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On A Nonlocal Schrödinger-Poisson System With Critical Exponent*

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Abstract

This work is concerned with a class of Kirchhoff-Schrödinger-Poisson systems involving the critical Sobolev exponent. By means of the variational method and the concentration compact argument, for each positive integer k, the existence of k pairs of nontrivial solutions is established.

1 Introduction

Let Ω be a bounded smooth domain of $\mathbb{R}^3$. Consider the system

$$\begin{cases} -M\left(\int_\Omega |\nabla u|^2 dx\right)\Delta u + \phi u = \lambda |u|^4 u + f(x,u) & \text{in } \Omega, \\ -\Delta\phi = u^2 & \text{in } \Omega, \\ u = \phi = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\lambda > 0$ and M, f are continuous functions satisfying some conditions which will be given later.

The presence of the nonlocal term $M\left(\int_\Omega |\nabla u|^2 dx\right)$ in (1) causes some mathematical difficulties and so the study of such a class of problems is of much interest. This type of problems are closely related to the following hyperbolic equation

$$\rho\frac{\partial^2 u}{\partial t^2} - \left(\frac{\rho_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx\right)\frac{\partial^2 u}{\partial x^2} = 0, \tag{2}$$

which was proposed by Kirchhoff [13] as a model to describe the transversal vibrations of a stretched string by considering the subsequent change in string length during the vibrations. Recently, the Kirchhoff type problems with or without critical growth have been investigated by many researchers, we cite here [1, 3, 8, 9, 10, 11, 12, 16, 21].

When $M \equiv 1$, system (1) reduces to Schrödinger-Poisson system. Due to its importance in many physical applications (see[5, 18]), it has received much attention in the recent years. Many interesting papers have been devoted to the existence or multiplicity of solutions for (1), when $\lambda \in \{0, 1\}$ see for instance [7, 14, 15, 20]. In [4], Batkam and Júnior studied the following Kirchhoff-Schrödinger-Poisson system

$$\begin{cases} -\left(a + b\int_\Omega |\nabla u|^2 dx\right)\Delta u + \mu\phi u = f(x,u) & \text{in } \Omega, \\ -\Delta\phi = u^2 & \text{in } \Omega, \\ u = \phi = 0 & \text{on } \partial\Omega, \end{cases} \tag{3}$$

with $\mu = 1$. The authors proved that (3) has at least three solutions. Furthermore, in case f is odd with respect to u, they established the existence of unbounded sequence of solutions. Under the general singular assumptions on f and by using the variational arguments, Li et al. [17], proved the existence and the uniqueness of solutions for (3). In the critical case, to my knowledge, the existence of multiple solutions for

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system (1) has not investigated until now. Motivated by the above results, in this note we are interested in finding multiple solutions by using the variational method and the concentration compact argument.

Throughout the paper, we assume the following conditions on the Kirchhoff function and the nonlinearity:

(m_1) $M : [0, +\infty) \to [M_0, +\infty)$ is continuous for some $M_0 > 0$;

(m_2) $2\widehat{M}(t) \geq M(t)t$ for all $t \geq 0$, where $\widehat{M}(t) = \int_0^t M(s)ds$;

(m_3) There exist $a > 0$ and $b \geq 0$ such that $M(t) \leq a + bt$ for all $t \geq 0$;

(f_1) $f(x,t) \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$ is odd in t;

(f_2) $f(x,t) = o(|t|^5)$ as $|t| \to \infty$, uniformly in Ω;

(f_3) There exists an open $\Omega_0 \subset \Omega$ of positive measure such that $\liminf_{|t|\to\infty} \frac{F(x,t)}{t^4} = 0$, uniformly in Ω_0, where $F(x,t) := \int_0^t f(x,s)ds$;

(f_4) There exist $q \in [0,2)$ and $a_1, a_2 > 0$ such that

$$F(x,t) - \frac{1}{4}f(x,t)t \leq a_1 + a_2|t|^q \quad \text{for all } x \in \Omega \text{ and } t \in \mathbb{R};$$

(f_5) There exist $r \in (2,6)$ and $b_1, b_2 > 0$ such that

$$F(x,t) \leq b_1 + b_2|t|^r \quad \text{for all } x \in \Omega \text{ and } t \in \mathbb{R}.$$

(f_6) $\xi(x) := \limsup_{t\to 0} \frac{F(x,t)}{t^2}$ is such that $\max\{0, \xi(x)\} =: \xi^+(x) \in L^\infty(\Omega)$.

