbf{k}$ and $Q_{w,n}=pQ_{w,n-1}+qQ_{w,n-2}$, for $n\geq 2$ which we call the generalized Fibonacci or Horadam quaternions. So, each term of the generalized Fibonacci sequence $\{Q_{w,n}\}_{n\geq 0}$ is called generalized Fibonacci quaternion.

The Binet formula for generalized Fibonacci quaternion $Q_{w,n}$, see [10], is given by

$$Q_{w,n}=\frac{A\underline{\alpha}\alpha^n-B\underline{\beta}\beta^n}{\alpha-\beta}, \tag{4}$$

where $A=b-a\beta$, $B=b-a\alpha$, and α,β are roots of the characteristic equation $t^2-pt-q=0$, $\underline{\alpha}=1+\alpha\mathbf{i}+\alpha^2\mathbf{j}+\alpha^3\mathbf{k}$ and $\underline{\beta}=1+\beta\mathbf{i}+\beta^2\mathbf{j}+\beta^3\mathbf{k}$. If $a=0$ and $b=1$, we get the classical (p,q)-Fibonacci quaternion $Q_{\mathcal{F},n}$. If $a=2$ and $b=p$, we get the (p,q)-Lucas quaternion $Q_{\mathcal{L},n}$. For more details and identities of this type of numbers, see [19].

Recently, Özdemir [16] defined a new generalization of complex, hyperbolic and dual numbers. In this generalization, the author gave a system of such numbers that consists of all three number systems together. This set was called hybrid numbers, denoted by $\mathbb{K}$, is defined as

$$\mathbb{K}=\left\{\mathbf{z}=a+b\mathbf{i}+c\boldsymbol{\varepsilon}+d\mathbf{h}:a,b,c,d\in\mathbb{R},\quad \begin{array}{c}\mathbf{i}^2=-1,\ \boldsymbol{\varepsilon}^2=0,\ \mathbf{h}^2=1,\\ \mathbf{ih}=-\mathbf{hi}=\boldsymbol{\varepsilon}+\mathbf{i}\end{array}\right\}. \tag{5}$$

Two hybrid numbers are equal if all their components are equal, one by one. The sum of two hybrid numbers is defined by summing their components. Addition operation in the hybrid numbers is both commutative and associative. Zero is the null element. With respect to the addition operation, the inverse element of $\mathbf{z}$ is $-\mathbf{z}$, which is defined as having all the components of $\mathbf{z}$ changed in their signs. This implies that, $(\mathbb{K},+)$ is an Abelian group.

The hybridian product is obtained by distributing the terms on the right as in ordinary algebra, preserving that the multiplication order of the units and then writing the values of followings replacing each product of units by the equalities $\mathbf{i}^2=-1$, $\boldsymbol{\varepsilon}^2=0$, $\mathbf{h}^2=1$ and $\mathbf{ih}=-\mathbf{hi}=\boldsymbol{\varepsilon}+\mathbf{i}$. Using these equalities we can find the product of any two hybrid units. For example, let's find $\mathbf{i}\boldsymbol{\varepsilon}$. For this, let's multiply $\mathbf{ih}=\boldsymbol{\varepsilon}+\mathbf{i}$ by $\mathbf{i}$ from the left. Thus, we get $\mathbf{i}\boldsymbol{\varepsilon}=1-\mathbf{h}$. If we continue in a similar way, we get the following multiplication table.

Table 1: The multiplication table for the basis of $\mathbb{K}$.

$\times$	1	$\mathbf{i}$	$\boldsymbol{\varepsilon}$	$\mathbf{h}$
1	1	$\mathbf{i}$	$\boldsymbol{\varepsilon}$	$\mathbf{h}$
$\mathbf{i}$	$\mathbf{i}$	-1	$1-\mathbf{h}$	$\boldsymbol{\varepsilon}+\mathbf{i}$
$\boldsymbol{\varepsilon}$	$\boldsymbol{\varepsilon}$	$1+\mathbf{h}$	0	$-\boldsymbol{\varepsilon}$
$\mathbf{h}$	$\mathbf{h}$	$-(\boldsymbol{\varepsilon}+\mathbf{i})$	$\boldsymbol{\varepsilon}$	1

The table 1 shows us that the multiplication operation in the hybrid numbers is not commutative. But it has the property of associativity. The conjugate of a hybrid number $\mathbf{z}=a+b\mathbf{i}+c\boldsymbol{\varepsilon}+d\mathbf{h}$, denoted by $\overline{\mathbf{z}}$, is defined as $\mathbf{z}=a-b\mathbf{i}-c\boldsymbol{\varepsilon}-d\mathbf{h}$ as in the quaternions. The conjugate of the sum of hybrid numbers is equal to the sum of their conjugates. Also, according to the hybridian product, we have $\mathbf{z}\overline{\mathbf{z}}=\overline{\mathbf{z}}\mathbf{z}$. The real number

$$\mathcal{C}(\mathbf{z})=\mathbf{z}\overline{\mathbf{z}}=\overline{\mathbf{z}}\mathbf{z}=a^2+(b-c)^2-c^2-d^2$$

is called the character of the hybrid number $\mathbf{z}=a+b\mathbf{i}+c\boldsymbol{\varepsilon}+d\mathbf{h}$. The real number $\sqrt{|\mathcal{C}(\mathbf{z})|}$ will be called the norm of the hybrid number $\mathbf{z}$ and will be denoted by $\|\mathbf{z}\|_{\mathbb{K}}$.

