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Generalized Stability Of Thermistor Problem*

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Abstract

In this paper, we first establish Hyers-Ulam and Hyers-Ulam-Rassias stability for the fractional order Caputo nonlocal thermistor problem. Secondly, we prove the accompanying equation is Mittag-Leffler-Hyers-Ulam and Mittag-Leffler-Hyers-Ulam-Rassias stable.

1 Introduction

The stability theory for functional equations started with a problem related to the stability of group homomorphism that was considered by Ulam in 1940 ([23]). The first answer to the question of Ulam was given by Hyers in 1941 in the case of Banach spaces in [4]. Thereafter, this type of stability is called the Hyers-Ulam stability. In 1978, Th. M. Rassias [18] generalized the Hyers Theorem by considering the stability problem with unbounded Cauchy differences. In fact, he introduced a new type of stability which is called the Hyers-Ulam-Rassias stability.

Alsina and Ger were the first authors who investigated the Hyers-Ulam stability of a differential equation [1]. Recently some authors ([5], [6], [22], [25] and [26]) extended the Ulam stability problem from an integer-order differential equation to a fractional-order differential equation.

Fractional differential and integral equations can serve as an excellent tool for the description of mathematical modeling of systems and processes in the fields of economics, physics, chemistry, aerodynamics, and polymerrheology. It also serves as an excellent tool for the description of hereditary properties of various materials and processes. For more details on the fractional calculus theory, one can see the monographs of Kilbas et al. [7], Miller and Ross [16], Podlubny [17].

Thermistor is a thermo-electric device constructed from a ceramic material whose electrical conductivity depends strongly on the temperature. This makes thermistor problems highly nonlinear [20]. They can be used as a switching device in many electronic circuits. A broad application spectrum of thermistor problems in heating processes and current flow can be found in several areas of electronics and its related industries [21]. Generally, there are two kinds of thermistors: the first have an electrical conductivity that decreases with the increasing of temperature; the second have an electrical conductivity that increases with the increasing of temperature [8, 15]. Here we consider a prototype of electrical conductivity that depends strongly in both time and temperature. The global existence of solutions for a fractional Caputo nonlocal thermistor problem was proved in [19]. Here, precisely we consider the following fractional order initial value problem:

$$\begin{cases} {}^{C}D_{0,t}^{2\alpha}u(t) = \dfrac{\lambda f(t,u(t))}{\left(\int_0^t f(x,u(x))dx\right)^2}, \quad t \in (0,\infty), \\ u(t)|_{t=0} = u_0, \end{cases} \tag{1}$$

where ${}^{C}D_{0,t}^{2\alpha}$ is the fractional Caputo derivative operator of order 2α with $0 < \alpha < \frac{1}{2}$ a real parameter and $f : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function. The function u denotes the temperature and λ is a positive real.

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Choosing λ such that

$$0 < \lambda < \left(\frac{L_f h^{2\alpha}}{(C_1^2 \Gamma(2\alpha+1)} + \frac{2C_2^2 L_f h^{2\alpha}}{C_1^4 \Gamma(2\alpha+1)} \right)^{-1}$$

is discussed in section 3.

It is seen that the equation (1) is equivalent to the following equation:

$$u(t) = u_0 + \frac{\lambda}{\Gamma(2\alpha)} \int_0^t (t-s)^{2\alpha-1} \frac{f(s,u(s))}{\left(\int_0^t f(x,u(x))dx\right)^2} ds. \tag{2}$$

D. Vivek, K. Kanagarajan and Seenith Sivasundaram [24] studied dynamic and stability results for Hilfer fractional type thermistor problem. In [2], the authors by defining all types of Mittag-Leffler-Hyers-Ulam stability of a fractional integral equation proved that every mapping of this type can be somehow approximated by an exact solution of the considered equation. In this paper we present similar definitions to that of [2] and prove the stability results for equation (1).

The paper is organized as follows. In section 2, basic deffinitions, notations and lemmas are given. In section 3, the Hyers-Ulam and Hyers-Ulam-Rassias stability of fractional order initial value problem (1) are proven. Section 4 is devoted to the Mittag-Leffler-Hyers-Ulam and Mittag-Leffler-Hyers-Ulam-Rassias stability of problem (1).

2 Preliminaries

In this section, we introduce notations, definitions and theorems which are used throughout this paper from the references [7, 9, 10, 11, 13, 16]. Let $C[a,b]$ be the Banach space of all real valued continuous functions on $[a,b]$ endowed with the norm $\|x\|_{[a,b]} := \max_{t\in[a,b]} |x(t)|$.

