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Classifications Of Conformal Rotational Surfaces In Euclidean 3-Space*

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Abstract

In this paper the conformal rotational surfaces are classified under the conditions $\Delta^{I,II}\Psi = A\Psi$, where $\Delta^{I,II}$ are the Laplace operators with respect to the first and the second fundamental forms and A is a real 3×3 matrix.

1 Introduction

Surfaces of revolution constitute the most easily recognized surface class. The use of surfaces of revolution is fundamental in physics and engineering. Since ancient times, surfaces of revolution are not only well known also common in geometric modelling, because they can be found everywhere in nature. For instance, in human artefacts, in technical practise and also in mathematics. Moreover, many objects from everday life such as cans, table glasses and furniture legs. They are examples of surfaces of revolution. Also, the process of lathing wood produces surfaces of revolution by its very nature ([1, 24]).

Conformality means the Gauss map maintains infinitely small rectangular proportions. If we plot a surface in the usual way with a sufficiently fine mesh, then it is divided into lots of small, thereabout rectangular shapes. The ratios of the lengths of the sides of these rectangles, and also the angles between the sides must be preserved by the Gauss map ([10]).

Mladenov reproduced a few explicit formulas on condition that conformal coordinates of the axially symmetric constant mean curvature surfaces introduced by Delaunay and their duals. Also, he gives new examples in a long line of research related with finding isothermic immersions of surfaces and their duals ([18]). Moreover, Ragan studied the properties and the relationships between isothermal, harmonic and characteristic coordinate systems ([20]). Fernandez debated some new results about conformal surface parametrizations defined by very simple functions. Conformal surfaces of revolution in the different ambient spaces are studied by Lee, Zarske, Martin and Lambert. ([13, 14, 15, 16]).

The notion of finite type immersion of submanifolds of a Euclidean space has been used in classifying and characterizing the well known Riemannian submanifolds. The problem of classifying the finite type surfaces in the 3-dimensional Euclidean space $\mathbb{E}^3$ is posed by Chen. A Euclidean submanifold is said to be of Chen finite type if its coordinate functions are a finite sum of eigenfunctions of its Laplacian Δ. Further, the notion of finite type can be extended to any smooth function on a submanifold of a Euclidean space or a pseudo-Euclidean space. The theory of submanifolds of finite type has been studied by many geometers ([6, 12]).

Let $\Psi : S \to \mathbb{E}^m$ be an isometric immersion of a connected n-dimensional manifold in the m-dimensional Euclidean space $\mathbb{E}^m$. Let us denote by H and Δ the mean curvature and the Laplacian of S with respect to the Riemannian metric on S induced from that of $\mathbb{E}^m$, respectively. The submanifolds in $\mathbb{E}^m$ satisfying $\Delta\Psi = \lambda\Psi$, that is, all coordinate functions are eigenfunctions of the Laplacian with the same eigenvalue

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$\lambda \in \mathbb{R}$ are either the minimal submanifolds of $\mathbb{E}^m$ or the minimal submanifolds of hypersphere $\mathbb{S}^{m-1}$ in $\mathbb{E}^m$ was proved by Takahashi ([4, 25]).

As an extension of Takahashi theorem, In ([10]) hypersurfaces in $\mathbb{E}^m$ whose coordinate functions are eigenfunctions of the Laplacian, but not necessarily associated with the same eigenvalue was studied by Garay. He considered hypersurfaces in $\mathbb{E}^m$ satisfying the condition

$$\Delta x = Ax, \tag{1}$$

where $A \in Mat\,(m, \mathbb{R})$ is an $m \times m$- diagonal matrix, and proved that such hypersurfaces are minimal $(H = 0)$ in $\mathbb{E}^m$and open pieces of either round hyperspheres or generalized right spherical cylinders. Related to this, Dillen, Pas and Verstraelen ([7]) investigated surfaces in $\mathbb{E}^3$ whose immersions satisfy the condition

$$\Delta x = Ax + B,$$

where $A \in Mat\,(3, \mathbb{R})$ is a 3×3-real matrix and $B \in \mathbb{R}^3$. In other words, each coordinate function is of 1-type in the sense of Chen ([6]). The notion of an isometric immersion Ψ is naturally extended to smooth functions on submanifolds of Euclidean space or pseudo-Euclidean space. The most natural one of them is the Gauss map of the submanifold. In particular, if the submanifold is a hypersurface, the Gauss map can be identified with the unit normal vector field to it ([4, 25]). Dillen, Pas and Verstraelen ([8]) studied surfaces of revolution in the three dimensional Euclidean space $\mathbb{E}^3$ such that its Gauss map G satisfies the condition

