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Certain Subclasses Of Analytic Functions Defined By Fractional q-Calculus Operators*

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Abstract

Applying the concept of fractional q-calculus, we introduce the subclass $\mathcal{TS}_q^m(\delta,\lambda,\alpha,\beta)$ of β-uniformly starlike and β-uniformly convex functions involving a linear multiplier fractional q-differintegral operator. A characterization of those functions belonging to the newly-introduced subclass $\mathcal{TS}_q^m(\cdot)$ is provided. Results on growth and distortion theorems, extreme points and other interesting properties of the subclass are also investigated.

1 Introduction and Preliminaries

The fractional q-calculus is the q-extension of the ordinary fractional calculus. The theories of q-calculus operators have been applied in such many areas of mathematics, statistics and physics as (for example) ordinary fractional calculus, optimal control problems, and solutions of the q-difference. For more other results involving the fractional q-calculus and its applications, one may refer to such works as [1, 3, 4, 5, 8, 13].

A firm footing of the usage of the q-calculus in the context of geometric function theory was provided in the book [18]. Recently, many authors have introduced new classes of analytic functions using q-calculus operators. For some recent investigations on the classes of analytic functions defined by using q-calculus operators and related topics, we refer the reader to [10, 11, 12, 15, 16] and the references cited therein. In this paper, we aim to introduce a new generalized class of β-uniformly starlike functions and β-uniformly convex functions defined by fractional q-calculus operators, which is analytic in the open unit disk. Some interesting and (potentially) useful properties of those functions are also investigated. It is also noted that the results presented here are general enough to reduce to yield many simpler ones. Throughout this paper, let $\mathbb{C}$ and $\mathbb{N}$ denote the sets of complex numbers and positive integers, respectively, and let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

The following notations and definitions are recalled (see, e.g. [19, Chapter 6]): The q-shifted factorial $(a;q)_n$ is defined by

$$(\alpha;q)_n := \prod_{k=0}^{n-1} \left(1-\alpha\, q^k\right) \quad (n\in\mathbb{N}), \tag{1}$$

where $\alpha,\, q\in\mathbb{C}$ and it is assumed that $\alpha \neq q^{-m}$ $(m\in\mathbb{N}_0)$. It is noted that the limiting case as $q\to 1^-$ of $(\alpha;q)_n$ in (1) yields the familiar Pochhammer symbol $(\alpha)_n$ defined by $(\alpha)_0 = 1$ and $(\alpha)_n = \alpha(\alpha+1)\cdots(\alpha+n-1)$ $(n\in\mathbb{N})$. We also write

$$(\alpha;q)_\infty := \prod_{k=0}^{\infty} \left(1-\alpha\, q^k\right) \quad (\alpha,\, q\in\mathbb{C};\ |q|<1). \tag{2}$$

When $\alpha\neq 0$ and $|q|\geq 1$, the infinite product in (2) diverges. So, whenever $(\alpha;q)_\infty$ is involved in a given formula, the constraint $|q|<1$ will be tacitly assumed. The $(\alpha;q)_n$ in (1) can be expressed in terms of the

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q-Gamma function defined by

$$\Gamma_q(x) = \frac{(q;q)_\infty (1-q)^{1-x}}{(q^x;q)_\infty} \quad (0 < q < 1) \tag{3}$$

as follows:

$$(q^\alpha;q)_n = \frac{\Gamma_q(\alpha+n)(1-q)^n}{\Gamma_q(\alpha)} \quad (n \in \mathbb{N}). \tag{4}$$

The q-derivative of a function $f(t)$ is defined by

$$D_q f(t) := \frac{d_q}{d_q\, t} f(t) = \frac{f(qt) - f(t)}{(q-1)t}. \tag{5}$$

From (5) we find

$$\lim_{q \to 1} D_q f(t) = \frac{d}{dt} f(t),$$

if $f(t)$ is differentiable. For example, the q-derivative of $f(z) = z^n$ $(n \in \mathbb{N})$ is

$$D_q z^n = \frac{z^n - (zq)^n}{(1-q)z} = [n]_q\, z^{n-1},$$

where

$$[n]_q := \frac{1-q^n}{1-q} = 1 + q + \cdots + q^{n-1} \to n$$

as $q \to 1$. So the notation $[n]_q$ $(n \in \mathbb{N})$ is called q-analogue of $n \in \mathbb{N}$.