Definition 1 *The Mittag-Leffler function of one parameter is denoted by $E_\alpha(z)$ where $z, \alpha \in \mathbb{C}, Re(\alpha) > 0$, and defined by,*

$$E_\alpha(z) = \sum_{k=o}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)}$$

where the Euler Gamma function $\Gamma : (0,\infty) \to \mathbb{R}$ is defined by

$$\Gamma(\alpha) = \int_0^\infty s^{\alpha-1} \exp(-s) ds.$$

Definition 2 *The Riemann-Liouville integral of a function g with order $\alpha > 0$ is defined by*

$${}_{RL}I_{0,t}^{\alpha} g(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s) ds, t > 0,$$

where $\Gamma(.)$ is the (Euler's) Gamma function defined by

$$\Gamma(\xi) = \int_0^\infty t^{\xi-1} e^{-t} dt, \xi > 0.$$

Definition 3 *The Riemann-Lioville fractional derivative of order $\alpha > 0$ of a function g is defined by*

$${}_{RL}D_{0,t}^{\alpha} g(s) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-s)^{n-\alpha-1} g(s) ds, \quad t > 0,$$

where $n-1 < \alpha < n \in \mathbb{Z}^+$.

Definition 4 *The Caputo fractional derivative of order $\alpha \in (0,1]$ of a function $h \in L^1\{R_+\}$ is defined by*

$$({}^cD_{0,t}^{\alpha}g(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{\alpha-1}g^{(n)}(s)ds,$$

where $n-1 < \alpha < n \in \mathbb{Z}^+$.

Lemma 1 ([12]) *Let M be a subset of $C([0,T])$. Then M is precompact if and only if the following conditions hold:*

(i) $\{u(t) : u \in M\}$ is uniformly bounded,

(ii) $\{u(t) : u \in M\}$ is equicontinuous on $[0,T]$.

Theorem 1 (Schauder fixed point theorem [12]) *Let U be closed, convex and nonempty subset of a Banach space X. Let $T : U \to U$ be a continuous mapping such that $T(U)$ is relatively compact subset of X. Then, T has at least one fixed point in U.*

Lemma 2 (Generalized Gronwall inequality [3]) *Let $v : [0,b] \to [0,+\infty)$ be a real function and $w(.)$ be a nonnegative, locally integrable function on $[0,b]$. Suppose that there exist $a > 1$ and $0 < \alpha < 1$ such that*

$$v(t) \le w(t) + a\int_0^t \frac{v(s)}{(t-s)^{\alpha}}ds.$$

Then, there exists a constant $K = K(\alpha)$ such that

$$v(t) \le w(t) + Ka\int_0^t \frac{w(s)}{(t-s)^{\alpha}}ds$$

for all $t \in [0,b]$.

3 Hyers-Ulam Stability and Hyers-Ulam-Rassias Stability

Consider the following fractional order initial value problem

$$^CD_{0,t}^{2\alpha}u(t) = \frac{\lambda f(t,u(t))}{\left(\int_0^t f(x,u(x))dx\right)^2}, \quad t > 0, \tag{3}$$

and the following fractional inequalities:

$$\left| ^CD_{0,t}^{2\alpha}z(t) - \frac{\lambda f(t,z(t))}{\left(\int_0^t f(x,z(x))dx\right)^2} \right| \le \epsilon, \quad t > 0, \tag{4}$$

$$\left| ^CD_{0,t}^{2\alpha}z(t) - \frac{\lambda f(t,z(t))}{\left(\int_0^t f(x,z(x))dx\right)^2} \right| \le \epsilon\varphi(t), \quad t > 0, \tag{5}$$

$$\left| ^CD_{0,t}^{2\alpha}z(t) - \frac{\lambda f(t,z(t))}{\left(\int_0^t f(x,z(x))dx\right)^2} \right| \le \varphi(t), \quad t > 0. \tag{6}$$