In this study, we consider the generalized Fibonacci hybrid numbers. We give the generating functions and Binet formulas for these numbers. Moreover, the well-known properties e.g. Cassini and Catalan identities have been obtained for these numbers.

2 Generalized Fibonacci Hybrid Numbers

We define the n-th (p,q)-Fibonacci and (p,q)-Lucas hybrid numbers, respectively, by the following recurrence relations

$$\mathbb{H}\mathcal{F}_n = \mathcal{F}_n + \mathcal{F}_{n+1}\mathbf{i} + \mathcal{F}_{n+2}\boldsymbol{\varepsilon} + \mathcal{F}_{n+3}\mathbf{h} \tag{6}$$

and

$$\mathbb{H}\mathcal{L}_n = \mathcal{L}_n + \mathcal{L}_{n+1}\mathbf{i} + \mathcal{L}_{n+2}\boldsymbol{\varepsilon} + \mathcal{L}_{n+3}\mathbf{h}, \tag{7}$$

where $\mathcal{F}_n$ and $\mathcal{L}_n$ are the n-th (p,q)-Fibonacci and (p,q)-Lucas numbers defined by

$$\mathcal{F}_n = p\mathcal{F}_{n-1} + q\mathcal{F}_{n-2}, \quad \mathcal{F}_0 = 0, \quad \mathcal{F}_1 = 1$$

and

$$\mathcal{L}_n = p\mathcal{L}_{n-1} + q\mathcal{L}_{n-2}, \quad \mathcal{L}_0 = 2, \quad \mathcal{L}_1 = p,$$

respectively. Here $\{\mathbf{i}, \boldsymbol{\varepsilon}, \mathbf{h}\}$ satisfies the multiplication rule given in the Table 1.

By some elementary calculations we find the following recurrence relations for the (p,q)-Fibonacci and (p,q)-Lucas hybrid numbers from (6) and (7):

$$\begin{aligned} p\mathbb{H}\mathcal{F}_n + q\mathbb{H}\mathcal{F}_{n-1} &= p(\mathcal{F}_n + \mathcal{F}_{n+1}\mathbf{i} + \mathcal{F}_{n+2}\boldsymbol{\varepsilon} + \mathcal{F}_{n+3}\mathbf{h}) + q(\mathcal{F}_{n-1} + \mathcal{F}_n\mathbf{i} + \mathcal{F}_{n+1}\boldsymbol{\varepsilon} + \mathcal{F}_{n+2}\mathbf{h}) \\ &= (p\mathcal{F}_n + q\mathcal{F}_{n-1}) + (p\mathcal{F}_{n+1} + q\mathcal{F}_n)\mathbf{i} + (p\mathcal{F}_{n+2} + q\mathcal{F}_{n+1})\boldsymbol{\varepsilon} + (p\mathcal{F}_{n+3} + q\mathcal{F}_{n+2})\mathbf{h} \\ &= \mathcal{F}_{n+1} + \mathcal{F}_{n+2}\mathbf{i} + \mathcal{F}_{n+3}\boldsymbol{\varepsilon} + \mathcal{F}_{n+4}\mathbf{h} \\ &= \mathbb{H}\mathcal{F}_{n+1} \end{aligned}$$

and similarly $\mathbb{H}\mathcal{L}_{n+1} = p\mathbb{H}\mathcal{L}_n + q\mathbb{H}\mathcal{L}_{n-1}$, for $n \geq 1$ (see [18]).

In this paper, following Halici and Karataş [10], we define the generalized Fibonacci hybrid numbers as

$$\mathbb{H}\mathcal{J}_n = p\mathbb{H}\mathcal{J}_{n-1} + q\mathbb{H}\mathcal{J}_{n-2}, \ n \geq 2, \tag{8}$$

where $\mathbb{H}\mathcal{J}_0 = a + b\mathbf{i} + (pb+qa)\boldsymbol{\varepsilon} + ((p^2+q)b + pqa)\mathbf{h}$ and $\mathbb{H}\mathcal{J}_1 = b + (pb+qa)\mathbf{i} + ((p^2+q)b+pqa)\boldsymbol{\varepsilon} + ((p^3 + 2pq)b + q(p^2+q)a)\mathbf{h}$.

So, each term of the generalized Fibonacci hybrid sequence $\{\mathbb{H}\mathcal{J}_n\}_{n\geq 0}$ is called generalized Fibonacci hybrid number. Furthermore, if $a = 0$ and $b = 1$, we get the (p,q)-Fibonacci hybrid number $\mathbb{H}\mathcal{F}_n$. If $a = 2$ and $b = p$, we get the (p,q)-Lucas hybrid number $\mathbb{H}\mathcal{L}_n$.

Generating functions for the generalized Fibonacci hybrid numbers are given in the next theorem.

Theorem 1 ([18]) *The generating function for the generalized Fibonacci hybrid number is*

$$\sum_{r=0}^{\infty} \mathbb{H}\mathcal{J}_r t^r = \frac{\left\{ \begin{array}{c} a + b\mathbf{i} + (pb+qa)\boldsymbol{\varepsilon} + ((p^2+q)b + pqa)\mathbf{h} \\ +t((b-pa) + qa\mathbf{i} + qb\boldsymbol{\varepsilon} + (pqb + q^2a)\mathbf{h}) \end{array} \right\}}{1 - pt - qt^2}. \tag{9}$$

The next theorem gives the Binet formulas for the generalized Fibonacci hybrid numbers in a different way than Theorem 1 in [18].