The main result is the following theorem.

Theorem 1 *Suppose that $(m_1) - (m_3)$, and $(f_1) - (f_4)$ hold, furthermore, one of conditions (f_5) or (f_6) is verified. Then for each positive integer k, there exists $\lambda_k^* > 0$ such that system (1) admits at least k pairs of nontrivial solutions for all $\lambda \in (0, \lambda_k^*)$.*

2 Auxiliary Results

We look for solutions in the Sobolev space $H_0^1(\Omega)$ with the norm $||u||^2 = \int_\Omega |\nabla u|^2 dx$. Denote

$$|u|_p = \left(\int_\Omega |u|^p dx \right)^{\frac{1}{p}} \quad \text{for } u \in L^p(\Omega).$$

Thanks to the Lax-Milgram theorem, for any $u \in H_0^1(\Omega)$ the Poisson problem $-\Delta\phi = u^2$ in Ω, $\phi = 0$ on $\partial\Omega$, has a unique solution $\phi_u \in H_0^1(\Omega)$. Moreover, we recall the following lemma (see [15]).

Lemma 1 *Let $u \in H_0^1(\Omega)$. Then*

1. *$\phi_u(x) \geq 0, x \in \Omega$;*
2. *For all $t \geq 0$, $\phi_{tu} = t^2\phi_u$;*
3. *There exists $A_\phi > 0$ such that $\int_\Omega \phi_u u^2 dx = \int_\Omega |\nabla\phi_u|^2 dx \leq A_\phi ||u||^4$;*
4. *If $u_n \rightharpoonup u$ in $H_0^1(\Omega)$, then $\phi_{u_n} \to \phi_u$ in $H_0^1(\Omega)$.*

By substituting ϕ_u, system (1) reduces to following problem

$$\begin{cases} -M\left(\int_\Omega |\nabla u|^2 dx\right)\Delta u + \phi_u u = \lambda |u|^4 u + f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4}$$

The energy functional associated to (4) is given by

$$I_\lambda(u) = \frac{1}{2}\widehat{M}\left(\int_\Omega |\nabla u|^2 dx\right) + \frac{1}{4}\int_\Omega \phi_u u^2 dx - \frac{\lambda}{6}\int_\Omega |u|^6 dx - \int_\Omega F(x,u)dx.$$

By assumptions of Theorem 1, $I_\lambda \in C^1(H_0^1(\Omega))$ and for all $u, v \in H_0^1(\Omega)$

$$\langle I'_\lambda(u), v\rangle = M\left(\int_\Omega |\nabla u|^2 dx\right)\int_\Omega \nabla u \nabla v dx + \int_\Omega \phi_u uv dx - \lambda\int_\Omega |u|^4 uv dx - \int_\Omega f(x,u)v dx. \tag{5}$$

Lemma 2 *Suppose that $(m_1)-(m_2)$, $(f_1)-(f_2)$, (f_4) and one of conditions (f_5) or (f_6) hold. Then, for any $\beta > 0$, there is $\Lambda > 0$ such that for all $\lambda \in (0, \Lambda)$, the functional I_λ satisfies the $(PS)_c$ condition at every level $c < \beta$.*

Proof. Let $\{u_n\} \subset H_0^1(\Omega)$ be a sequence such that $I_\lambda(u_n) \to c < \beta$ and $I'_\lambda(u_n) \to 0$.
We claim that $\{u_n\}$ is bounded. Indeed, we have $\frac{|t|^q}{t^6} \to 0$ as $|t| \to \infty$, thus for given $\varepsilon > 0$, there exists $C_\varepsilon >$ such that

$$|t|^q \le \varepsilon t^6 + C_\varepsilon \text{ for all } t \in \mathbb{R}. \tag{6}$$

Therefore assumptions (m_2) and (f_4) yield

$$\begin{aligned} c + o_n(1) + o_n(1)||u_n|| &\ge I_\lambda(u_n) - \frac{1}{4}\langle I'_\lambda(u_n), u_n\rangle \\ &\ge \frac{\lambda}{12}|u_n|_6^6 - \int_\Omega \left(F(x,u_n) - \frac{1}{4}f(x,u_n)u_n\right)dx \\ &\ge \frac{\lambda}{12}|u_n|_6^6 - a_1|\Omega| - a_2\int_\Omega |u_n|^q dx \\ &\ge \left(\frac{\lambda}{12} - \varepsilon a_2\right)|u_n|_6^6 - (a_1 + a_2 C_\varepsilon)|\Omega|. \end{aligned}$$