Definition 5 *Equation (3) is Hyers-Ulam stable if there exists a real number $c > 0$ such that for each $\epsilon > 0$ and for each solution $z \in C[0,h]$ of inequality (4), there exists a solution $u \in C[0,h]$ of equation (3) with*

$$|z(t) - u(t)| \leq c\epsilon, t > 0.$$

Definition 6 *Equation (3) is generalized Hyers-Ulam stable if there exists $\theta \in C([0,\infty), [0,\infty))$, $\theta(0) = 0$ such that, for each solution $z \in C[0,h]$ of inequality (4), there exists a solution $u \in C[0,h]$ of equation (3) with*

$$|z(t) - u(t)| \leq \theta(\epsilon), t > 0.$$

Definition 7 *Equation (3) is Hyers-Ulam-Rassias stable if there exists a real number $c_\varphi > 0$ such that for each $\epsilon > 0$ and for each solution $z \in C[0,h]$ of inequality (5), there exists a solution $u \in C[0,h]$ of equation(4) with*

$$|z(t) - u(t)| \leq c_\varphi \epsilon \varphi(t), t > 0.$$

Definition 8 *Equation (3) is generalized Hyers-Ulam-Rassias stable if there exists a real number $c_\varphi > 0$ such that for each solution $z \in C[0,h]$ of inequality (6), there exists a solution $u \in C[0,h]$ of equation (3) with*

$$|z(t) - u(t)| \leq c_\varphi \varphi(t), t > 0.$$

Remark 1 *A function $z \in C[0,h]$ is a solution of inequality (4) if and only if there exists a function $g \in C[0,h]$ (which depends on solution z) such that*

1. $|g(t)| \leq \epsilon, t > 0.$
2. ${}^C D^{2\alpha}_{0,t} z(t) = \dfrac{\lambda f(t,z(t))}{(\int_0^t f(x,z(x))dx)^2} + g(t), t > 0.$

We consider the following hypotheses:

(H_1) $f : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is a Lipschitz continuous function with Lipschitz constant L_f with respect to the second variable such that $c_1 \leq f(s,u) \leq c_2$ with c_1 and c_2 two positive constants;

(H_2) there exists a positive constant M such that $f(s,u) \leq Ms^2$;

(H_3) $|f(s,u) - f(s,v)| \leq s^2|u - v|$ or, in a more general manner, there exists a constant $w \geq 2$ such that $|f(s,u) - f(s,v)| \leq s^w|u - v|$;

(H_4) There exists an increasing function $\varphi \in C[0,h]$ and there exists $\lambda_\varphi > 0$ such that for any $t > 0$, ${}^C I^{2\alpha}_{0,t}\varphi(t) \leq \lambda_\varphi \varphi(t)$.

Lemma 3 *Suppose that assumptions (H_1)–(H_3) are satisfied. Then the initial value problem (1) is equivalent to the integral equation (2) .*

Proof. It is a simple exercise to see that u is a solution of the integral equation (2) if and if only it is also a solution of the IVP (1). ■

Theorem 2 ([19]) *Suppose that conditions (H_1)–(H_3) are verified. Then (1) has at least one solution $u \in C[0,h]$ for some $T \geq h > 0$.*

Lemma 4 *(Uniqueness). Assume that the assumptions (H_1)–(H_3) are satisfied. If*

$$\frac{\lambda h^{2\alpha} L_f}{c_1^2 \Gamma(2\alpha+1)} + \frac{2\lambda c_2^2 h^{2\alpha} L_f}{c_1^4 \Gamma(2\alpha+1)} < 1, \tag{7}$$

and

$$0 < \lambda < \left(\frac{L_f h^{2\alpha}}{(C_1^2 \Gamma(2\alpha+1)} + \frac{2C_2^2 L_f h^{2\alpha}}{C_1^4 \Gamma(2\alpha+1)} \right)^{-1},$$

then the problem (1) has a unique solution.

Proof. Consider the operator $A : C\,[0,h] \to C\,[0,h]$

$$(Au)(t) = u_0 + \frac{\lambda}{\Gamma(2\alpha)} \int_0^t (t-s)^{2\alpha-1} \frac{f(s,u(s))}{\left(\int_0^t f(x,u(x))dx\right)^2} ds. \tag{8}$$

It is clear that the fixed points of A are solutions of problem (1). Let $u, v \in C\,[0,h]$ and $t > 0$. Then we have

$$\begin{aligned} |(Av)(t) - (Au)(t)| &\leq \frac{\lambda}{\Gamma(2\alpha)} \int_0^t (t-s)^{2\alpha-1} \left| \frac{f(s,v(s))}{\left(\int_0^t f(x,v(x))dx\right)^2} - \frac{f(s,u(s))}{\left(\int_0^t f(x,u(x))dx\right)^2} \right| ds \\ &\leq \left(\frac{\lambda h^{2\alpha} L_f}{(c_1)^2 \Gamma(2\alpha+1)} + \frac{2\lambda c_2^2 h^{2\alpha} L_f}{(c_1)^4 \Gamma(2\alpha+1)} \right) \|v-u\|_{C[0,h]}. \end{aligned}$$