Theorem 2 *For any integer $n \geq 0$, the n-th generalized Fibonacci hybrid number is*

$$\mathbb{H}\mathcal{J}_n = \frac{A\underline{\boldsymbol{\alpha}}\alpha^n - B\underline{\boldsymbol{\beta}}\beta^n}{\alpha - \beta}, \tag{10}$$

where $A = b - a\beta$, $B = b - a\alpha$, and α, β are roots of the characteristic equation $t^2 - pt - q = 0$, $\underline{\boldsymbol{\alpha}} = 1 + \alpha\mathbf{i} + \alpha^2\boldsymbol{\varepsilon} + \alpha^3\mathbf{h}$ and $\underline{\boldsymbol{\beta}} = 1 + \beta\mathbf{i} + \beta^2\boldsymbol{\varepsilon} + \beta^3\mathbf{h}$. If $a = 0$ and $b = 1$, we get the (p,q)-Fibonacci hybrid number $\mathbb{H}\mathcal{F}_n$. If $a = 2$ and $b = p$, we get the (p,q)-Lucas hybrid number $\mathbb{H}\mathcal{L}_n$.

Proof. For the Eq. (10), we have

$$\begin{aligned}\alpha\mathbb{H}\mathcal{J}_{n+1}+q\mathbb{H}\mathcal{J}_{n} &= \alpha(\mathcal{J}_{n+1}+\mathcal{J}_{n+2}\mathbf{i}+\mathcal{J}_{n+3}\boldsymbol{\varepsilon}+\mathcal{J}_{n+4}\mathbf{h})+q(\mathcal{J}_{n}+\mathcal{J}_{n+1}\mathbf{i}+\mathcal{J}_{n+2}\boldsymbol{\varepsilon}+\mathcal{J}_{n+3}\mathbf{h})\\ &= (\alpha\mathcal{J}_{n+1}+q\mathcal{J}_{n})+(\alpha\mathcal{J}_{n+2}+q\mathcal{J}_{n+1})\mathbf{i}+(\alpha\mathcal{J}_{n+3}+q\mathcal{J}_{n+2})\boldsymbol{\varepsilon}+(\alpha\mathcal{J}_{n+4}+q\mathcal{J}_{n+3})\mathbf{h}.\end{aligned}$$

From the identity $\alpha\mathcal{J}_{n+1}+q\mathcal{J}_{n}=\alpha^{n}(\alpha b+qa)$, we obtain

$$\alpha\mathbb{H}\mathcal{J}_{n+1}+q\mathbb{H}\mathcal{J}_{n}=\underline{\boldsymbol{\alpha}}\alpha^{n}(\alpha b+qa). \tag{11}$$

Similarly, we have

$$\beta\mathbb{H}\mathcal{J}_{n+1}+q\mathbb{H}\mathcal{J}_{n}=\underline{\boldsymbol{\beta}}\beta^{n}(\beta b+qa). \tag{12}$$

Subtracting Eq. (12) from Eq. (11) gives

$$(\alpha-\beta)\mathbb{H}\mathcal{J}_{n+1}=A\underline{\boldsymbol{\alpha}}\alpha^{n+1}-B\underline{\boldsymbol{\beta}}\beta^{n+1},$$

where $A=b-a\beta$, $B=b-a\alpha$ and α,β are roots of the characteristic equation $t^2-pt-q=0$. Furthermore, $\underline{\boldsymbol{\alpha}}=1+\alpha\mathbf{i}+\alpha^2\boldsymbol{\varepsilon}+\alpha^3\mathbf{h}$ and $\underline{\boldsymbol{\beta}}=1+\beta\mathbf{i}+\beta^2\boldsymbol{\varepsilon}+\beta^3\mathbf{h}$. So, the theorem is proved. ■

There are three well-known identities for generalized Fibonacci numbers, namely, Catalan's, Cassini's, and d'Ocagne's identities (see [1]). The proofs of these identities are based on Binet formulas. We can obtain these types of identities for generalized Fibonacci hybrid numbers using the Binet formula for $\mathbb{H}\mathcal{J}_n$. Then, we require $\underline{\boldsymbol{\alpha}}\underline{\boldsymbol{\beta}}$ and $\underline{\boldsymbol{\beta}}\underline{\boldsymbol{\alpha}}$. These products are given in the next lemma.

Lemma 3 *We have*

$$\underline{\boldsymbol{\alpha}}\underline{\boldsymbol{\beta}}=\mathbb{H}\mathcal{L}_0-(q^3+pq-q+1)+q(\alpha-\beta)(\mathbb{H}\mathcal{F}_0-\boldsymbol{\omega}), \tag{13}$$

and

$$\underline{\boldsymbol{\beta}}\underline{\boldsymbol{\alpha}}=\mathbb{H}\mathcal{L}_0-(q^3+pq-q+1)-q(\alpha-\beta)(\mathbb{H}\mathcal{F}_0-\boldsymbol{\omega}), \tag{14}$$

where $\boldsymbol{\omega}=(1-p)\mathbf{i}-q\boldsymbol{\varepsilon}+(p^2+q+1)\mathbf{h}$ *and* $\alpha-\beta=\sqrt{p^2+4q}$.