$$\Delta G = AG, \tag{2}$$

where $A \in Mat\,(3, \mathbb{R})$. The surfaces of revolution satisfying the condition (2) in the three dimensional Minkowski space $\mathbb{E}^3_1$ was completely classified by Choi ([5]). Surfaces of revolution and helicoidal surfaces satisfying (1) in Minkowski 3-space was classified by the authors ([2, 21]). The authors ([13]) classified surfaces of revolution satisfying

$$\Delta^{III} r = Ar$$

in Lorentz-Minkowski 3-space. The main purpose of this paper is to complete the classification of the conformal rotational surfaces in $\mathbb{E}^3$ in terms of the position vector field and the Laplacian operators.

2 Preliminaries

Let $\mathbb{E}^3$ be a Euclidean 3-space with the scalar product given by

$$\langle , \rangle = dx^2 + dy^2 + dz^2,$$

where (x, y, z) is a rectangular coordinate system of $\mathbb{E}^3$, the norm of a vector $V \in \mathbb{E}^3$ is given by

$$\|V\| = \sqrt{\langle V, V \rangle}.$$

If $V = (v_1, v_2, v_3)$ and $W = (y_1, y_2, y_3)$ are arbitrary vectors in $\mathbb{E}^3$, the vector product of V and W is given by

$$V \wedge W = (v_2 w_3 - v_3 w_2, v_3 w_1 - v_1 w_3, v_1 w_2 - v_2 w_1).$$

Let $S : \Psi := \Psi(u, v)$ be a surface in Euclidean 3-space. The unit normal vector field of S can be defined by

$$U = \frac{\Psi_u \wedge \Psi_v}{\|\Psi_u \wedge \Psi_v\|}.$$

The first fundamental form I of the surface S is

$$I = E du^2 + 2F du dv + G dv^2,$$

with the coefficients

$$E=\langle \Psi_u,\Psi_u\rangle\,,\quad G=\langle \Psi_v,\Psi_v\rangle\,,\quad F=\langle \Psi_u,\Psi_v\rangle\,.$$

The second fundamental form II of the surface S is given by

$$II=Ldu^2+2Mdudv+Ndv^2,$$

with the coefficients

$$L=\langle \Psi_{uu},U\rangle\,,\quad N=\langle \Psi_{vv},U\rangle\,,\quad M=\langle \Psi_{uv},U\rangle\,.$$

Under this parametrization of the surface S, the Gaussian curvature K and the mean curvature H are given by

$$K=\frac{LN-M^2}{EG-F^2}\quad\text{and}\quad H=\frac{EN+GL-2FM}{2\left(EG-F^2\right)},$$

respectively.

Now we introduce the notion of isothermal coordinates which is a useful tool in differential geometry.

Definition 1 *Let S be a surface in $\mathbb{R}^3$. Then local coordinates $(u,v):D\subset S\to\mathbb{R}^2$ on S are said to be isothermal coordinates if there exists a strictly positive function, called the dilation, $\lambda:U\to\mathbb{R}$ such that*

$$E=\langle \Psi_u,\Psi_u\rangle=\lambda^2\left(u,v\right)=\langle \Psi_v,\Psi_v\rangle=G,\quad F=\langle \Psi_u,\Psi_v\rangle\,. \tag{3}$$

These are also called conformal coordinates ([10]).

Equations (3) implies that a surface parametrization, as a conformal map from $\mathbb{R}^2$ to the ambient space, preserves the angles. The tangent vectors to the cartesian coordinates on the plane, $\{\partial_u,\partial_v\}$ are transformed by the parametrization conformally into surface tangent vectors $\{\Psi_u,\Psi_v\}$ with the same module and orthogonal between them on each surface point. The Laplacian operator of Ψ respect to the first fundamental form on S satisfies

$$\Delta\Psi=2\lambda^2HU. \tag{4}$$

Corollary 1 *An isothermal regular parametrized surface $\Psi:D\to S$ is minimal if and only if the coordinate functions of Ψ are harmonic functions on D* ([10]).

