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Fixed Point Theorems For Nonlinear Contraction In Controlled Metric Type Space*

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Abstract

In this article, we introduced the notion of controlled comparison function and obtained some fixed point theorems involving such function in the setting of controlled metric type space. Our presented results extend, generalize and improve results on the topic in the literature. As a consequence, our fixed point results generalize the corresponding results in the Hicks and Rhoades (Math. Japonica, 24 (1979), 327-330) and B. Samet at el. (Nonlinear Anal., 75 (2012), 2145-2165). Finally, we provide an illustratire application.

1 Introduction

Fixed point theory is an interesting subject, with a vast number of applications in various fields of mathematics. In metric spaces, this theory started with the Banach fixed-point theorem which guarantees the existence and uniqueness of fixed points of certain self-maps of complete metric spaces and provides a constructive method of finding those fixed points. There are enormous amount of literature dealing with generalizations of this remarkable theorem (see [3, 6, 8, 10, 19] and references therein). Many researchers [3, 5, 8, 13] generalized the structure of metric space by weakening triangle inequality and proved some fixed point for these spaces. Bourbaki [6] and Bakhtin [3] give the idea of b-metric space. Czerwik [8] generalized Banach contraction theorem for b-metric space. Since then several articles appeared which dealt with fixed points theorems for single-valued and multi-valued mapping in b-metric space. By further weakening the triangular inequality Kamran et al. [13] introduced the concept of extended b-metric space. Recently, N. Mlaiki et al. [17] introduced the new extension of b-metric space, called controlled metric type spaces, by defining the triangular inequality as $d(x, z) \leq \alpha(x, y)d(x, y) + \alpha(y, z)d(y, z)$. They gave an example to show that controlled metric type spaces are not extended b-metric spaces in the sense of Kamran et al. They also proved the corresponding Banach fixed point theorem on controlled metric type spaces. In present work, we introduce the notion of controlled comparison function and prove some fixed point theorems involving such function in the setting of controlled metric type space. Our presented results extend, generalize and improve results on the topic in the literature. The obtained results for such setting become more useful in different avenues of applications.

2 Preliminaries

In the following, we will recall some basic definitions and known results needed in the sequel.

Definition 1 ([8]) *Let X be a non empty set and $s \geq 1$ be a given real number. A function $d : X \times X \to [0, \infty)$ is called b-metric if for all $x, y, z \in X$ it satisfies:*

(i) $d(x, y) = 0$ *if and only if* $x = y$;

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(ii) $d(x, y) = d(y, x)$;

(iii) $d(x, z) \leq s[d(x, y) + d(y, z)]$.

The pair (X, d) *is called* b*-metric space.*

The above definition shows that the class of metric spaces is contained in the class of b-metric spaces. In fact, for $s = 1$ a b-metric space is reduced into a standard metric space.

Example 1 ([13]) *Let* $X := l_p(\mathbb{R})$ *with* $0 < p < 1$*, where* $l_p(\mathbb{R}) := \{x = \{x_n\} \subset \mathbb{R} : \sum_{n=1}^{\infty} |x_n|^p < \infty\}$*. Then* $d : X \times X \to [0, \infty)$ *is defined by* $d(x, y) = (\sum_{n=1}^{\infty} |x_n - y_n|^p)^{\frac{1}{p}}$ *is a* b*-metric on* X *with* $s = 2^{1/p}$

Example 2 ([13]) *Let* $X = L_p[0, 1]$ *be the space of all functions* $x(t), t \in [0, 1]$ *such that* $\int_0^1 |x(t)|^p dt < \infty$ *with* $0 < p < 1$*.Then* (X, d) *is a* b*-metric space with* $s = 2^{1/p}$ *where* $d : X \times X \to [0, \infty)$ *is defined by* $d(x, y) = \left(\int_0^1 |x(t) - y(t)|^p dt\right)^{\frac{1}{p}}$.