Proof. From the definitions of $\underline{\boldsymbol{\alpha}}$ and $\underline{\boldsymbol{\beta}}$, and using $\mathbf{i}^2=-1$, $\boldsymbol{\varepsilon}^2=0$, $\mathbf{h}^2=1$ and $\mathbf{ih}=-\mathbf{hi}=\boldsymbol{\varepsilon}+\mathbf{i}$ in Table 1, we have

$$\begin{aligned}\underline{\boldsymbol{\alpha}}\underline{\boldsymbol{\beta}} &= (1+\alpha\mathbf{i}+\alpha^2\boldsymbol{\varepsilon}+\alpha^3\mathbf{h})(1+\beta\mathbf{i}+\beta^2\boldsymbol{\varepsilon}+\beta^3\mathbf{h})\\ &= 2+(\alpha+\beta)\mathbf{i}+(\alpha^2+\beta^2)\boldsymbol{\varepsilon}+(\alpha^3+\beta^3)\mathbf{h}-1+\alpha\beta(-1+\alpha+\beta+\alpha^2\beta^2)\\ &\quad -\alpha\beta(\alpha^2-\beta^2)\mathbf{i}-\alpha\beta(\alpha^2-\beta^2-\alpha^2\beta+\alpha\beta^2)\boldsymbol{\varepsilon}+\alpha\beta(\alpha-\beta)\mathbf{h}\\ &= 2+p\mathbf{i}+(p^2+2q)\boldsymbol{\varepsilon}+(p^3+3pq)\mathbf{h}-(q^3+pq-q+1)\\ &\quad +q(\alpha-\beta)(p\mathbf{i}+(p+q)\boldsymbol{\varepsilon}-\mathbf{h})\\ &= \mathbb{H}\mathcal{L}_0-(q^3+pq-q+1)+q(\alpha-\beta)(p\mathbf{i}+(p+q)\boldsymbol{\varepsilon}-\mathbf{h})\\ &= \mathbb{H}\mathcal{L}_0-(q^3+pq-q+1)+q(\alpha-\beta)(\mathbb{H}\mathcal{F}_0-\boldsymbol{\omega}),\end{aligned}$$

where $\boldsymbol{\omega}=(1-p)\mathbf{i}-q\boldsymbol{\varepsilon}+(p^2+q+1)\mathbf{h}$ and the final equation gives Eq. (13). The other identity can be computed similarly. ■

This lemma gives us the following useful identity:

$$\underline{\boldsymbol{\alpha}}\underline{\boldsymbol{\beta}}+\underline{\boldsymbol{\beta}}\underline{\boldsymbol{\alpha}}=2(\mathbb{H}\mathcal{L}_0-(q^3+pq-q+1)). \tag{15}$$

Catalan's identities for generalized Fibonacci hybrid numbers are given in the next theorem.

Theorem 4 *For any integers* $m\geq r\geq 0$, *we have*

$$\mathbb{H}\mathcal{J}_m^2-\mathbb{H}\mathcal{J}_{m+r}\mathbb{H}\mathcal{J}_{m-r}=-AB(-q)^m\mathcal{F}_{-r}\left\{\begin{array}{c}(\mathbb{H}\mathcal{L}_0-(q^3+pq-q+1))\mathcal{F}_r\\ +q(\mathbb{H}\mathcal{F}_0-\boldsymbol{\omega})\mathcal{L}_r\end{array}\right\}, \tag{16}$$

where $A=b-a\beta$, $B=b-a\alpha$, $\boldsymbol{\omega}=(1-p)\mathbf{i}-q\boldsymbol{\varepsilon}+(p^2+q+1)\mathbf{h}$ *and* $\mathcal{F}_r$, $\mathcal{L}_r$ *are the* r*-th* (p,q)*-Fibonacci and* (p,q)*-Lucas numbers, respectively.*

Proof. From the Binet formula for generalized Fibonacci hybrid numbers $\mathbb{H}\mathcal{J}_m$ in (10) and $(\alpha-\beta)^2 = p^2+4q$, we have

$$\begin{aligned}
& (p^2+4q)\left(\mathbb{H}\mathcal{J}_m^2 - \mathbb{H}\mathcal{J}_{m+r}\mathbb{H}\mathcal{J}_{m-r}\right) \\
= \ & \left(A\underline{\boldsymbol{\alpha}}\alpha^m - B\underline{\boldsymbol{\beta}}\beta^m\right)^2 - \left(A\underline{\boldsymbol{\alpha}}\alpha^{m+r} - B\underline{\boldsymbol{\beta}}\beta^{m+r}\right)\left(A\underline{\boldsymbol{\alpha}}\alpha^{m-r} - B\underline{\boldsymbol{\beta}}\beta^{m-r}\right) \\
= \ & AB(-q)^{m-r}\left(\underline{\boldsymbol{\alpha}}\underline{\boldsymbol{\beta}}\alpha^{2r} + \underline{\boldsymbol{\beta}}\underline{\boldsymbol{\alpha}}\beta^{2r} - (-q)^r\left(\underline{\boldsymbol{\alpha}}\underline{\boldsymbol{\beta}} + \underline{\boldsymbol{\beta}}\underline{\boldsymbol{\alpha}}\right)\right).
\end{aligned}$$