For a detailed proof of this and other conformal transformation properties see ([1, 3, 9, 10, 11, 19, 20, 22]). The Laplacian operators Δ^I and Δ^{II} on S according to local coordinates $\{u,v\}$ of S are defined by ([3, 4, 5, 7, 10])

$$\Delta^I\Psi=-\frac{1}{\sqrt{|EG-F^2|}}\left[\frac{\partial}{\partial u}\left(\frac{G\Psi_u-F\Psi_v}{\sqrt{|EG-F^2|}}\right)-\frac{\partial}{\partial v}\left(\frac{F\Psi_u-E\Psi_v}{\sqrt{|EG-F^2|}}\right)\right],$$

$$\Delta^{II}\Psi=-\frac{1}{\sqrt{|LN-M^2|}}\left[\frac{\partial}{\partial u}\left(\frac{N\Psi_u-M\Psi_v}{\sqrt{|LN-M^2|}}\right)-\frac{\partial}{\partial v}\left(\frac{M\Psi_u-L\Psi_v}{\sqrt{|LN-M^2|}}\right)\right].$$

3 The Conformal Rotational Surface

In ([18]), the author presented the result of the classification theory of the surfaces of revolution (the Delaunay surfaces) and some of their properties. That is, the author investigated the rotational surface and their dual such as

$$\Psi(u,v)=\left(e^{f(u)}\cos v,e^{f(u)}\sin v,\int\limits_0^u\left(e^{f(t)}\sin g(t)\right)dt\right), \tag{5}$$

and

$$\Psi(u,v) = \left(-e^{-f(u)} \cos v, -e^{-f(u)} \sin v, \int_0^u \left(e^{-f(t)} \sin g(t) \right) dt \right),$$

respectively. He showed that, if

$$e^{f(u)} = \cos hu, \ \sin g(u) = \sec u, \ e^{f(u)} \sin g(u) = u,$$

then the rotational surface is conformal (Figure 1).

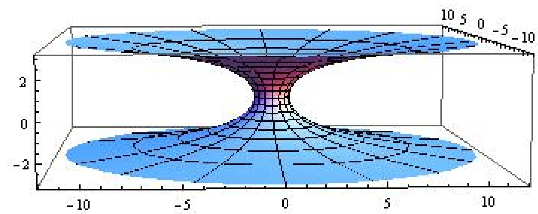

Figure 1:

In this paper, we will investigate the solutions of the function f under the conformal condition. Now, we concentrate on the central subject of this paper and introduce the rotational surface given by (5). The coefficients of the first and the second fundamental forms are given by

$$E = -\frac{1}{2} e^{2f(u)} \left(-1 + \cos(g(u)) - f'^2(u) \right), \quad G = e^{2f(u)}, \quad F = 0$$

and

$$L = -\frac{e^{f(u)} \left(f'g' \cos(g) - f'' \sin(g) \right)}{\sqrt{f'^2 + \sin^2(g)}}, \quad N = \frac{e^{f(u)}}{\sqrt{f'^2 + \sin^2(g)}}, M = 0,$$

respectively. If we require $\Psi(u,v)$ to be conformal, then we have

$$-\frac{1}{2} e^{2f(u)} \left(-1 + \cos(g(u)) - f'^2(u) \right) = e^{2f(u)} \tag{6}$$

or

$$e^{f(u)} \left(-1 + \cos(g(u)) - f'^2(u) \right) = 0. \tag{7}$$

The general solution of (6) respect to $g(u)$ is given by

$$g(u) = \pm \frac{1}{2} \arccos\left(-1 + 2f'^2 \right). \tag{8}$$

Substituting the solution (8) into (5), the rotational surface turns out to be the conformal rotational surface:

$$\Psi(u,v) = \left(e^{f(u)} \cos v, e^{f(u)} \sin v, \int_0^u e^{f(t)} \sin\left(\frac{1}{2} \arccos\left(-1 + 2f'^2(t) \right) \right) dt \right). \tag{9}$$

The coefficients of the fundamental forms I and II are given by.

$$E = G = e^{2f(u)}, \ F = 0,$$

$$L = -\frac{e^{f(u)} f''(u)}{\sqrt{1 - f'^2(u)}}, \quad N = e^{f(u)} \sqrt{1 - f'^2(u)}, \quad M = 0.$$