The q-integral of a function $f(t)$ is defined by

$$\int_0^z f(t) d_q t = z(1-q) \sum_{k=0}^{\infty} q^k f(zq^k). \tag{6}$$

We also recall the fractional q-calculus operators of a complex-valued function $f(z)$ (see [11]). The fractional q-integral operator $I_{q,z}^{\delta}$ of a function $f(z)$ of order δ is defined by

$$I_{q,z}^{\delta} f(z) \equiv D_{q,z}^{-\delta} f(z) = \frac{1}{\Gamma_q(\delta)} \int_0^z (z-tq)_{\delta-1} f(t)\, d_q t \quad (\delta > 0), \tag{7}$$

where $f(z)$ is analytic in a simply connected region of the z-plane containing the origin and the q-binomial function $(z-tq)_{\delta-1}$ is given by

$$(z-tq)_{\delta-1} = z^{\delta-1}\, {}_1\Phi_0\left(q^{-\delta+1}; -; q, tq^\delta/z\right).$$

The series ${}_1\Phi_0$ is defined by

$${}_1\Phi_0(a;-;q,z) := \sum_{k=0}^{\infty} \frac{(a;q)_k}{(q;q)_k} z^k = \frac{(az;q)_\infty}{(z;q)_\infty} \quad (|q| < 1;\ |z| < 1), \tag{8}$$

which was proven by several mathematicians and whose last equality is called q-binomial theorem (see [19, Section 6.3]). The series ${}_1\Phi_0(a;-;q,z)$ is single valued when $|\arg(z)| < \pi$ and $|z| < 1$ and so the function $(z-tq)_{\delta-1}$ in (7) is single valued when $|\arg(-tq^\delta/z)| < \pi$, $|tq^\delta/z| < 1$ and $|\arg(z)| < \pi$.

The fractional q-derivative operator $D_{q,z}^{\delta}$ of a function $f(z)$ of order δ is defined by

$$D_{q,z}^{\delta} f(z) \equiv D_{q,z}\, I_{q,z}^{1-\delta} f(z) = \frac{1}{\Gamma_q(1-\delta)}\, D_{q,z} \int_0^z (z-tq)_{-\delta}\, f(t)\, d_q t \quad (0 \leq \delta < 1), \tag{9}$$

where $f(z)$ is suitably constrained and the multiplicity of $(z-tq)_{-\delta}$ is removed as in (7).

Let $\delta > 0$ and m be the smallest integer that exceeds δ. Then the extended fractional q-derivative of a function $f(z)$ of order δ is defined by

$$D_{q,z}^{\delta} f(z) = D_{q,z}^{m}\, I_{q,z}^{m-\delta} f(z), \tag{10}$$

provided that it exists.

We find from (10) that

$$D_{q,z}^{\delta}\, z^n = \frac{\Gamma_q(n+1)}{\Gamma_q(n+1-\delta)}\, z^{n-\delta} \quad (\delta \geq 0;\ n > -1). \tag{11}$$

2 The Class $\mathcal{TS}_q^m(\delta,\lambda,\alpha,\beta)$

Let $\mathcal{A}$ denote the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{12}$$

which are analytic in the open unit disk $\mathtt{D} := \{z \in \mathbb{C} : |z| < 1\}$. Also let $\mathcal{T}$ denote a subclass of $\mathcal{A}$ consisting of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \quad (a_n \geq 0;\ n \in \mathbb{N} \setminus \{1\}). \tag{13}$$

Let $\mathcal{S}$, $\mathcal{S}^*$, $\mathcal{K}$ and $\mathcal{C}$ denote the class of all functions in $\mathcal{A}$ which are, respectively, univalent, starlike, convex and close-to-convex. A function $f(z) \in \mathcal{A}$ is said to be β-uniformly starlike of order α $(0 \leq \alpha < 1)$, the class of such functions is denoted by $\mathcal{UST}(\alpha,\beta)$, if and only if

$$\Re\left\{\frac{zf'(z)}{f(z)} - \alpha\right\} \geq \beta\left|\frac{zf'(z)}{f(z)} - 1\right| \quad (\beta \geq 0;\ z \in \mathtt{D}).$$

A function $f(z) \in \mathcal{A}$ is said to be β-uniformly convex of order α $(0 \leq \alpha < 1)$, the class of those functions is denoted by $\mathcal{UCV}(\alpha,\beta)$, if and only if