We require Eqs. (13) and (14). Using this equations, we obtain

$$\begin{aligned}
& \mathbb{H}\mathcal{J}_m^2 - \mathbb{H}\mathcal{J}_{m+r}\mathbb{H}\mathcal{J}_{m-r} \\
= \ & \frac{AB(-q)^{m-r}}{p^2+4q}\left\{\begin{array}{c} (\mathbb{H}\mathcal{L}_0 - (q^3+pq-q+1))(\alpha^{2r}+\beta^{2r}-2(-q)^r) \\ +q(\alpha-\beta)(\mathbb{H}\mathcal{F}_0 - \boldsymbol{\omega})(\alpha^{2r}-\beta^{2r}) \end{array}\right\} \\
= \ & \frac{AB(-q)^{m-r}}{p^2+4q}\left\{\begin{array}{c} (\mathbb{H}\mathcal{L}_0 - (q^3+pq-q+1))(\mathcal{L}_{2r}-2(-q)^r) \\ +q(p^2+4q)(\mathbb{H}\mathcal{F}_0 - \boldsymbol{\omega})\mathcal{F}_{2r} \end{array}\right\}.
\end{aligned}$$

Using the identity $(p^2+4q)\mathcal{F}_r^2 = \mathcal{L}_{2r} - 2(-q)^r$ gives

$$\mathbb{H}\mathcal{J}_m^2 - \mathbb{H}\mathcal{J}_{m+r}\mathbb{H}\mathcal{J}_{m-r} = AB(-q)^{m-r}\left\{\begin{array}{c} (\mathbb{H}\mathcal{L}_0 - (q^3+pq-q+1))\mathcal{F}_r^2 \\ +q(\mathbb{H}\mathcal{F}_0 - \boldsymbol{\omega})\mathcal{F}_{2r} \end{array}\right\},$$

where $\mathcal{L}_r$, $\mathcal{F}_r$ are the r-th (p,q)-Lucas and (p,q)-Fibonacci numbers, respectively. With the help of the identities $\mathcal{F}_{2r} = \mathcal{F}_r\mathcal{L}_r$ and $\mathcal{F}_{-r} = -(-q)^{-r}\mathcal{F}_r$, we have Eq. (16). The proof is completed. ■

Taking $r = 1$ in the Theorem 4 and using the identity $\mathcal{F}_{-1} = \frac{1}{q}$, we obtain Cassini's identities for generalized Fibonacci hybrid numbers.

Corollary 5 *For any integer m, we have*

$$\mathbb{H}\mathcal{J}_m^2 - \mathbb{H}\mathcal{J}_{m+1}\mathbb{H}\mathcal{J}_{m-1} = AB(-q)^{m-1}\left\{\begin{array}{c} (\mathbb{H}\mathcal{L}_0 - (q^3+pq-q+1)) \\ +pq(\mathbb{H}\mathcal{F}_0 - \boldsymbol{\omega}) \end{array}\right\}, \tag{17}$$

where $A = b - a\beta$, $B = b - a\alpha$ and $\boldsymbol{\omega} = (1-p)\mathbf{i} - q\boldsymbol{\varepsilon} + (p^2+q+1)\mathbf{h}$.

The following theorem gives d'Ocagne's identities for generalized Fibonacci hybrid numbers.

Theorem 6 *For any integers r and m, we have*

$$\mathbb{H}\mathcal{J}_r\mathbb{H}\mathcal{J}_{m+1} - \mathbb{H}\mathcal{J}_{r+1}\mathbb{H}\mathcal{J}_m = (-q)^m AB\left\{\begin{array}{c} (\mathbb{H}\mathcal{L}_0 - (q^3+pq-q+1))\mathcal{F}_{r-m} \\ +q(\mathbb{H}\mathcal{F}_0 - \boldsymbol{\omega})\mathcal{L}_{r-m} \end{array}\right\}, \tag{18}$$

where $\mathcal{F}_r$, $\mathcal{L}_r$ are the r-th (p,q)-Fibonacci and (p,q)-Lucas numbers, respectively.

Proof. Using the Binet formula for the generalized Fibonacci hybrid numbers gives

$$\begin{aligned}
& (p^2+4q)(\mathbb{H}\mathcal{J}_r\mathbb{H}\mathcal{J}_{m+1} - \mathbb{H}\mathcal{J}_{r+1}\mathbb{H}\mathcal{J}_m) \\
= \ & \left(A\underline{\boldsymbol{\alpha}}\alpha^r - B\underline{\boldsymbol{\beta}}\beta^r\right)\left(A\underline{\boldsymbol{\alpha}}\alpha^{m+1} - B\underline{\boldsymbol{\beta}}\beta^{m+1}\right) - \left(A\underline{\boldsymbol{\alpha}}\alpha^{r+1} - B\underline{\boldsymbol{\beta}}\beta^{r+1}\right)\left(A\underline{\boldsymbol{\alpha}}\alpha^m - B\underline{\boldsymbol{\beta}}\beta^m\right) \\
= \ & (-q)^m AB(\alpha-\beta)\left(\underline{\boldsymbol{\alpha}}\underline{\boldsymbol{\beta}}\alpha^{r-m} - \underline{\boldsymbol{\beta}}\underline{\boldsymbol{\alpha}}\beta^{r-m}\right).
\end{aligned}$$