Also, the unit normal vector field of S is

$$U=\left(-\sqrt{1-f'^2(u)}\cos v,-\sqrt{1-f'^2(u)}\sin v,f'(u)\right).$$

So the Gaussian curvature K and the mean curvature H are calculated by

$$K=-e^{-2f(u)}f''(u),\quad H=\frac{-e^{-f(u)}\left(-1+f'^2(u)+f''(u)\right)}{2\sqrt{1-f'^2(u)}}. \tag{10}$$

Let $K=0$, we get from (10)

$$f''(u)=0.$$

So the general solution is given by $f(u)=c_1u+c_2$, where $c_1,c_2\in\mathbb{R}$ (Figure 2).

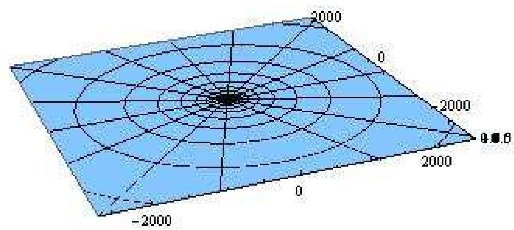

Figure 2:

Now we assume that the conformal rotational surface has nonzero constant Gaussian curvature K. Then the solution of the first part of (10) can be written as

$$f(u)=\ln\left|\frac{\sqrt{c_1-c_1\tanh^2\left(\sqrt{c_1(u+c_2)^2}\right)}}{\sqrt{K}}\right|,$$

where $c_1,c_2,K\in\mathbb{R}$ (Figure 3).

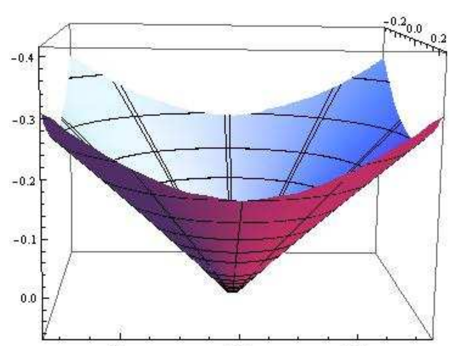

Figure 3:

Suppose that H satisfies the condition $H=0$. In this case, we define as a surface satisfying that minimal condition. Then, from (10) we can write

$$\left(-1+f'^2(u)+f''(u)\right)=0.$$

Its general solution is given by

$$f(u) = c_1 - u + \ln\left|e^{2u} + e^{2c_2}\right|, \tag{11}$$

where $c_1, c_2 \in \mathbb{R}$ (Figure 4).

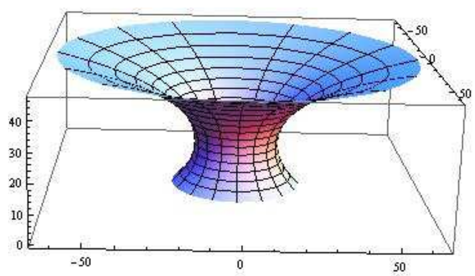

Figure 4:

Thus, we can give the following propositions:

Proposition 1 *The conformal rotational surface given by (9) in Euclidean 3-space $\mathbb{E}^3$ is flat iff $f(u) = c_1u + c_2$ for some constants c_1 and c_2. Then, the parametrization of $\mathbf{S}$ is given by*

$$\Psi(u,v) = \left(e^{(c_1u+c_2)}\cos v, e^{(c_1u+c_2)}\sin v, \int_0^u e^{(c_1t+c_2)}\sin\left(\frac{1}{2}\arccos\left(-1+2c_1^2\right)\right)dt\right).$$

Proposition 2 *The conformal rotational surface given by (9) in Euclidean 3-space $\mathbb{E}^3$ is minimal iff the parametrization of $\mathbf{S}$ is given by (9) with (11).*

Now, we classify the conformal rotational surface satisfying the equation

$$\Delta^I\Psi{=}\mathbf{A}\Psi, \tag{12}$$

where $\mathbf{A} = (\mathbf{a}_{ij}) \in Mat(3,R)$. By straightforward computation, the Laplacian operator respect to the first fundamental form on S turns out to be

$$\Delta^I\Psi = \begin{pmatrix} -e^{-f(u)}\cos v\left(-1+f'^2(u)+f''(u)\right) \\ -e^{-f(u)}\sin v\left(-1+f'^2(u)+f''(u)\right) \\ -\frac{e^{-f(u)}f'(u)\sqrt{1-f'^2(u)}\left(-1+f'^2(u)+f''(u)\right)}{f'^2(u)-1} \end{pmatrix}.$$