Definition 2 *Let* $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$*. Consider the following properties:*

$(i)_\varphi$ $t_1 \leq t_2$ *implies* $\varphi(t_1) \leq \varphi(t_2)$*, for all* $t_1, t_2 \in \mathbb{R}^+$*;*

$(ii)_\varphi$ $\varphi(t) < t$ *for* $t > 0$*;*

$(iii)_\varphi$ $\varphi(0) = 0$*;*

$(iv)_\varphi$ $\lim_{n\to\infty} \varphi^n(t) = 0$ *for all* $t \geq 0$*;*

$(v)_\varphi$ $\sum_{n=0}^{\infty} \varphi^n(t)$ *converges for all* $t > 0$*.*

A function φ satisfying $(i)_\varphi$ and $(iv)_\varphi$ is said to be a comparison function. A function φ satisfying $(i)_\varphi$ and $(v)_\varphi$ is known as (c)-comparison function. Any (c)-comparison function is a comparison function but converse may not be true. For example $\varphi(t) = \frac{t}{1+t}$, $t \in \mathbb{R}^+$ is a comparison function but not a (c)-comparison function. On the other hand define $\varphi(t) = \frac{t}{2}$, $0 \leq t \leq 1$ and $\varphi(t) = t - \frac{1}{2}$, $t > 1$, then φ is a (c)-comparison function

Berinde [5] introduced the class of b-comparison functions and also obtained some estimations for the rate of convergence of the proposed iterative process to the fixed point.

Definition 3 *Let* $s \geq 1$ *be a fixed real number. A function* $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ *is known as* b*-comparison function if it satisfies* $(i)_\varphi$ *and the following holds;*

$(vi)_\varphi$ $\sum_{n=0}^{\infty} s^n \varphi^n(t)$ *converges for all* $t \in \mathbb{R}^+$*.*

The definition of b-comparison function reduces to comparison function when $s = 1$. Let (X, d) be a b-metric space with coefficient $s \geq 1$, then $\varphi(t) = \kappa t$, $t \in \mathbb{R}^+$ with $0 < \kappa < \frac{1}{s}$ is a b-comparison function.

Recently, N. Mlaiki et al. [17] generalized b-metric space by introducing the controlled function on right side of the triangular inequality and called it a controlled metric type space. They also proved some fixed point theorems on these spaces.

Definition 4 ([17]) *Let* X *be a non empty set and* $\alpha : X \times X \to [1, \infty)$*. A function* $d : X \times X \to [0, \infty)$ *is called a controlled metric type space if for all* $x, y, z \in X$ *it satisfies.*

(i) $d(x, y) = 0$ *if and only if* $x = y$;

(ii) $d(x, y) = d(y, x)$;

(iii) $d(x, z) \leq \alpha(x, y)d(x, y) + \alpha(y, z)d(y, z)$.

Then the pair (X, d) is called a controlled metric type space.

Note that if for all $x, y \in X$ $\alpha(x, y) = s$ for $s \geq 1$, then triangular inequality in b-metric space is satisfied; however, it does not hold true in general. Thus the class of controlled metric type space is larger than that of b-metric space. That is, every b-metric space is a controlled metric type space, but converse may not be true. The following example shows the above remarks.

Example 3 ([17]) *Let $X = \mathbb{N}$ and define $d : X \times X \to \mathbb{R}^+$ as*

$$d(x,y) = \begin{cases} 0 & \text{if and only if } x = y, \\ \frac{1}{x} & \text{if } x \text{ is even and } y \text{ is odd,} \\ \frac{1}{y} & \text{if } y \text{ is even and } x \text{ is odd,} \\ 1 & \text{otherwise.} \end{cases}$$

Then (X, d) is a controlled metric type space with $\alpha : X \times X \to [1, \infty)$ defined by

$$\alpha(x,y) = \begin{cases} x & \text{if } x \text{ is even and } y \text{ is odd,} \\ y & \text{if } y \text{ is even and } x \text{ is odd,} \\ 1 & \text{otherwise.} \end{cases}$$

Definition 5 ([17]) *Let (X, d) be a controlled metric type space, $x \in X$ and let $\{x_n\}$ be a sequence in X . Then*

(i) $\{x_n\}$ converges to x, if for every $\varepsilon > 0$ there exists $N = N(\varepsilon) \in \mathbb{N}$ such that $d(x_n, x) < \varepsilon$, for all $n \geq N$.

(ii) $\{x_n\}$ is a Cauchy sequence, if for every $\varepsilon > 0$ there exists $N = N(\varepsilon) \in \mathbb{N}$ such that $d(x_m, x_n) < \varepsilon$ for all $m, n \geq N$.

(iii) (X, d) is a complete extended b-metric space if every Cauchy sequence is convergent.

In the following we recollect some generalizations of Banach contraction principal in case of a controlled metric type space.