Thus, choose $\varepsilon < \frac{\lambda}{12a_2}$, for some $c_1, c_2 > 0$ and for n large enough

$$|u_n|_6^6 \le c_1 + c_2||u_n||. \tag{7}$$

On the other hand, by (m_1), (f_5) and Lemma 1 we have

$$\frac{M_0}{2}||u_n||^2 \le \frac{1}{2}\widehat{M}(||u_n||^2) \le \frac{\lambda}{6}|u_n|_6^6 + b_1|\Omega| + b_2|u_n|_r^r + c + o_n(1).$$

Since $r \in (2,6)$, we can find $C'_\varepsilon > 0$ such that $|t|^r \le \varepsilon t^6 + C'_\varepsilon$ for all $t \in \mathbb{R}$. Then

$$\frac{M_0}{2}||u_n||^2 \le \left(\frac{\lambda}{6} + \varepsilon b_2\right)|u_n|_6^6 + (b_1 + b_2 C'_\varepsilon)|\Omega| + c + o_n(1).$$

It follows from (7) that for some $c_3, c_4 >$ and for n large enough

$$\frac{M_0}{2}||u_n||^2 \le c_3||u_n|| + c_4.$$

This shows that $\{u_n\}$ is bounded in $H_0^1(\Omega)$.

Now the hypothesis (f_5) will be replaced by condition (f_6). By (f_2) we have

$$\limsup_{|t|\to\infty}\frac{|F(x,t)|}{t^6}=0 \text{ uniformly in } \Omega.$$

This and (f_6) imply that for any $\varepsilon>0$, there exists $C_\varepsilon>$ such that

$$F(x,t)\le(|\xi^+|_\infty+\varepsilon)t^2+C_\varepsilon t^6. \tag{8}$$

Therefore, by similar arguments as above, we show that $\{u_n\}$ is bounded. Then, up to subsequence for some $u\in H_0^1(\Omega)$,

$$\begin{aligned}
&u_n\rightharpoonup u \text{ in } H_0^1(\Omega),\\
&u_n\to u \text{ a.e. in } \Omega,\\
&u_n\to u \text{ in } L^s(\Omega) \text{ for all } s\in[1,6),\\
&|\nabla u_n|^2\rightharpoonup\mu \text{ weakly in the sense of measures,}\\
&u_n^6\rightharpoonup\nu \text{ weakly in the sense of measures,}
\end{aligned}\tag{9}$$

where μ and ν are nonnegative bounded measures on $\overline{\Omega}$. Applying concentration compact result due to Lions [19], we can find at most countable index set J and elements $\{x_j\}_{j\in J}$ of $\overline{\Omega}$ such that

$$\begin{cases}\nu=|u|^6+\sum_{j\in J}\nu_j\delta_{x_j}, & \nu_j>0,\\ \mu\ge|\nabla u|^2+\sum_{j\in J}\mu_j\delta_{x_j}, & \mu_j>0\\ S_*\nu_j^{1/3}\le\mu_j & \text{for all } j\in J.\end{cases}\tag{10}$$

where

$$S_*:=\inf_{u\in H_0^1(\Omega)\setminus\{0\}}\frac{||u||^2}{|u|_6^{\frac{1}{3}}}.$$

We claim that $\nu_j\ge\left(\frac{M_0S_*}{\lambda}\right)^{\frac{3}{2}}$ for all $j\in J$. Let $j\in J$ be fixed and for an arbitrary $\varepsilon>0$, choosing ψ_ε of $C_0^\infty(\mathbb{R}^3)$ such that $0\le\psi_\varepsilon\le1$,

$$\psi_\varepsilon(x)=\begin{cases}1 & \text{if } x\in B(x_j,\varepsilon),\\ 0 & \text{if } x\notin B(x_j,2\varepsilon),\end{cases}$$

and $|\nabla\psi_\varepsilon|_\infty\le\frac{2}{\varepsilon}$. Clearly $\langle I'_\lambda(u_n),\psi_\varepsilon u_n\rangle=o_n(1)$, that is

$$\begin{aligned}
&M\left(\int_\Omega|\nabla u_n|^2dx\right)\left(\int_\Omega u_n\nabla u_n\nabla\psi_\varepsilon dx+\int_\Omega\psi_\varepsilon|\nabla u_n|^2dx\right)+\int_\Omega\phi_{u_n}u_n^2\psi_\varepsilon dx\\
=\;&\lambda\int_\Omega|u_n|^6\psi_\varepsilon dx+\int_\Omega f(x,u_n)u_n\psi_\varepsilon dx+o_n(1).
\end{aligned}\tag{11}$$