Then

$$\|Av - Au\|_{C[0,h]} \leq \left(\frac{\lambda h^{2\alpha} L_f}{(c_1)^2 \Gamma(2\alpha+1)} + \frac{2\lambda c_2^2 h^{2\alpha} L_f}{(c_1)^4 \Gamma(2\alpha+1)} \right) \|v-u\|_{C[0,h]}.$$

Choosing λ such that

$$0 < \lambda < \left(\frac{L_f h^{2\alpha}}{(C_1^2 \Gamma(2\alpha+1)} + \frac{2C_2^2 L_f h^{2\alpha}}{C_1^4 \Gamma(2\alpha+1)} \right)^{-1},$$

the map $A : C\,[0,h] \to C\,[0,h]$ is a contraction. From (7), it follows that A has a unique fixed point, which is a solution of problem (1). ■

Theorem 3 *If assumptions* (H_1)–(H_3) *and the equation (7) are satisfied, then the problem (1) is Hyers-Ulam stable.*

Proof. Let $\epsilon > 0$ and $z \in C\,[0,h]$ be a function that satisfies inequality in the inequality (4) and let $u \in C\,[0,h]$ be a unique solution of the problem

$$\begin{cases} {}^C D_{0,t}^{2\alpha} u(t) = \dfrac{\lambda f(t,u(t))}{\left(\int_0^t f(x,u(x))dx\right)^2}, \quad t > 0, \\ u(t)|_{t=0} = z(t)|_{t=0} = u_0, \end{cases}$$

where $0 < \alpha < \frac{1}{2}$. From Lemma 3, we have

$$u(t) = u_0 + \frac{\lambda}{\Gamma(2\alpha)} \int_0^t (t-s)^{2\alpha-1} \frac{f(s,u(s))}{\left(\int_0^t f(x,u(x))dx\right)^2} ds.$$

By using of (4), we get

$$\left| z(t) - u_0 - \frac{\lambda}{\Gamma(2\alpha)} \int_0^t (t-s)^{2\alpha-1} \frac{f(s,z(s))}{\left(\int_0^t f(x,z(x))dx\right)^2} ds \right| \leq \frac{\epsilon h^{2\alpha}}{\Gamma(2\alpha+1)} \tag{9}$$

for all $t > 0$. So it follows that

$$\begin{aligned}
|z(t) - u(t)| &\le \left| z(t) - u_0 - \frac{\lambda}{\Gamma(2\alpha)} \int_0^t (t-s)^{2\alpha-1} \frac{f(s,u(s))}{(\int_0^t f(x,u(x))dx)^2} ds \right| \\
&\le \left| z(t) - u_0 - \frac{\lambda}{\Gamma(2\alpha)} \int_0^t (t-s)^{2\alpha-1} \frac{f(s,z(s))}{(\int_0^t f(x,z(x))dx)^2} ds \right| \\
&+ \frac{\lambda}{\Gamma(2\alpha)} \int_0^t (t-s)^{2\alpha-1} \left| \frac{f(s,z(s))}{(\int_0^t f(x,z(x))dx)^2} - \frac{f(s,u(s))}{(\int_0^t f(x,u(x))dx)^2} \right| ds \\
&\le \frac{\epsilon h^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{\lambda}{\Gamma(2\alpha)} \int_0^t (t-s)^{2\alpha-1} \frac{1}{(\int_0^t f(x,z(x))dx)^2} |f(s,z(s)) - f(s,u(s))| ds \\
&+ \frac{\lambda}{\Gamma(2\alpha)} \int_0^t (t-s)^{2\alpha-1} |f(s,u(s))| \left| \frac{1}{(\int_0^t f(x,z(x))dx)^2} - \frac{1}{(\int_0^t f(x,u(x))dx)^2} \right| ds.
\end{aligned}$$