We require the Eqs. (13) and (14). Substituting these into the previous equation, we have

$$\begin{aligned}
& \mathbb{H}\mathcal{J}_r\mathbb{H}\mathcal{J}_{m+1} - \mathbb{H}\mathcal{J}_{r+1}\mathbb{H}\mathcal{J}_m \\
= \ & \frac{(-q)^m}{\alpha-\beta}AB\left\{\begin{array}{c} (\mathbb{H}\mathcal{L}_0 - (q^3+pq-q+1))(\alpha^{r-m}-\beta^{r-m}) \\ +q(\alpha-\beta)(\mathbb{H}\mathcal{F}_0 - \boldsymbol{\omega})(\alpha^{r-m}+\beta^{r-m}) \end{array}\right\} \\
= \ & (-q)^m AB\left((\mathbb{H}\mathcal{L}_0 - (q^3+pq-q+1))\mathcal{F}_{r-m} + q(\mathbb{H}\mathcal{F}_0 - \boldsymbol{\omega})\mathcal{L}_{r-m}\right).
\end{aligned}$$

The second identity in the above equality, can be proved using $\mathcal{L}_{r-m} = \alpha^{r-m} + \beta^{r-m}$ and $\mathcal{F}_{r-m} = \frac{\alpha^{r-m}-\beta^{r-m}}{\alpha-\beta}$. This proof is completed. ■

In particular, if $m = r - 1$ in this theorem and using the identity $\mathcal{L}_1 = p$, we obtain Cassini's identities for generalized Fibonacci hybrid numbers. Now, taking $m = r$ in the Theorem 6 and using the identities $\mathcal{F}_0 = 0$ and $\mathcal{L}_0 = 2$, we obtain the next identity.

Corollary 7 *For any integer $r \geq 0$, we have*

$$\mathbb{H}\mathcal{J}_{r+1}\mathbb{H}\mathcal{J}_r - \mathbb{H}\mathcal{J}_r\mathbb{H}\mathcal{J}_{r+1} = 2(-q)^{r+1}AB(\mathbb{H}\mathcal{F}_0 - \boldsymbol{\omega}), \tag{19}$$

where $A = b - a\beta$, $B = b - a\alpha$ and $\boldsymbol{\omega} = (1-p)\mathbf{i} - q\boldsymbol{\varepsilon} + (p^2+q+1)\mathbf{h}$.

After deriving these three famous identities, we present some other identities for the (p,q)-Fibonacci and (p,q)-Lucas hybrid numbers.

Theorem 8 *For any integers n, r and s, we have*

$$\mathbb{H}\mathcal{L}_{n+r}\mathbb{H}\mathcal{F}_{n+s} - \mathbb{H}\mathcal{L}_{n+s}\mathbb{H}\mathcal{F}_{n+r} = 2(-q)^{n+r}\mathcal{F}_{s-r}(\mathbb{H}\mathcal{L}_0 - (q^3+pq-q+1)). \tag{20}$$

Proof. The Binet formulas for the (p,q)-Lucas and (p,q)-Fibonacci hybrid numbers give

$$\begin{aligned}
& (\alpha-\beta)(\mathbb{H}\mathcal{L}_{n+r}\mathbb{H}\mathcal{F}_{n+s} - \mathbb{H}\mathcal{L}_{n+s}\mathbb{H}\mathcal{F}_{n+r}) \\
= \; & \left(\underline{\boldsymbol{\alpha}}\alpha^{n+r} + \underline{\boldsymbol{\beta}}\beta^{n+r}\right)\left(\underline{\boldsymbol{\alpha}}\alpha^{n+s} - \underline{\boldsymbol{\beta}}\beta^{n+s}\right) - \left(\underline{\boldsymbol{\alpha}}\alpha^{n+s} + \underline{\boldsymbol{\beta}}\beta^{n+s}\right)\left(\underline{\boldsymbol{\alpha}}\alpha^{n+r} - \underline{\boldsymbol{\beta}}\beta^{n+r}\right) \\
= \; & (\alpha\beta)^n(\alpha^s\beta^r - \alpha^r\beta^s)(\underline{\boldsymbol{\alpha}}\underline{\boldsymbol{\beta}} + \underline{\boldsymbol{\beta}}\underline{\boldsymbol{\alpha}}).
\end{aligned}$$

Using Eqs. (13) and (14), we have

$$\mathbb{H}\mathcal{L}_{n+r}\mathbb{H}\mathcal{F}_{n+s} - \mathbb{H}\mathcal{L}_{n+s}\mathbb{H}\mathcal{F}_{n+r} = 2(-q)^{n+r}\mathcal{F}_{s-r}(\mathbb{H}\mathcal{L}_0 - (q^3+pq-q+1)).$$

The proof is completed. ■

After deriving these famous identities, we present some other identities for the generalized Fibonacci hybrid numbers. In particular, when using the Binet formulas to obtain identities for the (p,q)-Fibonacci and (p,q)-Lucas hybrid numbers, we require $\underline{\boldsymbol{\alpha}}^2$ and $\underline{\boldsymbol{\beta}}^2$. These products are given in the next lemma.

Lemma 9 *We have*

$$\underline{\boldsymbol{\alpha}}^2 = (\mathbb{H}\mathcal{L}_0 + r_{p,q}) + (\alpha-\beta)(\mathbb{H}\mathcal{F}_0 + s_{p,q}) \tag{21}$$

and

$$\underline{\boldsymbol{\beta}}^2 = (\mathbb{H}\mathcal{L}_0 + r_{p,q}) - (\alpha-\beta)(\mathbb{H}\mathcal{F}_0 + s_{p,q}), \tag{22}$$

where $r_{p,q} = -1 + \frac{p}{2}(\mathcal{F}_6 + 2\mathcal{F}_3 - \mathcal{F}_2) + q(\mathcal{F}_5 + 2\mathcal{F}_2 - \mathcal{F}_1)$, $s_{p,q} = \frac{1}{2}(\mathcal{F}_6 + 2\mathcal{F}_3 - \mathcal{F}_2)$ and $\mathcal{F}_n$ is the n-th (p,q)-Fibonacci number.