Theorem 1 ([17]) *Let (X, d) be a complete controlled metric type space and suppose $T : X \to X$ satisfies:*

$$d(Tx, Ty) \leq kd(x, y) \text{ for all } x, y \in X$$

where $k \in (0, 1)$ is such that for each $x_0 \in X$

$$\sup_{m \geq 1} \lim_{j \to \infty} \frac{\alpha(x_{j+1}, x_{j+2})}{\alpha(x_j, x_{j+1})} \alpha(x_{j+1}, x_m) \leq \frac{1}{k}$$

where $x_n = T^n x_0$. In addition, assume that, for every $x \in X$, we have

$$\lim_{n \to +\infty} \alpha(x_n, x) \text{ and } \lim_{n \to +\infty} \alpha(x, x_n) \text{ exist and are finite.}$$

Then T has a unique fixed point.

Definition 6 *Let T be a function from a subset D of X to X. A set $\mathcal{O}(x_0) = \{x_0, Tx_0, T^2x_0, \cdots\}$ is called an orbit of $x_0 \in D$ if $\mathcal{O}(x_0) \subset D$ for some $x_0 \in D$. A function $G : D \to \mathbb{R}$ is said to be T-orbitally lower semi continuous at $s \in D$ if $\{x_n\} \subset \mathcal{O}(x_0)$ and $x_n \to s$ implises $G(s) \leq lim_{n\to\infty} \inf G(x_n)$.*

Corollary 1 ([17]) *Let (X, d) be a complete controlled metric type space and suppose $T : X \to X$ satisfies:*

$$d(Tx, T^2x) \leq kd_\theta(x, Tx) \text{ for all } x \in \mathcal{O}(x_0)$$

where $k \in (0, 1)$ is such that for each $x_0 \in X$

$$\sup_{m \geq 1} \lim_{j \to \infty} \frac{\alpha(x_{j+1}, x_{j+2})}{\alpha(x_j, x_{j+1})} \alpha(x_{j+1}, x_m) < \frac{1}{k},$$

where $x_n = T^n x_0$. Then, $x_n \to z \in X$. Furthermore z is a fixed point of T if and only if $G(x) = d(x, Tx)$ is T-orbitally lower semi-continuous at z.

3 Main Results

In this section we introduce the notion of controlled comparison function and prove some fixed point theorems involving such function in the setting of controlled metric type space. We start this section with following definition.

Definition 7 *Let (X,d) be a controlled metric type space. A function $\varphi:\mathbb{R}^+\to\mathbb{R}^+$ is called controlled comparison function if it satisfies:*

$(i)_\varphi$ $t_1\le t_2$ *implies* $\varphi(t_1)\le\varphi(t_2)$, *for all* $t_1,t_2\in\mathbb{R}^+$

$(ii)_\varphi$ $\lim_{n\to+\infty}\varphi^n(t)=0$ *for all* $t\ge 0$;

$(iii)_\varphi$ there exists a mapping $T:D\subset X\to X$ such that for some $x_0\in D, O(x_0)\subset D$ such that

$$\sum_{n=1}^{\infty}\varphi^n(t)\left(\prod_{i=1}^{n}\alpha(x_i,x_m)\right)\alpha(x_n,x_{n+1})$$

converges for all $t\in\mathbb{R}^+$ and for every $m\in\mathbb{N}$.

Here $x_n=T^nx_0$ for $n=1,2,\cdots$. We say that φ is a controlled comparison function for T at x_0.

Example 4 *Let (X,d) be a controlled metric type space. Define $\varphi:\mathbb{R}^+\to\mathbb{R}^+$ by*

$$\varphi(t)=kt \text{ such that } \sup_{m\ge1}\lim_{j\to\infty}\frac{\alpha(x_{j+1},x_{j+2})}{\alpha(x_j,x_{j+1})}\alpha(x_{j+1},x_m)<\frac{1}{k}.$$

Then φ is a controlled comparison function. Here for $x_o\in X$, $x_n=T^nx_o$, $n=1,2,3,\cdots$ and T is a self map on X.

Now we proceed to establish our main result.

Theorem 2 *Let (X,d) be a complete controlled metric type space. Let $T:D\subset X\to X$ be a mapping such that $\mathcal{O}(x_0)\subset D$. Suppose that*

$$d(Tx,T^2x)\le\varphi(d(x,Tx)) \text{ for each } x\in\mathcal{O}(x_0), \tag{1}$$

where φ is controlled comparison function for T at x_0. Then $T^nx_0\to z\in X$ (as $n\to+\infty$). Furthermore z is a fixed point of T if and only if $G(x)=d(x,Tx)$ is T-orbitally lower semi-continuous at z.