Also, the conformal rotational surface given by (9) satisfies the equation (4). Suppose that S satisfies (11). Then from (5), we have

$$\begin{aligned} a_{11}A + a_{12}B + a_{13}C &= -e^{-f(u)}\cos v\left(-1+f'^2+f''\right), \\ a_{21}A + a_{22}B + a_{23}C &= -e^{-f(u)}\sin v\left(-1+f'^2+f''\right), \\ a_{31}A + a_{32}eB + a_{33}C &= -\frac{e^{-f(u)}f'\sqrt{1-f'^2}\left(-1+f'^2+f''\right)}{f'^2-1}, \end{aligned} \tag{13}$$

where

$$A = e^{f(u)}\cos v, \quad B = e^{f(u)}\sin v \quad \text{and} \quad C = \int_0^u e^{f(t)}\sin\left(\frac{1}{2}\arccos\left(-1+2f'^2\right)\right)dt.$$

Since $\{\cos v, \sin v\}$ and the constant function are linearly independent by (13), we get

$$a_{12} = a_{13} = a_{21} = a_{23} = a_{31} = a_{32} = 0, \;\; a_{11} = a_{22} = \lambda, \;\; a_{33} = \mu.$$

Consequently the matrix $\mathbf{A}$ satisfies

$$\mathbf{A} = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \mu \end{bmatrix}.$$

The equation (13) is rewritten as the following:

$$\lambda e^{f(u)} = -e^{-f(u)}\left(-1 + f'^2 + f''\right), \tag{14}$$

$$\mu\left(\int_0^u e^{f(t)} \sin(\Phi)\, dt\right) = -\frac{e^{-f(u)} f' \sqrt{1 - f'^2}\left(-1 + f'^2 + f''\right)}{f'^2 - 1},$$

where $\Phi = \left(\frac{1}{2}\arccos\left(-1 + 2f'^2(u)\right)\right)$. If $\lambda = \mu = 0$, then we easily get $\mathbf{A} = diag(0,0,0)$ and

$$\left(-1 + f'^2(u) + f''(u)\right) = 0,$$

which implies the mean curvature H vanishes identically because of (10). Therefore, the conformal rotational surface is minimal.

Proposition 3 *The conformal rotational surface given by (9) Euclidean 3-space $\mathbb{E}^3$ is harmonic iff the surface $\mathbf{S}$ is minimal.*

Combining the first and the second equation of (14), we obtain

$$\mu\left(\int_0^u e^{f(t)} \sqrt{1 - f'^2(t)} dt\right) + \lambda \frac{e^{f(u)} f'(u)}{\sqrt{1 - f'^2(u)}} = 0.$$

If $\{\lambda \neq 0, \mu = 0\}$, then we have $f(u) = c_1$. If $\{\lambda = 0, \mu \neq 0\}$, then we have

$$\mu\left(\int_0^u e^{f(t)} \sqrt{1 - f'^2(t)} dt\right) = 0. \tag{15}$$

After differentiating (15) respect to u, we obtain

$$\mu e^{f(t)} \sqrt{1 - f'^2(t)} = 0 \tag{16}$$

and $f(t) = c_1 \pm t$, respectively. The remaining case $\{\lambda \neq 0, \mu \neq 0\}$ does not appear. Substituting the solutions $f(u) = c_1$ (Figure 5) and $f(u) = c_1 \pm u$ into (14), respectively, we can easly see they satisfy these equations for $\lambda = e^{-2c_1}$.