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)} - \alpha\right\} \geq \beta\left|\frac{zf''(z)}{f'(z)}\right| \quad (\beta \geq 0;\ z \in \mathtt{D}).$$

It is noted that $\mathcal{UST}(\alpha,0) = \mathcal{S}^*(\alpha)$ and $\mathcal{UCV}(\alpha,0) = \mathcal{K}(\alpha)$, where $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ are the familiar classes of starlike and convex functions of order α. It is also noted that $f(z) \in \mathcal{UCV}(\alpha,\beta)$ if and only if $zf'(z) \in \mathcal{UST}(\alpha,\beta)$. To know more about β-uniformly starlike and β-uniformly convex functions one may refer to the papers of Kanas and Wisniowska [6, 7].

We define a q-differintegral operator $\Omega_{q,z}^{\delta} : \mathcal{A} \longrightarrow \mathcal{A}$ as follows:

$$\Omega_{q,z}^{\delta} f(z) = \frac{\Gamma_q(2-\delta)}{\Gamma_q(2)}\, z^{\delta}\, D_{q,z}^{\delta}\, f(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma_q(2-\delta)\Gamma_q(n+1)}{\Gamma_q(2)\Gamma_q(n+1-\delta)}\, a_n z^n \tag{14}$$
$$(\delta < 2;\ 0 < q < 1;\ z \in \mathtt{D}).$$

Here $D_{q,z}^{\delta}\, f(z)$ represents, respectively, a fractional q-integral of $f(z)$ of order δ when $-\infty < \delta < 0$ and a fractional q-derivative of $f(z)$ of order δ when $0 \leq \delta < 2$. It is noted that $\Omega_{q,z}^{0} f(z) = f(z)$. We remark here that the operator $\Omega_{q,z}^{\delta}$ is the q-extension of Srivastava-Owa differintegral operator [9].

We also define a linear multiplier fractional q-differintegral operator $\mathscr{D}_{q,\lambda}^{\delta,m}$ as follows:

$$\mathscr{D}_{q,\lambda}^{\delta,m} f(z) := \mathscr{D}_{q,\lambda}^{\delta,1}\left(\mathscr{D}_{q,\lambda}^{\delta,m-1} f(z)\right) \quad (m \in \mathbb{N};\ \delta < 2;\ \lambda \geq 0;\ 0 < q < 1;\ z \in \mathtt{D}) \tag{15}$$

where

$$\mathscr{D}_{q,\lambda}^{\delta,0} f(z) := f(z)$$

and

$$\mathscr{D}_{q,\lambda}^{\delta,1} f(z) := (1-\lambda)\Omega_{q,z}^{\delta} f(z) + \lambda\, z\, D_q(\Omega_{q,z}^{\delta} f(z)).$$

It is seen from (15) that, for $f(z)$ in (12),

$$\begin{aligned}\mathscr{D}_{q,\lambda}^{\delta,m} f(z) &= z + \sum_{n=2}^{\infty} \left(\frac{\Gamma_q(2-\delta)\Gamma_q(n+1)}{\Gamma_q(2)\Gamma_q(n+1-\delta)} \big[1-\lambda+[n]_q\lambda\big] \right)^m a_n z^n \\ &= z + \sum_{n=2}^{\infty} A_q(\lambda,\delta,m,n) a_n z^n, \end{aligned} \tag{16}$$

where

$$A_q(\lambda,\delta,m,n) = \left(\frac{\Gamma_q(2-\delta)\Gamma_q(n+1)}{\Gamma_q(2)\Gamma_q(n+1-\delta)} \big[1-\lambda+[n]_q\lambda\big] \right)^m. \tag{17}$$

Remark 1 *By suitably specializing the parameters, the operator* $\mathscr{D}_{q,\lambda}^{\delta,m}$ *in* (15) *reduces to many known and new integral and differential operators. For example, when* $\delta = 0$ *and* $q \to 1$*, it reduces to the operator introduced by Al-Oboudi [2]; when* $\delta = 0$*,* $\lambda = 1$*, and* $q \to 1$*, it yields the operator given by Sălăgean [14].*