Proof. Let $x_0\in X$ be arbitrary, consider the sequence $T^nx_0=x_n$. By making full use of inequality (1), we yield

$$d(x_n,x_{n+1})\le\varphi^n(d(x_0,x_1)). \tag{2}$$

By triangular inequality, for $m>n$ we obtain

$$\begin{aligned}
& d(x_n,x_m)\\
\le\ & \alpha(x_n,x_{n+1})d(x_n,x_{n+1})+\alpha(x_{n+1},x_m)(d(x_{n+1},x_m))\\
\le\ & \alpha(x_n,x_{n+1})d(x_n,x_{n+1})+\alpha(x_{n+1},x_m)\alpha(x_{n+1},x_{n+2})d(x_{n+1},x_{n+2})\\
& +\alpha(x_{n+1},x_m)\alpha(x_{n+2},x_m)d(x_{n+2},x_m)\\
\le\ & \alpha(x_n,x_{n+1})d(x_n,x_{n+1})+\alpha(x_{n+1},x_m)\alpha(x_{n+1},x_{n+2})d(x_{n+1},x_{n+2})\\
& +\alpha(x_{n+1},x_m)\alpha(x_{n+2},x_m)\alpha(x_{n+2},x_{n+3})d(x_{n+2},x_{n+3})\\
& +\alpha(x_{n+1},x_m)\alpha(x_{n+2},x_m)\alpha(x_{n+3},x_m)d(x_{n+3},x_m)\\
& \vdots\\
\le\ & \alpha(x_n,x_{n+1})d(x_n,x_{n+1})+\sum_{i=n+1}^{m-2}\left(\prod_{j=n+1}^{i}\alpha(x_j,x_m)\right)\alpha(x_i,x_{i+1})d(x_i,x_{i+1})\prod_{k=n+1}^{m-1}\alpha(x_k,x_m)d(x_{m-1},x_m).
\end{aligned}$$

Now by using (2) , we have

$$\begin{aligned} d(x_n, x_m) &\leq \alpha(x_n, x_{n+1})\varphi^n(d(x_0, x_1)) + \sum_{i=n+1}^{m-2}\left(\prod_{j=n+1}^{i}\alpha(x_j, x_m)\right)\alpha(x_i, x_{i+1})\varphi^i(d(x_0, x_1)) \\ &\quad + \prod_{k=n+1}^{m-1}\alpha(x_k, x_m)\varphi^{m-1}(d(x_0, x_1)) \\ &\leq \alpha(x_n, x_{n+1})\varphi^n(d(x_0, x_1)) + \sum_{i=n+1}^{m-2}\left(\prod_{j=n+1}^{i}\alpha(x_j, x_m)\right)\alpha(x_i, x_{i+1})\varphi^i(d(x_0, x_1)) \\ &\quad + \left(\prod_{k=n+1}^{m-1}\alpha(x_k, x_m)\right)\alpha(x_{m-1}, x_m)\varphi^{m-1}(d(x_0, x_1)) \\ &= \alpha(x_n, x_{n+1})\varphi^n(d(x_0, x_1)) + \sum_{i=n+1}^{m-1}\left(\prod_{j=n+1}^{i}\alpha(x_j, x_m)\right)\alpha(x_i, x_{i+1})\varphi^i(d(x_0, x_1)) \\ &\leq \alpha(x_n, x_{n+1})\varphi^n(d(x_0, x_1)) + \sum_{i=n+1}^{m-2}\left(\prod_{j=0}^{i}\alpha(x_j, x_m)\right)\alpha(x_i, x_{i+1})\varphi^i(d(x_0, x_1)). \end{aligned}$$

Since the series

$$\sum_{n=1}^{\infty}\varphi^n(d(x_0, x_1))\left(\prod_{i=1}^{n}\alpha(x_i, x_m)\right)\alpha(x_n, x_{n+1})$$

converges for every $m \in \mathbb{N}$. Let

$$S = \sum_{n=1}^{\infty}\varphi^n(d(x_0, x_1))\left(\prod_{i=1}^{n}\alpha(x_i, x_m)\right)\alpha(x_n, x_{n+1}),$$

$$S_n = \sum_{i=0}^{n}\varphi^i(d(x_0, x_1))\left(\prod_{j=0}^{i}\alpha(x_j, x_m)\right)\alpha(x_i, x_{i+1}).$$