By the Hölder inequality, we have

$$\begin{aligned}
\limsup_{n\to\infty}\left|\int_\Omega u_n\nabla u_n\nabla\psi_\varepsilon dx\right| &\le \limsup_{n\to\infty}\left(\int_\Omega|\nabla u_n|^2\right)^{\frac{1}{2}}\left(\int_\Omega|u_n|^2|\nabla\psi_\varepsilon|^2dx\right)^{\frac{1}{2}}\\
&\le C\left(\int_\Omega|u|^2|\nabla\psi_\varepsilon|^2dx\right)^{\frac{1}{2}}\\
&\le C\left(\int_{B(x_j,\varepsilon)}|\nabla\psi_\varepsilon|^3dx\right)^{\frac{1}{3}}\left(\int_{B(x_j,\varepsilon)}|u|^6dx\right)^{\frac{1}{6}}\\
&\le C|\nabla\psi_\varepsilon|_\infty w_6^{\frac{1}{3}}\varepsilon\left(\int_{B(x_j,\varepsilon)}|u|^6dx\right)^{\frac{1}{6}}\to0 \text{ as } \varepsilon\to0,
\end{aligned}\tag{12}$$

where w_6 is the volume of $B(0,1)$. In view of Lemma 1, $\phi_{u_n} \to \phi_u$ in $H_0^1(\Omega)$, thus

$$\lim_{n\to\infty} \int_\Omega \phi_{u_n} u_n^2 dx = \int_\Omega \phi_u u^2 dx.$$

Since ψ_ε is bounded in Ω,

$$\lim_{n\to\infty} \int_\Omega \phi_{u_n} u_n^2 \psi_\varepsilon dx = \int_\Omega \phi_u u^2 \psi_\varepsilon dx \to 0 \quad \text{as } \varepsilon \to 0. \tag{13}$$

Using (f_1) and that $f(x,u_n)u_n\psi_\varepsilon \to f(x,u)u\psi_\varepsilon$ a.e. in Ω, we have by compactness Lemma of Strauss [6]

$$\lim_{n\to\infty} \int_\Omega f(x,u_n)u_n\psi_\varepsilon dx = \int_\Omega f(x,u)u\psi_\varepsilon dx \to 0 \quad \text{as } \varepsilon \to 0. \tag{14}$$

Letting $n \to \infty$ and $\varepsilon \to 0$ in (11), from (9) and (12)–(14) we obtain $M_0\mu_j \le \lambda\nu_j$. Therefore (10) implies

$$\nu_j \ge \left(\frac{M_0 S_*}{\lambda}\right)^{\frac{3}{2}}.$$

Now we prove that J is empty. Suppose by contradiction that there is $j \in J$. Then

$$\begin{aligned} \beta + o_n(1) &> c + o_n(1) = I_\lambda(u_n) - \frac{1}{4}\langle I'_\lambda(u_n), u_n\rangle \\ &\ge \frac{\lambda}{12}|u_n|_6^6 - a_1|\Omega| - a_2\int_\Omega |u_n|^q dx \\ &\ge \frac{\lambda}{12}|u_n|_6^6 - a_1|\Omega| - a_2|\Omega|^{\frac{6-q}{6}}\left(\int_\Omega u_n^6 dx\right)^{\frac{q}{6}} \end{aligned}$$

therefore letting $n \to +\infty$ and using (9), we get

$$\frac{\lambda}{12}\nu(\overline{\Omega}) \le \beta + a_1|\Omega| + a_2|\Omega|^{\frac{6-q}{6}}\nu(\overline{\Omega})^{\frac{q}{6}}. \tag{15}$$