By using the linear multiplier q-fractional differintegral operator $\mathscr{D}_{q,\lambda}^{\delta,m}$ in (15), we introduce another subclass $\mathcal{S}_q^m(\delta,\lambda,\alpha,\beta)$ of analytic functions: A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{S}_q^m(\delta,\lambda,\alpha,\beta)$ if and only if

$$\Re\left\{ \frac{z\, D_q\big(\mathscr{D}_{q,\lambda}^{\delta,m} f(z)\big)}{\mathscr{D}_{q,\lambda}^{\delta,m} f(z)} - \alpha \right\} \geq \beta \left| \frac{z\, D_q\big(\mathscr{D}_{q,\lambda}^{\delta,m} f(z)\big)}{\mathscr{D}_{q,\lambda}^{\delta,m} f(z)} - 1 \right| \tag{18}$$

where $m \in \mathbb{N};\ 0 \leq \alpha < 1;\ \beta \geq 0;\ \delta < 2;\ \lambda \geq 0;\ 0 < q < 1;\ z \in \mathbb{D}$. It is interesting to note that the class $\mathcal{S}_q^m(\delta,\lambda,\alpha,\beta)$ is a generalization of β-uniformly starlike and convex functions.

Let $\mathcal{TS}_q^m(\delta,\lambda,\alpha,\beta) := \mathcal{S}_q^m(\delta,\lambda,\alpha,\beta) \cap \mathcal{T}$. It is seen that $\mathcal{TS}_q^m(\delta,\lambda,\alpha,\beta)$ extends the classes of starlike, convex, β-uniformly starlike and convex functions with suitable choices of the parameters as shown in the following examples: As $q \to 1$,

(i) $\mathcal{TS}_q^0(\delta,\lambda,\alpha,\beta) \to \mathcal{UST}(\alpha,\beta)$;

(ii) $\mathcal{TS}_q^1(0,1,\alpha,\beta) \to \mathcal{UCV}(\alpha,\beta)$;

(iii) $\mathcal{TS}_q^0(\delta,\lambda,\alpha,0) \to \mathcal{S}^*(\alpha)$;

(iv) $\mathcal{TS}_q^1(0,1,\alpha,0) \to \mathcal{K}(\alpha)$ as $q \to 1$.

3 Coefficient Estimates

Here a necessary condition for a function to belong to the class $\mathcal{S}_q^m(\delta,\lambda,\alpha,\beta)$ and a characterization of the class $\mathcal{TS}_q^m(\delta,\lambda,\alpha,\beta)$ are provided.

Theorem 1 *A function* $f(z)$ *in* (12) *is in* $\mathcal{S}_q^m(\delta,\lambda,\alpha,\beta)$ *if*

$$\sum_{n=2}^{\infty} \big[(1+\beta)[n]_q - (\alpha+\beta)\big] A_q(\lambda,\delta,m,n)|a_n| \leq 1-\alpha \tag{19}$$

where $0 \leq \alpha < 1,\ \beta \geq 0, \lambda \geq 0,\ \delta < 2,\ 0 < q < 1,\ m \in \mathbb{N}$ *and* $A_q(\lambda,\delta,m,n)$ *is given as in* (17).

Proof. It suffices to show that

$$L(z,q,\delta,\lambda,m) \leq 1-\alpha,$$

where

$$L(z,q,\delta,\lambda,m) := \beta\left|\frac{z\,D_q\big(\mathscr{D}_{q,\lambda}^{\delta,m}f(z)\big)}{\mathscr{D}_{q,\lambda}^{\delta,m}f(z)}-1\right| - \Re\left\{\frac{z\,D_q\big(\mathscr{D}_{q,\lambda}^{\delta,m}f(z)\big)}{\mathscr{D}_{q,\lambda}^{\delta,m}f(z)}-1\right\}.$$

We find that

$$L(z,q,\delta,\lambda,m) \leq (1+\beta)\left|\frac{z\,D_q\big(\mathscr{D}_{q,\lambda}^{\delta,m}f(z)\big)}{\mathscr{D}_{q,\lambda}^{\delta,m}f(z)}-1\right| \leq \frac{(1+\beta)\sum_{n=2}^{\infty}\big([n]_q-1\big)A_q(\lambda,\delta,m,n)|a_n|}{1-\sum_{n=2}^{\infty}A_q(\lambda,\delta,m,n)|a_n|},$$

where $A_q(\lambda,\delta,m,n)$ is given in (17). Now it is seen that the last quantity is bounded above by $1-\alpha$ under the given condition (19). This completes the proof. ■

A necessary and sufficient condition that a given function $f(z)\in\mathcal{T}$ belongs to the class $\mathcal{TS}_q^m(\delta,\lambda,\alpha,\beta)$ is provided, which is asserted by Theorem 2.