Hence for $m > n$ we have

$$d(x_n, x_m) \leq [\alpha(x_n, x_{n+1})\varphi^n(d(x_0, x_1)) + (S_{m-1} - S_n)].$$

Letting $n \to \infty$ we conclude that $\{x_n\}$ is Cauchy sequence. Since X is complete so there exists $z \in X$ such that $x_n = T^n x_0 \to z$. Assume that G is lower semi continuous at $z \in X$, then

$$d(z, Tz) \leq \liminf_{n\to\infty} d(T^n x_0, T^{n+1}x_0) \leq \liminf_{n\to\infty}\varphi^n(d(x_0, x_1)) = 0.$$

Conversely, assume that $Tz = z$ and $x_n \in \mathcal{O}(x)$ with $x_n \to z$. Then

$$G(z) = d(z, Tz) = 0 \leq \liminf_{n\to\infty} G(x_n) = d(T^n x_0, T^{n+1}x_0).$$

■

4 Consequences

In this section,we give some consequences of our main results. We will show that many existing results in the literature can be deduced easily from our theorems. In the following we include an analogue of the theorem of Hicks and Rhoades [12], in the setting of controlled metric type space.

Theorem 3 *Let (X,d) be a complete controlled metric type space and suppose $T : X \to X$ satisfies:*

$$d(Tx, T^2x) \leq kd_\theta(x, Tx) \text{ for all } x \in \mathcal{O}(x_0)$$

where $k \in (0,1)$ is such that for each $x_0 \in X$

$$\sup_{m\geq 1} \lim_{j\to\infty} \frac{\alpha(x_{j+1}, x_{j+2})}{\alpha(x_j, x_{j+1})} \alpha(x_{j+1}, x_m) < \frac{1}{k} \quad \text{where } x_n = T^n x_0.$$

Then, $x_n \to z \in X$. Furthermore z is a fixed point of T if and only if $G(x) = d(x.Tx)$ is T-orbitally lower semi continuous at z.

Proof. Define $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ by $\varphi(t) = kt$. Example 4 implies that φ is controlled comparison function. Hence the result follows from Theorem 2. ∎

Remark 1 *When $\alpha(x, y) = \alpha(y, z) = 1$ for all x, y, z, then Theorem 3 reduces to main result of Hicks and Rhoades ([12, Theorem 1]).*

Definition 8 ([21]) *Let $\alpha : X \times X \to [0,\infty)$. A map $T : X \to X$ is said to be α-admissible if*

$$\alpha(x, y) \geq 1 \text{ implies } \alpha(Tx, Ty) \geq 1$$

for all $x, y \in X$.

Theorem 4 *Let (X,d) be a complete controlled metric type space and suppose $T : X \to X$ satisfies*

$$\alpha(x, y)d(Tx, Ty) \leq \varphi(d(x, y) \text{ for all } x, y \in X, \tag{3}$$

where φ is controlled comparison function for T at x_0. Assume that:

(i) T is α-admissible .

(ii) $\alpha(x_0, Tx_0) \geq 1$ for $x_0 \in X$.

Then $T^n x_0 \to z \in X$. Furthermore $Tz = z$ if and only if $G(x) = d(x, Tx)$ is T-orbitally lower semi-continuous at z.

Proof. From (i) and (ii) we get

$$\alpha(T^n x_0, T^{n+1}x_0) \geq 1, \; n = 1, 2, \cdots .$$

By using the contraction condition (4) we have

$$d(T^n x_0, T^{n+1}x_0) \leq \alpha(T^n x_0, T^{n+1}x_0)d(T^n x_0, T^{n+1}x_0) \leq \varphi(d(T^{n-1}x_0, T^n x_0)).$$

The above inequality is equivalent to (1). Thus all the conditions of Theorem 2 are satisfied. Hence the result follows. ∎

Corollary 2 ([21, Theorem 2.1]) *Let (X,d) be a complete metric space and suppose $T : X \to X$ satisfies*

$$\alpha(x, y)d(Tx, Ty) \leq \varphi(d(x, y) \text{ for all } x, y \in X, \tag{4}$$

where φ is controlled comparison function for T at x_0. Assume that:

(i) T is α-admissible .

(ii) $\alpha(x_0, Tx_0) \geq 1$ for $x_0 \in X$.

(iii) T is continuous

Then T has a fixed point.