If $\nu(\overline{\Omega}) > 1$, from the last inequality, we can write

$$\nu(\overline{\Omega}) \le \frac{12}{\lambda}\left(\beta + a_1|\Omega| + a_2|\Omega|^{\frac{6-q}{6}}\right)\nu(\overline{\Omega})^{\frac{q}{6}},$$

hence

$$\nu(\overline{\Omega}) \le \left(\frac{12\left(\beta + a_1|\Omega| + a_2|\Omega|^{\frac{6-q}{6}}\right)}{\lambda}\right)^{\frac{6}{6-q}} \to 0 \quad \text{as } \lambda \to 0^+.$$

If $\nu(\overline{\Omega}) \le 1$, then

$$\nu(\overline{\Omega}) \le \frac{12\left(\beta + a_1|\Omega| + a_2|\Omega|^{\frac{6-q}{6}}\right)}{\lambda} \quad \text{as } \lambda \to 0^+.$$

Since $\max\left(1, \frac{6}{6-q}\right) < \frac{3}{2}$, in both above cases, there exists $\Lambda > 0$ such

$$\nu_j \le \nu(\overline{\Omega}) < \left(\frac{M_0 S_*}{\lambda}\right)^{\frac{3}{2}} \quad \text{for all } \lambda \in (0,\Lambda).$$

which is impossible and hence $J = \emptyset$ for all $\lambda \in (0,\Lambda)$. It follows that $u_n \to u$ in $L^6(\Omega)$, thus

$$\lim_{n\to\infty} \int_\Omega u_n^5(u_n - u)dx = 0. \tag{16}$$

On the other hand, we see that

$$\lim_{n\to\infty}\int_\Omega f(x,u_n)(u_n-u)dx=0. \tag{17}$$

According to Lemma 1, we have

$$\lim_{n\to\infty}\int_\Omega \phi_{u_n}u_n(u_n-u)dx=0. \tag{18}$$

Since $\langle I'(u_n),u_n-u\rangle=o_n(1)$, by (m_1) and (16)-(18) we deduce that

$$\lim_{n\to\infty}\int_\Omega \nabla u_n\nabla(u_n-u)dx=0.$$

Similarly, we have

$$\lim_{n\to\infty}\int_\Omega \nabla u\nabla(u_n-u)dx=0.$$

So that $u_n\to u$ in $H_0^1(\Omega)$. ■

3 Proof of Theorem 1

To this end, we need to ensure that I_λ satisfies the conditions of the following version of Symmetric Mountain Pass theorem [2].

Theorem 2 *Let $H=V\oplus W$ be a real Banach space with $dimV<\infty$. Assume that $I\in C^1(H,\mathbb{R})$ is an even functional verifying $I(0)=0$ and*

(i) there exist $\alpha,\rho>0$ such that

$$\inf_{u\in\partial B_\rho(0)\cap W} I(u)\geq\alpha;$$

(ii) there exists a subspace $E\subset H$ such that $dimV<dimE$ and

$$\max_{u\in E} I(u)\leq\beta \ \ \textit{for some } \beta>0;$$

(iii) the functional I satisfies $(PS)_c$ for every $c\in(0,\beta)$.

Then I admits at least $dimE-dimV$ nontrivial critical points.

Let $\{e_1,e_2,....\}$ be a system of the normalized eigenfunctions of $(-\Delta,H_0^1(\Omega))$. Consider the k-subspace

$$V_k=\{e_1,e_2,...,e_k\}.$$

Then $H_0^1(\Omega)=V_k\oplus V_k^\perp$. Moreover, for any $s\in[2,6)$ and $\sigma>0$ there exists $k_{s,\sigma}\in\mathbb{N}$ such that for all $k\geq k_{s,\sigma}$

$$|u|_s^s\leq\sigma||u||^s \text{ for all } u\in V_k^\perp. \tag{19}$$

Lemma 3 *Assume that (m_1), $(f_1)-(f_2)$ and one of conditions (f_5) or (f_6) hold. Then, there exist $\sigma,\rho=\rho(\sigma),\alpha,\lambda_*>0$ such that $\forall k\geq k_{r,\sigma}$ and $\forall\lambda\in(0,\lambda_*)$,*

$$\inf_{u\in\partial B_\rho(0)\cap V_k^\perp} I_\lambda(u)\geq\alpha.$$

Proof. By (f_5), (m_1) and (19), for all $u \in V_k^{\perp}$,

$$\begin{aligned} I_\lambda(u) &\geq \frac{M_0}{2}||u||^2 - \frac{\lambda}{6}|u|_6^6 - b_1|\Omega| - b_2|u|_r^r \\ &\geq \left(\frac{M_0}{2} - b_2\sigma||u||^{r-2}\right)||u||^2 - \frac{\lambda c_5}{6}||u||^6 - b_1|\Omega|. \end{aligned}$$