Theorem 2 *A function $f(z)$ in (13) is in the class $\mathcal{TS}_q^m(\delta,\lambda,\alpha,\beta)$ if and only if*

$$\sum_{n=2}^{\infty}\big[(1+\beta)[n]_q-(\alpha+\beta)\big]A_q(\lambda,\delta,m,n)a_n \leq 1-\alpha \tag{20}$$

where $0\leq\alpha<1$, $\beta\geq 0, \lambda\geq 0$, $\delta<2$, $0<q<1$, $m\in\mathbb{N}$ and $A_q(\lambda,\delta,m,n)$ is given in (17).

Proof. In view of Theorem 1, we need only to prove the sufficient part. Let $f(z)\in\mathcal{TS}_q^m(\delta,\lambda,\alpha,\beta)$ and z be real. Then it follows from (18) that

$$\Re\left\{\frac{z\,D_q\big(\mathscr{D}_{q,\lambda}^{\delta,m}f(z)\big)}{\mathscr{D}_{q,\lambda}^{\delta,m}f(z)}-\alpha\right\} \geq \beta\left|\frac{z\,D_q\big(\mathscr{D}_{q,\lambda}^{\delta,m}f(z)\big)}{\mathscr{D}_{q,\lambda}^{\delta,m}f(z)}-1\right|,$$

which implies

$$\frac{1-\sum_{n=2}^{\infty}[n]_qA_q(\lambda,\delta,m,n)a_nz^{n-1}}{1-\sum_{n=2}^{\infty}A_q(\lambda,\delta,m,n)a_nz^{n-1}}-\alpha \geq \beta\left|\frac{\sum_{n=2}^{\infty}\big([n]_q-1\big)A_q(\lambda,\delta,m,n)a_nz^{n-1}}{1-\sum_{n=2}^{\infty}A_q(\lambda,\delta,m,n)a_nz^{n-1}}\right|.$$

Taking the limit as $z\to 1$ along the real axis, we obtain the desired inequality. The result (20) is sharp for

$$f(z)=z-\sum_{n=2}^{\infty}\frac{1-\alpha}{\big[(1+\beta)[n]_q-(\alpha+\beta)\big]A_q(\lambda,\delta,m,n)}z^n.$$

■

We demonstrate only an application of Theorem 2 in the following corollary.

Corollary 1 *Assume that a function $f(z)$ in (13) is in $\mathcal{TS}_q^m(\delta,\lambda,\alpha,\beta)$. Then we have*

$$\sum_{n=2}^{\infty}a_n \leq \frac{1-\alpha}{\big[(1+\beta)q+(1-\alpha)\big]A_q(\lambda,\delta,m,2)}, \tag{21}$$

where $A_q(\lambda,\delta,m,2)$ is given in (17) and

$$A_q(\lambda,\delta,m,2)=\left[\frac{(1-q^2)(1+q\lambda)}{1-q^{2-\delta}}\right]^m. \tag{22}$$

Proof. Since $f(z) \in \mathcal{TS}_q^m(\delta, \lambda, \alpha, \beta)$, we can use Theorem 2 to obtain

$$\begin{aligned} & \big[(1+\beta)q + (1-\alpha)\big]\, A_q(\lambda, \delta, m, 2) \sum_{n=2}^{\infty} a_n \\ \leq \; & \sum_{n=2}^{\infty} \big[(1+\beta)[n]_q - (\alpha+\beta)\big]\, A_q(\lambda, \delta, m, n) a_n \\ \leq \; & 1-\alpha. \end{aligned}$$

Thus we find that

$$\sum_{n=2}^{\infty} a_n \leq \frac{1-\alpha}{\big[(1+\beta)q + (1-\alpha)\big] A_q(\lambda, \delta, m, 2)},$$

which is the desired inequality. ■

4 Growth and Distortion Bounds

Distortion bounds for functions $f(z) \in \mathcal{TS}_q^m(\delta, \lambda, \alpha, \beta)$ involving q-derivative are given in the following theorem.