Proof. The existence of fixed point follows immediately from Theorem 4 by letting $\alpha(x, y) = 1 = \alpha(y, z)$. ∎

5 Application

In this section, we will show that how our proved results can be used to prove the exixtence of solution of nonlinear integral equation. Consider the following

$$x(t) = f(t) + \int_0^t K(t,u)g(u,x(u))du, \ t \in [0,1], \tag{5}$$

where $f : [0,1] \to \mathbb{R}$, $g : [0,1] \times \mathbb{R} \to \mathbb{R}$ are two bounded continuous functions and $S : [0,1] \times [0,1] \to [0,\infty]$ is a function such that $K(t,.) \in L^1([0,1])$ for all $t \in [0,1]$. Before we state and prove the theorem for the existance of the solution to (5) we have the following simple lemma.

Lemma 1 *Let (X,d) be a controlled metric type space, $T : X \to X$ be a self map, $x_0 \in X$ and*

$$\sup_{m \geq 1} \lim_{j \to \infty} \frac{\alpha(x_{j+1}, x_{j+2})}{\alpha(x_j, x_{j+1})} \alpha(x_{j+1}, x_m) < \frac{1}{k}$$

for $k \in (0,1)$. Suppose that Ψ is comparison function then $\varphi(t) = k\Psi$ is a controller comparison function for T at x_0.

Theorem 5 *Let $T : X \to X$ be the integral operator given by*

$$Tx(t) = f(t) + \int_0^t K(t,u)g(u,x(u))du.$$

Assume that the following conditions hold:

(i) for $x \in X$ and for every $u \in [0,1]$, we have

$$0 \leq g(u,x(u)) - g(u,Tx(u)) \leq \frac{1}{4}\sqrt{\ln(1 + |x(u) - Tx(u)|^2)};$$

(ii) for every $u \in [0,1]$, we have

$$\left\| \int_0^1 K(t,u)du \right\|_\infty < 1.$$

Then T has a fixed point.

Proof. Let X be the set of all continuous real valued functions defined on $[0,1]$. Define $d : X \times X \to [0,\infty)$ by

$$d(x,y) = \|x - y\|_\infty = \sup_{t \in [a,b]} |x(t) - y(t)|^2,$$

then (X,d) is complete controlled metric type space with $\alpha(x,y) = \alpha(y,z) = 2$. Now from condition (ii), we have

$$\begin{aligned}
\left|Tx(t) - T^2x(t)\right|^2 &= \left| \int_0^t K(t,u)[g(u,x(u)) - g(u,Tx(u))]du \right|^2 \\
&\leq \int_0^t |K(t,u)|^2 \left|g(u,x(u)) - g(u,Tx(u))\right|^2 du \\
&\leq \int_0^t |K(t,u)|^2 \frac{\ln(1 + |x(u) - Tx(u)|^2)}{16} du \\
&\leq \frac{\ln(1 + \|x - Tx\|_\infty)}{16}.
\end{aligned}$$

Thus we obtain

$$\left\|Tx - T^2x\right\|_\infty \leq \frac{\ln(1 + \|x - Tx\|_\infty)}{16}.$$

Hence

$$d(Tx, T^2x) \leq k\varphi(d(x, Tx)).$$

where $k = \frac{1}{8}$ and $\varphi(r) = \frac{\ln(1+r)}{2}$. Since φ is comparison function so from Lemma 1, its follows that $k\varphi$ is controlled comparison function. Moreover,

$$\sup_{m\geq 1} \lim_{j\to\infty} \frac{\alpha(x_{j+1}, x_{j+2})}{\alpha(x_j, x_{j+1})}\alpha(x_{j+1}, x_m) = 2$$

for $x_0 \in X$. Thus all the conditions of Theorem 2 are satisfied hence T has fixed point i.e. the integral equation (5) has a solution. ■

Corollary 3 *Let $T : X \to X$ be the integral operator given by*

$$Tx(t) = f(t) + \int_0^t \frac{1}{\Gamma(\alpha)}(t-u)^{\alpha-1} g(u, x(u))du,\ \ t \in [0, 1],\ \ \alpha \in (0, 1),$$

where Γ is the Euler gamma function given by $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1}e^{-t}dt$. Suppose that for $x \in X$ and for every $u \in [0, 1]$, we have

$$0 \leq g(u, x(u)) - g(u, Tx(u)) \leq \frac{\Gamma(\alpha - 1)}{8}\sqrt{\ln(1 + |x(u) - Tx(u)|^2)}.$$

Then T has a fixed point.

The above result is a special case of Theorem 5 for a fractional order integral equation.

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