We set

$$I_1 = \frac{\lambda}{\Gamma(2\alpha)} \int_0^t (t-s)^{2\alpha-1} \frac{1}{\left(\int_0^t f(x,z(x))dx\right)^2} |f(s,z(s)) - f(s,u(s))| ds,$$

$$I_2 = \frac{\lambda}{\Gamma(2\alpha)} \int_0^t (t-s)^{2\alpha-1} |f(s,u(s))| \left| \frac{1}{\left(\int_0^t f(x,z(x))dx\right)^2} - \frac{1}{\left(\int_0^t f(x,u(x))dx\right)^2} \right| ds. \tag{10}$$

Now, we estimate I_1, I_2 terms separately. We have

$$\begin{aligned}
I_1 &\le \frac{\lambda}{\Gamma(2\alpha)} \int_0^t (t-s)^{2\alpha-1} \frac{1}{\left(\int_0^t f(x,z(x))dx\right)^2} |f(s,z(s)) - f(s,u(s))| ds \\
&\le \frac{\lambda}{(c_1 t)^2 \Gamma(2\alpha)} \int_0^t (t-s)^{2\alpha-1} |f(s,z(s)) - f(s,u(s))| ds \\
&\le \frac{\lambda L_f}{c_1^2 \Gamma(2\alpha)} \int_0^t (t-s)^{2\alpha-1} |z(s) - u(s)| ds, \qquad (11)
\end{aligned}$$

and

$$\begin{aligned}
I_2 &\le \frac{\lambda}{\Gamma(2\alpha)} \int_0^t (t-s)^{2\alpha-1} |f(s,u(s))| \left| \frac{1}{\left(\int_0^t f(x,z(x))dx\right)^2} - \frac{1}{\left(\int_0^t f(x,u(x))dx\right)^2} \right| \\
&\le \frac{\lambda}{\Gamma(2\alpha)} \int_0^t (t-s)^{2\alpha-1} |f(s,u(s))| \frac{\left| \left(\int_0^t f(x,z(x))dx\right)^2 - \left(\int_0^t f(x,u(x))dx\right)^2 \right|}{\left(\int_0^t f(x,z(x))dx\right)^2 \left(\int_0^t f(x,u(x))dx\right)^2} ds \\
&\le \frac{2\lambda c_2^2 t^4 L_f}{(c_1 t)^4 \Gamma(2\alpha)} \|z-u\|_{C[0,h]} \int_0^t (t-s)^{2\alpha-1} ds \\
&\le \frac{2\lambda c_2^2 L_f}{c_1^4 \Gamma(2\alpha)} \int_0^t (t-s)^{2\alpha-1} |z(s) - u(s)| ds. \qquad (12)
\end{aligned}$$

Substituting (11) and (12) into (10), we get

$$\begin{aligned}|z(t)-u(t)| &\le \frac{\epsilon h^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{\lambda L_f}{c_1^2\Gamma(2\alpha)}\int_0^t (t-s)^{2\alpha-1}|z(s)-u(s)|ds \\ &+ \frac{2\lambda c_2^2 L_f}{c_1^4\Gamma(2\alpha)}\int_0^t (t-s)^{2\alpha-1}|z(s)-u(s)|ds \\ &\le \frac{\epsilon h^{2\alpha}}{\Gamma(2\alpha+1)} + \left(\frac{\lambda L_f}{c_1^2}+\frac{2\lambda c_2^2 L_f}{c_1^4}\right)\frac{1}{\Gamma(2\alpha)}\int_0^t (t-s)^{2\alpha-1}|z(s)-u(s)|ds.\end{aligned}$$

By Lemma 2, we have

$$|z(t)-u(t)| \le \frac{h^{2\alpha}}{\Gamma(2\alpha+1)}[1+\frac{Kh^{2\alpha}}{\Gamma(2\alpha+1)}(\frac{\lambda L_f}{c_1^2}+\frac{2\lambda c_2^2 L_f}{c_1^4})]\epsilon := c\epsilon,$$

where $K = K(\alpha)$ is a constant, which completes the proof of the theorem. ■

Theorem 4 *If assumptions (H_1)–(H_4) and the equation (7) are satisfied, then the problem (1) is Hyers-Ulam-Rassias stable.*