Theorem 3 *Let $f(z) \in \mathcal{TS}_q^m(\delta, \lambda, \alpha, \beta)$. Then we find that, for $z \in \mathtt{D}$,*

$$\begin{aligned} & |z| - \frac{(1-\alpha)|z|^2}{\big[(1+\beta)q + (1-\alpha)\big] A_q(\lambda, \delta, m, 2)} \\ \leq \; & |f(z)| \leq |z| + \frac{(1-\alpha)|z|^2}{\big[(1+\beta)q + (1-\alpha)\big] A_q(\lambda, \delta, m, 2)} \end{aligned} \tag{23}$$

and

$$\begin{aligned} & 1 - \frac{(1+q)(1-\alpha)|z|}{\big[(1+\beta)q + (1-\alpha)\big] A_q(\lambda, \delta, m, 2)} \\ \leq \; & |D_q f(z)| \leq 1 + \frac{(1+q)(1-\alpha)|z|}{\big[(1+\beta)q + (1-\alpha)\big] A_q(\lambda, \delta, m, 2)} \end{aligned} \tag{24}$$

where $A_q(\lambda, \delta, m, 2)$ is the same as in (22).

Proof. For $f(z) \in \mathcal{TS}_q^m(\delta, \lambda, \alpha, \beta)$, we find from Corollary 1 that

$$\sum_{n=2}^{\infty} a_n \leq \frac{1-\alpha}{\big[(1+\beta)q + (1-\alpha)\big] A_q(\lambda, \delta, m, 2)}$$

which implies

$$|f(z)| \leq |z| + |z|^2 \sum_{n=2}^{\infty} a_n \leq |z| + \frac{(1-\alpha)|z|^2}{\big[(1+\beta)q + (1-\alpha)\big] A_q(\lambda, \delta, m, 2)} \quad (z \in \mathtt{D})$$

and

$$|f(z)| \geq |z| - |z|^2 \sum_{n=2}^{\infty} a_n \geq |z| - \frac{(1-\alpha)|z|^2}{\big[(1+\beta)q + (1-\alpha)\big] A_q(\lambda, \delta, m, 2)} \quad (z \in \mathtt{D}).$$

Thus the assertion (23) is seen to follow at once. In a similar manner, for the q-derivative $D_q\big(f(z)\big)$, we find that the following inequalities:

$$|D_q f(z)| \leq 1 + \sum_{n=2}^{\infty} [n]_q a_n |z|^{n-1} \leq 1 + |z| \sum_{n=2}^{\infty} [n]_q a_n \quad (z \in \mathtt{D})$$

and

$$\sum_{n=2}^{\infty}[n]_q a_n \leq \frac{(1+q)(1-\alpha)}{\big[(1+\beta)q+(1-\alpha)\big]A_q(\lambda,\delta,m,2)}$$

lead to the assertion (24). The distortion bounds in Theorem 3 are sharp for

$$f(z) = z - \frac{(1-\alpha)}{\big[(1+\beta)q+(1-\alpha)\big]A_q(\lambda,\delta,m,2)}z^2 \quad (z \in \mathbb{D}). \tag{25}$$

■

Theorem 4 below follows easily from Theorem 2.

Theorem 4 *Let*

$$f_k(z) = z - \sum_{n=2}^{\infty} a_{n,k} z^n \in \mathcal{TS}_q^m(\delta,\lambda,\alpha,\beta) \quad (k=1,\,2).$$

Then

$$f(z) = (1-\xi)f_1(z) + \xi f_2(z) = z - \sum_{n=2}^{\infty} a_n z^n \in \mathcal{TS}_q^m(\delta,\lambda,\alpha,\beta) \quad (0 \leq \xi \leq 1).$$

Proof. Since $f_k(z) \in \mathcal{TS}_q^m(\delta,\lambda,\alpha,\beta)$ $(k=1,\,2)$, we use Theorem 2 to get the following coefficient inequalities:

$$\sum_{n=2}^{\infty}\big[(1+\beta)[n]_q - (\alpha+\beta)\big]A_q(\lambda,\delta,m,n)a_{n,1} \leq 1-\alpha$$

and

$$\sum_{n=2}^{\infty}\big[(1+\beta)[n]_q - (\alpha+\beta)\big]A_q(\lambda,\delta,m,n)a_{n,2} \leq 1-\alpha.$$