Proof. From the definition of $\underline{\boldsymbol{\alpha}}$ and using $\mathbf{i}^2 = -1, \boldsymbol{\varepsilon}^2 = 0, \mathbf{h}^2 = 1$, $\mathbf{ih} = -\mathbf{hi} = \boldsymbol{\varepsilon} + \mathbf{i}$ in Table 1 and $\alpha^n = \mathcal{F}_n\alpha + q\mathcal{F}_{n-1}$ for $n \geq 1$, we have

$$\begin{aligned}
\underline{\boldsymbol{\alpha}}^2 &= (1 + \alpha\mathbf{i} + \alpha^2\boldsymbol{\varepsilon} + \alpha^3\mathbf{h})(1 + \alpha\mathbf{i} + \alpha^2\boldsymbol{\varepsilon} + \alpha^3\mathbf{h}) \\
&= 2(1 + \alpha\mathbf{i} + \alpha^2\boldsymbol{\varepsilon} + \alpha^3\mathbf{h}) + (\alpha^6 + 2\alpha^3 - \alpha^2 - 1) \\
&= 2 + 2\alpha\mathbf{i} + (2p\alpha + 2q)\boldsymbol{\varepsilon} + ((2p^2+2q)\alpha + 2pq)\mathbf{h}) + (\alpha^6 + 2\alpha^3 - \alpha^2 - 1) \\
&= 2 + p\mathbf{i} + (p^2+2q)\boldsymbol{\varepsilon} + (p^3+3pq)\mathbf{h} + (\alpha-\beta)(\mathbf{i} + p\boldsymbol{\varepsilon} + (p^2+q)\mathbf{h}) \\
&\quad + ((\mathcal{F}_6\alpha + q\mathcal{F}_5) + 2(\mathcal{F}_3\alpha + q\mathcal{F}_2) - (\mathcal{F}_2\alpha + q\mathcal{F}_1) - 1) \\
&= (\mathbb{H}\mathcal{L}_0 + r_{p,q}) + (\alpha-\beta)(\mathbb{H}\mathcal{F}_0 + s_{p,q}),
\end{aligned}$$

where $r_{p,q} = -1 + \frac{p}{2}(\mathcal{F}_6 + 2\mathcal{F}_3 - \mathcal{F}_2) + q(\mathcal{F}_5 + 2\mathcal{F}_2 - \mathcal{F}_1)$ and $s_{p,q} = \frac{1}{2}(\mathcal{F}_6 + 2\mathcal{F}_3 - \mathcal{F}_2)$ and the final equation gives Eq. (21). The other can be computed similarly. ■

We present some interesting identities for (p,q)-Fibonacci hybrid numbers, (p,q)-Lucas hybrid numbers and generalized Fibonacci hybrid numbers.

Theorem 10 *For any integer $n \geq 0$, we have*

$$\mathbb{H}\mathcal{L}_n^2 - \mathbb{H}\mathcal{F}_n^2 = \left\{ \begin{array}{c} \frac{p^2+4q-1}{p^2+4q}(\mathbb{H}\mathcal{L}_0 + r_{p,q})\mathcal{L}_{2n} + (p^2+4q-1)(\mathbb{H}\mathcal{F}_0 + s_{p,q})\mathcal{F}_{2n} \\ +\frac{2(p^2+4q+1)(-q)^n}{p^2+4q}(\mathbb{H}\mathcal{L}_0 - (q^3+pq-q+1)). \end{array} \right\}. \tag{23}$$

Proof. Using the Binet formulas for the (p,q)-Fibonacci and (p,q)-Lucas hybrid numbers, we obtain

$$\begin{aligned} (p^2+4q)(\mathbb{H}\mathcal{L}_n^2 - \mathbb{H}\mathcal{F}_n^2) &= (p^2+4q)\left(\underline{\boldsymbol{\alpha}}\alpha^n + \underline{\boldsymbol{\beta}}\beta^n\right)^2 - \left(\underline{\boldsymbol{\alpha}}\alpha^n - \underline{\boldsymbol{\beta}}\beta^n\right)^2 \\ &= (p^2+4q-1)(\underline{\boldsymbol{\alpha}}^2\alpha^{2n} + \underline{\boldsymbol{\beta}}^2\beta^{2n}) + (p^2+4q+1)(\alpha\beta)^n(\underline{\boldsymbol{\alpha}}\underline{\boldsymbol{\beta}} + \underline{\boldsymbol{\beta}}\underline{\boldsymbol{\alpha}}). \end{aligned}$$

Substituting Eqs. (13) and (14) into the last equation, we have

$$(p^2+4q)(\mathbb{H}\mathcal{L}_n^2 - \mathbb{H}\mathcal{F}_n^2) = (p^2+4q-1)(\underline{\boldsymbol{\alpha}}^2\alpha^{2n} + \underline{\boldsymbol{\beta}}^2\beta^{2n}) + 2(p^2+4q+1)(\alpha\beta)^n(\mathbb{H}\mathcal{L}_0 - (q^3+pq-q+1)). \tag{24}$$