Proof. Let $z \in C[0,h]$ be a solution of inequality (5) and let $u \in C[0,h]$ be a unique solution of the following thermistor problem

$$\begin{cases} {}^C D_{0,t}^{2\alpha} u(t) = \dfrac{\lambda f(t,u(t))}{\left(\int_0^t f(x,u(x))dx\right)^2}, & t>0, \\ u(t)|_{t=0} = z(t)|_{t=0} = u_0, \end{cases}$$

where $0<\alpha<\frac{1}{2}$. From Lemma 3, we have

$$u(t) = u_0 + \frac{\lambda}{\Gamma(2\alpha)}\int_0^t (t-s)^{2\alpha-1}\frac{f(s,u(s))}{(\int_0^t f(x,u(x))dx)^2}ds.$$

By using of (5) and assumption (H_4), we get

$$\left|z(t)-u_0-\frac{\lambda}{\Gamma(2\alpha)}\int_0^t (t-s)^{2\alpha-1}\frac{f(s,z(s))}{(\int_0^t f(x,z(x))dx)^2}ds\right| \le \epsilon\lambda_\varphi\varphi(t) \tag{13}$$

for all $t>0$. From above relations, it follows that:

$$\begin{aligned}|z(t)-u(t)| &\le \left|z(t)-u_0-\frac{\lambda}{\Gamma(2\alpha)}\int_0^t (t-s)^{2\alpha-1}\frac{f(s,u(s))}{(\int_0^t f(x,u(x))dx)^2}ds\right| \\ &\le \left|z(t)-u_0-\frac{\lambda}{\Gamma(2\alpha)}\int_0^t (t-s)^{2\alpha-1}\frac{f(s,z(s))}{(\int_0^t f(x,z(x))dx)^2}ds\right| \\ &+ \frac{\lambda}{\Gamma(2\alpha)}\int_0^t (t-s)^{2\alpha-1}\left|\frac{f(s,z(s))}{(\int_0^t f(x,z(x))dx)^2}-\frac{f(s,u(s))}{(\int_0^t f(x,u(x))dx)^2}\right|ds \\ &\le \epsilon\lambda_\varphi\varphi(t) + \frac{\lambda}{\Gamma(2\alpha)}\int_0^t (t-s)^{2\alpha-1}\frac{1}{(\int_0^t f(x,z(x))dx)^2}|f(s,z(s))-f(s,u(s))|ds \\ &+ \frac{\lambda}{\Gamma(2\alpha)}\int_0^t (t-s)^{2\alpha-1}|f(s,u(s))|\left|\frac{1}{(\int_0^t f(x,z(x))dx)^2}-\frac{1}{(\int_0^t f(x,u(x))dx)^2}\right|ds. \end{aligned} \tag{14}$$

Substituting (11) and (12) into (14), we get

$$\begin{aligned}|z(t)-u(t)| &\leq \epsilon\lambda_{\varphi}\varphi(t)+\frac{\lambda L_f}{c_1^2\Gamma(2\alpha)}\int_0^t (t-s)^{2\alpha-1}|z(s)-u(s)|ds \\ &+\frac{2\lambda c_2^2 L_f}{c_1^4\Gamma(2\alpha)}\int_0^t (t-s)^{2\alpha-1}|z(s)-u(s)|ds \\ &\leq \epsilon\lambda_{\varphi}\varphi(t)+\left(\frac{\lambda L_f}{c_1^2}+\frac{2\lambda c_2^2 L_f}{c_1^4}\right)\frac{1}{\Gamma(2\alpha)}\int_0^t (t-s)^{2\alpha-1}|z(s)-u(s)|ds.\end{aligned}$$

By Lemma 2, we have

$$|z(t)-u(t)| \leq \left[(1+K_1\lambda_{\varphi}(\frac{\lambda L_f}{c_1^2}+\frac{2\lambda c_2^2 L_f}{c_1^4}))\lambda_{\varphi}\right]\epsilon\varphi(t) := c\epsilon\varphi(t),$$

where $K_1 = K_1(\alpha)$ is a constant, which completes the proof of the theorem. ■

4 Mittag-Leffler-Hyers-Ulam and Mittag-Leffler-Hyers-Ulam-Rassias stability

In this section we study the Mittag-Leffler-Hyers-Ulam and Mittag-Leffler-Hyers-Ulam-Rassias stability for thermistor problem.