Furthermore, in view of the following relationship

$$a_n = (1-\xi)a_{n,1} + \xi a_{n,2} \quad (n \in \mathbb{N}/\{1\};\, 0 \leq \xi \leq 1),$$

we find that

$$\begin{aligned}
& \sum_{n=2}^{\infty}\big[(1+\beta)[n]_q - (\alpha+\beta)\big]A_q(\lambda,\delta,m,n)a_n \\
= & \sum_{n=2}^{\infty}\big[(1+\beta)[n]_q - (\alpha+\beta)\big]A_q(\lambda,\delta,m,n)\left[(1-\xi)a_{n,1} + \xi a_{n,2}\right] \\
= & \sum_{n=2}^{\infty}(1-\xi)\big[(1+\beta)[n]_q - (\alpha+\beta)\big]A_q(\lambda,\delta,m,n)a_{n,1} \\
& + \sum_{n=2}^{\infty}\xi\big[(1+\beta)[n]_q - (\alpha+\beta)\big]A_q(\lambda,\delta,m,n)a_{n,2} \\
\leq & \ (1-\xi)(1-\alpha) + \xi(1-\alpha) = (1-\alpha).
\end{aligned}$$

Thus, by Theorem 2 again, we finally obtain $f(z) \in \mathcal{TS}_q^m(\delta,\lambda,\alpha,\beta)$. This completes the proof of Theorem 4. ■

5 Extreme Points

Theorem 5 *Let* $f_1(z) = z$ *and*

$$f_n(z) = z - \frac{(1-\alpha)}{\big[(1+\beta)[n]_q - (\alpha+\beta)\big] A_q(\lambda,\delta,m,n)} z^n \quad (n \in \mathbb{N} \setminus \{1\}.$$

Then $f(z)$ *is in* $\mathcal{TS}_q^m(\delta,\lambda,\alpha,\beta)$ *if and only if it can be expressed in the form* $f(z) = \sum_{n=1}^{\infty} \lambda_n f_n(z)$*, where* $\lambda_n \geq 0$ *and* $\sum_{n=1}^{\infty} \lambda_n = 1$.

Proof. Adopting the same technique used by Silverman [17], we first assume that

$$f(z) = \sum_{n=1}^{\infty} \lambda_n f_n(z) = z - \sum_{n=2}^{\infty} \lambda_n \frac{(1-\alpha)}{\big[(1+\beta)[n]_q - (\alpha+\beta)\big] A_q(\lambda,\delta,m,n)} z^n.$$

Then we have

$$\begin{aligned} & \sum_{n=2}^{\infty} \lambda_n \frac{(1-\alpha)}{\big[(1+\beta)[n]_q - (\alpha+\beta)\big] A_q(\lambda,\delta,m,n)} \\ & \times \Big(\big[(1+\beta)[n]_q - (\alpha+\beta)\big] A_q(\lambda,\delta,m,n)\Big) \\ = \; & (1-\alpha) \sum_{n=2}^{\infty} \lambda_n = (1-\alpha)(1-\lambda_1) \leq (1-\alpha). \end{aligned}$$

Hence it follows from Theorem 2 that $f \in \mathcal{TS}_q^m(\delta,\lambda,\alpha,\beta)$.

Conversely, suppose $f \in \mathcal{TS}_q^m(\delta,\lambda,\alpha,\beta)$. Then we find from Corollary 1 that

$$a_n \leq \frac{1-\alpha}{\big[(1+\beta)[n]_q - (\alpha+\beta)\big] A_q(\lambda,\delta,m,n)} \quad (n \in \mathbb{N} \setminus \{1\}).$$

Now, by letting

$$\lambda_n = \left\{ \frac{\big[(1+\beta)[n]_q - (\alpha+\beta)\big] A_q(\lambda,\delta,m,n)}{1-\alpha} \right\} a_n \quad (n \in \mathbb{N} \setminus \{1\})$$

and $\lambda_1 = 1 - \sum_{n=2}^{\infty} \lambda_n$, we prove our assertion, since

$$f(z) = \sum_{n=1}^{\infty} \lambda_n f_n = \lambda_1 f_1(z) + \sum_{n=2}^{\infty} \lambda_n f_n(z).$$

■

We conclude this paper by remarking that, by suitably specializing the parameters involved in each of the results presented here, further numerous corollaries and consequences can be deduced.

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