Then, using Eqs. (21) and (22), we obtain

$$\underline{\boldsymbol{\alpha}}^2\alpha^{2n} + \underline{\boldsymbol{\beta}}^2\beta^{2n} = (\alpha^{2n}+\beta^{2n})(\mathbb{H}\mathcal{L}_0 + r_{p,q}) + (\alpha-\beta)(\mathbb{H}\mathcal{F}_0 + s_{p,q})(\alpha^{2n} - \beta^{2n}). \tag{25}$$

Substituting Eq. (25) into Eq. (24) gives Eq. (23). ■

Theorem 11 *For any integers $m \geq n \geq 0$, we have*

$$\mathbb{H}\mathcal{F}_n\mathbb{H}\mathcal{J}_m - \mathbb{H}\mathcal{J}_m\mathbb{H}\mathcal{F}_n = 2(-q)^{n+1}\mathcal{J}_{m-n}(\mathbb{H}\mathcal{F}_0 - \boldsymbol{\omega}), \tag{26}$$

where $\boldsymbol{\omega} = (1-p)\mathbf{i} - q\boldsymbol{\varepsilon} + (p^2+q+1)\mathbf{h}$ and $\mathcal{J}_n = \frac{A\alpha^n - B\beta^n}{\alpha-\beta}$ is the n-th generalized Fibonacci number.

Proof. The Binet formulas for the (p,q)-Fibonacci hybrid numbers and generalized Fibonacci hybrid numbers give

$$\begin{aligned} (p^2+4q)(\mathbb{H}\mathcal{F}_n\mathbb{H}\mathcal{J}_m - \mathbb{H}\mathcal{J}_m\mathbb{H}\mathcal{F}_n) &= \left(\underline{\boldsymbol{\alpha}}\alpha^n - \underline{\boldsymbol{\beta}}\beta^n\right)\left(A\underline{\boldsymbol{\alpha}}\alpha^m - B\underline{\boldsymbol{\beta}}\beta^m\right) - \left(A\underline{\boldsymbol{\alpha}}\alpha^m - B\underline{\boldsymbol{\beta}}\beta^m\right)\left(\underline{\boldsymbol{\alpha}}\alpha^n - \underline{\boldsymbol{\beta}}\beta^n\right) \\ &= (A\alpha^m\beta^n - B\alpha^n\beta^m)(\underline{\boldsymbol{\alpha}}\underline{\boldsymbol{\beta}} - \underline{\boldsymbol{\beta}}\underline{\boldsymbol{\alpha}}). \end{aligned}$$

Using Eqs. (13) and (14), we have

$$\begin{aligned} \mathbb{H}\mathcal{F}_n\mathbb{H}\mathcal{J}_m - \mathbb{H}\mathcal{J}_m\mathbb{H}\mathcal{F}_n &= \frac{2q(\alpha\beta)^n}{p^2+4q}(A\alpha^{m-n} - B\beta^{m-n})(\alpha-\beta)(\mathbb{H}\mathcal{F}_0 - \boldsymbol{\omega}) \\ &= -2(-q)^{n+1}\mathcal{J}_{m-n}(\mathbb{H}\mathcal{F}_0 - \boldsymbol{\omega}), \end{aligned}$$

where $\boldsymbol{\omega} = (1-p)\mathbf{i} - q\boldsymbol{\varepsilon} + (p^2+q+1)\mathbf{h}$ and $\mathcal{J}_n$ is the n-th generalized Fibonacci number defined by $\mathcal{J}_n = \frac{A\alpha^n - B\beta^n}{\alpha-\beta}$. So, the theorem is proved. ■

Taking $m = n$ in the Theorem 11 and using $\mathcal{J}_0 = a$, we obtain the next identity.

Corollary 12 *For any integer $n \geq 0$, we have*

$$\mathbb{H}\mathcal{F}_n\mathbb{H}\mathcal{J}_n - \mathbb{H}\mathcal{J}_n\mathbb{H}\mathcal{F}_n = 2a(-q)^{n+1}(\mathbb{H}\mathcal{F}_0 - \boldsymbol{\omega}), \tag{27}$$

where $A = b - a\beta$, $B = b - a\alpha$ and $\boldsymbol{\omega} = (1-p)\mathbf{i} - q\boldsymbol{\varepsilon} + (p^2+q+1)\mathbf{h}$.

3 Conclusions

There are differences between the generalized Fibonacci hybrid numbers and the coefficient generalized Fibonacci quaternions. For example, the usual coefficient generalized Fibonacci quaternionic units are $\mathbf{i}^2 =$

$\mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$ whereas the generalized Fibonacci hybrid units are $\mathbf{i}^2 = -1$, $\boldsymbol{\varepsilon}^2 = 0$, $\mathbf{h}^2 = 1$ and $\mathbf{ih} = -\mathbf{hi} = \boldsymbol{\varepsilon} + \mathbf{i}$.

In this work, we have examined a new type of numbers, which are non-commutative. We named this number set as generalized Fibonacci hybrid numbers because it is a linear combination of well-known complex, hyperbolic and dual Fibonacci numbers. We have given the relation $\mathbf{ih} = -\mathbf{hi} = \boldsymbol{\varepsilon} + \mathbf{i}$ between the units $\{\mathbf{i}, \boldsymbol{\varepsilon}, \mathbf{h}\}$ of these three number systems, and we have seen the algebraic consistency of this relation. Thus, we have obtained some properties of the generalized Fibonacci hybrid numbers.

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