Thus, we can give following theorem:

Theorem 1 *Let S be a non-harmonic conformal rotational surface given by (9) in Euclidean 3-space $\mathbb{E}^3$. If the surface S satisfies the condition $\Delta^I \Psi = \mathbf{A}\Psi$, then it is congruent to an open part of the following surface:*

$$\Psi(u, v) = \left(e^{c_1} \cos v, e^{c_1} \sin v, \int_0^u e^{c_1} \sin\left(\frac{1}{2}\arccos(-1)\right) dt\right),$$

$$\Psi(u, v) = \left(e^{c_1 \pm u} \cos v, e^{c_1 \pm u} \sin v, \int_0^u e^{c_1 \pm t} \sin\left(\frac{1}{2}\arccos(-1 + 2(\pm 1))\right) dt\right).$$

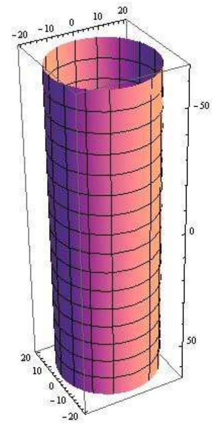

Figure 5:

Let's classify conformal rotational surface with non-degenerate second fundamental form in $\mathbb{E}^3$ satisfying the equation

$$\Delta^{II}\Psi = \mathbf{A}\Psi,$$

where $\mathbf{A} = (\mathbf{a}_{ij}) \in Mat(3, R)$. By straightforward computation, the Laplacian operator Δ^{II} on S turns out to be

$$\Delta^{II}\Psi = \begin{pmatrix} \frac{\sqrt{1-f'^2}\cos v\left(2f'^2 f'' + 4f''^2 - f' f'''\right)}{2f''^2}, \\ \frac{\sqrt{1-f'^2}\sin v\left(2f'^2 f'' + 4f''^2 - f' f'''\right)}{2f''^2}, \\ -\frac{f'\left(2f'^3 f'' + 2f' f''\left(f''-1\right) + 2f' f''^2 + f''' - f'^2 f'''\right)}{2f' f''^2} \end{pmatrix}.$$

By making similar calculations, we have

$$\frac{\sqrt{1-f'^2}\left(2f'^2 f'' + 4f''^2 - f' f'''\right)}{2f''^2} = \lambda e^{f(u)}, \tag{17}$$

$$-\frac{f'\left(2f'^3 f'' + 2f' f''\left(f''-1\right) + 2f' f''^2 + f''' - f'^2 f'''\right)}{2f' f''^2} = \mu\left(\int_0^u e^{f(t)}\sin\left(\Omega\right)dt\right),$$

where $\Omega = \frac{1}{2}\arccos\left(1 - 2f'^2\right)$. Combining the first and the second equation of (17), we obtain

$$-2f' f'' + f''' + \lambda\frac{2e^f f' f''^2}{\sqrt{1-f'^2}} + 2\mu f''^2\left(\int_0^u e^{f(t)}\sqrt{1-f'^2}dt\right) = 0. \tag{18}$$

In the cases $\{\lambda \neq 0, \mu \neq 0\}$, $\{\lambda \neq 0, \mu = 0\}$ and $\{\lambda = 0, \mu \neq 0\}$, we can not obtain any conformal rotational surface in $\mathbb{E}^3$. Substituting the solution $f(u) = c_1 u \pm c_2$ into (17), we can easily see it satisfies (17). But, we observe that the solution $f(u) = c_1 u \pm c_2$ is a contradiction to non-degenerate property with respect to (10). In the case $\{\lambda = 0, \mu = 0\}$, the general solution of (18) is given by

$$f(u) = c_1 - \ln\left|\cos\left(\sqrt{c_2}\left(u + c_3\right)\right)\right|, \tag{19}$$

where $c_i \in \mathbb{R}$ (Figure 6). In this case S is parameterized by (5) with (19).

Thus we can give the following theorems:

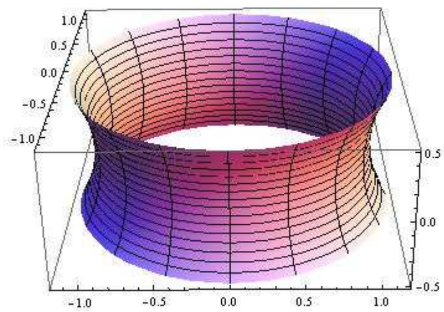

Figure 6:

Theorem 2 *Let S be a conformal rotational surface with non-degenerate second fundamental form given by (5) in $\mathbb{E}^3$. Then, there is no surface S satisfying the condition $\Delta^{II}\Psi = \mathbf{A}\Psi$.*

Theorem 3 *Let S be a conformal rotational surface with non-degenerate second fundamental form given by (5) in $\mathbb{E}^3$. If the surface S satisfies the condition $\Delta^{II}\Psi = \mathbf{0}$, then it is parameterized by (5) with (19).*

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