Let $||u|| = \rho = \rho(\sigma) := \left(\frac{M_0}{4b_2\sigma}\right)^{\frac{1}{r-2}}$. Then

$$I_\lambda(u) \geq \frac{M_0}{4}\rho^2 - \frac{\lambda c_5}{6}\rho^6 - b_1|\Omega|.$$

Since $\rho(\sigma) \to +\infty$ as $\sigma \to 0^+$, we can find $\sigma > 0$ and so $\rho > 0$ such that $b_1|\Omega| \leq \frac{M_0}{8}\rho^2$. Therefore

$$I_\lambda(u) \geq \frac{M_0}{8}\rho^2 - \frac{\lambda c_5}{6}\rho^6$$

and hence there exists λ_* such that for all $\lambda \in (0, \lambda_*)$

$$I_\lambda(u) \geq \alpha > 0 \ \text{ for all } u \in \partial B(0,\rho) \cap V_k^{\perp}. \tag{20}$$

Now assume that (f_6) holds. Then, from (m_1) and (8) we have

$$I_\lambda(u) \geq \frac{M_0}{2}||u||^2 - \left(\frac{\lambda}{6} + C_\varepsilon\right)|u|_6^6 - (|\xi^+|_\infty + \varepsilon)|u|_2^2.$$

Arguing as in above, also (20) follows in this case. ■

Lemma 4 *Assume that (m_3), (f_1) and (f_3) hold. Then, for each positive integer k, there is a k-subspace $V_k^0 \subset H_0^1(\Omega)$ and $\beta_k > 0$ such that for all $\lambda > 0$.*

$$\max_{u \in V_k^0} I_\lambda(u) < \beta_k.$$

Proof. Let $\{e_1^0, e_2^0, ...\}$ be a system of eigenfunctions of $(-\Delta, H_0^1(\Omega_0))$ and consider the k-subspace

$$V_k^0 = \{e_1^0, e_2^0, ..., e_k^0\} \subset H_0^1(\Omega) \ \text{ with } e_i^0 = 0 \text{ in } \Omega\backslash\Omega_0.$$

Since $dim V_k^0 < \infty$, there exists $C_k > 0$ such that

$$|u|_4^4 \geq C_k||u||^4 \ \text{ for all } u \in V_k^0. \tag{21}$$

By (f_3) and the continuity of F on $\overline{\Omega} \times \mathbb{R}$, for $\varepsilon > \frac{b+A_\phi}{4C_k}$ there exists $C_\varepsilon > 0$ such that

$$F(x,t) \geq \varepsilon t^4 - C_\varepsilon \ \text{ for all } (x,t) \in \Omega_0 \times \mathbb{R}.$$

Combining this last inequality with (m_3), (21) and Lemma 1, we get

$$I_\lambda(u) \leq \frac{a}{2}||u||^2 - \left(\varepsilon C_k - \frac{b + A_\phi}{4}\right)||u||^4 + C_\varepsilon|\Omega| \to -\infty \ \text{ as } ||u|| \to \infty.$$

Therefore, there exists $\beta_k > 0$ such that

$$\max_{u \in V_k^0} I_\lambda(u) \leq \beta_k \ \text{ for all } \lambda > 0.$$

■

Proof of Theorem 1. Obviously I_λ is an even functional and $I_\lambda(0) = 0$. Let k be a positive integer. In view of Lemma 3, we can choose $k_0 \in \mathbb{N}$ sufficiently large, $V = V_{k_0}$ and $W = V_{k_0}^\perp$ such that $H_0^1(\Omega) = V \oplus W$ and condition Theorem 2 (i) holds for all $\lambda \in (0, \lambda_*)$. By Lemma 4, for the positive integer $k + k_0$, there exists a subspace $V_{k+k_0}^0 \subset H_0^1(\Omega)$ with $dim V_{k+k_0}^0 = k + k_0$ and $\beta_k > 0$ such that condition Theorem 2 (ii). follows. According to lemma 2, there exists Λ_k such that for all $\lambda \in (0, \Lambda_k)$, I_λ verifies $(PS)_c$ for all $c < \beta_k$. Let $\lambda_k^* := \min(\Lambda_k, \lambda_*)$. Then, by applying Theorem 2, I_λ has $k + k_0 - k_0 = k$ nontrivial critical points, for all $\lambda \in (0, \lambda_k^*)$. ■

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References

[1] C. O. Alves, F. J. S. A. Corrêa and G.M. Figueiredo, On class of nonlocal elliptic problem with critical growth, Differ. Equ. Appl., 2(2010), 409–417.

[2] A. Ambrosetti and P. H. Rabinowitz, Dual variational methods in critical ponit theory and applications, J. Funct. Anal., 14(1973), 349–381.

[3] A. Azzollini, The elliptic Kirchhoff equation in $\mathbb{R}^N$ perturbed by a local nonlinearity, Differ. Integral Equ., 25(2012), 543–554.

[4] C. J. Batkam and J. R. S. Júnior, Schrödinger-Kirchhoff-Poisson type systems, Commun. Pure Appl. Anal., 15(2016), 429–444.

[5] I. Catto and P. Lions, Binding of atoms and stability of molecules in Hartree and Thomas-Fermi type theories. Part 1: A necessary and sufficient condition for the stability of general molecular system, Comm. Partial Differential Equations, 17(1992), 1051–1110.

[6] J. Chabrowski, Weak Convergence Methods for Semilinear Elliptic Equations, World Scientific Publishing Co. Inc. River Edge, NJ, 1999.

[7] J. Chen, Multiple positive solutions of a class of non autonomous Schrödinger-Poisson systems, Nonlinear Anal. Real World Appl., 21(2015), 13–26.

[8] Y. Duan, X. Sun and J. F. Liao, Multiplicity of positive solutions for a class of critical Sobolev exponent problems involving Kirchhoff-type nonlocal term, Comput. Math. Appl., 75(2018), 4427–4437.

[9] G. M. Figueiredo, Existence of a positive solution for a Kirchhoff problem type with critical growth via truncation argument, J. Math. Anal. Appl., 401(2013), 706–713.

[10] G. M. Figueiredo and J. R. S. Junior, Multiplicity of solutions for a Kirchhoff equation with subcritical and critical growth, Differ. Integral Equ., 25(2012), 853–868.

[11] M. F. Furtado, L.D. de Oliveira,and J. P. P. da Silva, Multiple solutions for a Kirchhoff equation with critical growth, Z. Angew. Math. Phys., 70(2019), 1–15.

[12] A. Hamydy, M. Massar and N. Tsouli, Existence of solutions for p-Kirchhoff type problems with critical exponent, Electron. J. Differ. Equ., 105(2011), 1–8.

[13] G. Kirchhoff, Mechanik. Teubner, Leipzig 1883.

[14] S. Khoutir and H. Chen, Multiple nontrivial solutions for a nonhomogeneous Schrödinger-Poisson system in $\mathbb{R}^3$, Electron. J. Qual. Theo. Differ. Equ., 28(2017), 1–17.

[15] C. Y. Lei and J. F. Liao, Multiple positive solutions for Schrödinger-Poisson system involving singularity and critical exponent, Math. Methods Appl. Sci., 42(2019), 1–14.

[16] H. Y. Li and J. F. Liao, Existence and multiplicity of solutions for superlinear Kirchhoff-type equations with critical Sobolev exponent in $\mathbb{R}^N$, Comput. Math. Appl., 72(2016), 2900–2907.

[17] F. Li, Z. Song and Q. Zhang, Existence and uniqueness results for Kirchhoff-Schrödinger-Poisson system with general singularity, Appl. Anal., 96(2017), 2906–2916.

[18] E. H. Lìeb, Thomas-Fermi and related theories and molecules, Rev. Modern Phys., 53(1981), 603–641.

[19] P. L. Lions, The concentration-compactness principle in the calculus of variations, The limit case, part 1, 2, Rev. Mat. Iberoamericana, 1(1985), 145–201, 45–121.

[20] A. Mao, L. Yang, A. Qian and S. Luan, Existence and concentration of solutions of Schrödinger-Poisson system, Appl. Math. Lett., 68(2017), 8–12.

[21] D. Naimen, Positive solutions of Kirchhoff type elliptic equations involving a critical Sobolev exponent, NoDEA Nonlinear Differ. Equ. Appl., 21(2014), 885–914.