Definition 9 *Equation (3) is Mittag-Leffler-Hyers-Ulam stable with respect to E_α if there exists a real number $c > 0$ such that for each $\epsilon > 0$ and for each solution $z \in C[0,h]$ of inequality (4), there exists a solution $u \in C[0,h]$ of equation (3) with*

$$|z(t)-u(t)| \leq c\epsilon E_\alpha(t), t > 0.$$

Definition 10 *Equation (3) is generalized Mittag-Leffler-Hyers-Ulam stable with respect to E_α if there exists $\theta \in C([0,\infty),[0,\infty))$, $\theta(0) = 0$ such that, for each solution $z \in C[0,h]$ of inequality (4), there exists a solution $u \in C[0,h]$ of equation (3) with*

$$|z(t)-u(t)| \leq \theta(\epsilon)E_\alpha(t), t > 0.$$

Definition 11 *Equation (3) is Mittag-Leffler-Hyers-Ulam-Rassias stable with respect to φE_α if there exists a real number $c_\varphi > 0$ such that for each $\epsilon > 0$ and for each solution $z \in C[0,h]$ of inequality (5), there exists a solution $u \in C[0,h]$ of equation (3) with*

$$|z(t)-u(t)| \leq c_\varphi\epsilon\varphi(t)E_\alpha(t), t > 0.$$

Definition 12 *Equation (3) is generalized Mittag-Leffler-Hyers-Ulam-Rassias stable with respect to φE_α if there exists a real number $c_\varphi > 0$ such that for each solution $z \in C[0,h]$ of inequality (6), there exists a solution $u \in C[0,h]$ of equation (3) with*

$$|z(t)-u(t)| \leq c_\varphi\varphi(t)E_\alpha(t), t > 0.$$

Theorem 5 ([14]) *For any $t \in [0,b)$, if*

$$u(t) \leq a(t)+\sum_{i=1}^{n} b_i(t)\int_0^t (t-s)^{\beta_i-1}u(s)ds,$$

where all the functions are not negative and continuous and the constants $\beta_i > 0$. $b_i(i = 1,2,...,n)$ are the bounded and monotonic increasing functions on $[0,b)$, then

$$u(t) \leq a(t)+\sum_{k=1}^{\infty}\left(\sum_{1',2',\ldots k'=1}^{n}\frac{\prod_{i=1}^{k}[b_{i'}(t)\Gamma(\beta_{i'})]}{\Gamma(\sum_{i=1}^{k}\beta_{i'})}\int_0^t (t-s)^{\sum_{i=1}^{k}\beta_i-1}a(s)ds\right).$$

Remark 2 *If the constants $b_1 \geq 0$, $\beta_1 > 0$, $a(t)$ are nonnegative and locally integrable on $0 \leq t < b$ and $u(t)$ is nonnegative and locally integrable on $0 \leq t < b$ with*

$$u(t) \leq a(t) + b_1 \int_0^t (t-s)^{\beta_1 - 1} u(s) ds,$$

then we have

$$u(t) \leq a(t) + \sum_{k=1}^{\infty} \left[\frac{(b_1 \Gamma(\beta_1))^k}{\Gamma(k\beta_1)} \int_0^t (t-s)^{k\beta_1 - 1} a(s) ds \right].$$

Remark 3 *Under the hypotheses of Remark 2, let $a(t)$ is a nondecreasing function on $0 \leq t < b$. We have*

$$u(t) \leq a(t)(E_{\beta_1}[b_1 \Gamma(\beta_1) t^{\beta_1}]),$$

where E_α is the Mittag-Leffler function [17] defined by $E_\alpha[z] = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+1)}$, $z \in \mathbb{C}$.

Theorem 6 *If assumptions (H_1)–(H_3) are satisfied and*

$$\frac{\lambda h^{2\alpha} L_f}{c_1^2 \Gamma(2\alpha+1)} + \frac{2\lambda c_2^2 h^{2\alpha} L_f}{c_1^4 \Gamma(2\alpha+1)} < 1,$$

then the problem (1) is Mittag-Leffler- Hyers-Ulam stable.

Proof. Let $\epsilon > 0$ and let $z \in C[0,h]$ be a function that satisfies inequality (4) and let $u \in C[0,h]$ be a unique solution of the following thermistor problem

$$\begin{cases} {}^C D_{0,t}^{2\alpha} u(t) = \dfrac{\lambda f(t, u(t))}{\left(\int_0^t f(x, u(x)) dx\right)^2}, \quad t > 0, \\ u(t)|_{t=0} = z(t)|_{t=0} = u_0, \end{cases}$$

where $0 < \alpha < \frac{1}{2}$. From Lemma 3, we have

$$u(t) = u_0 + \frac{\lambda}{\Gamma(2\alpha)} \int_0^t (t-s)^{2\alpha-1} \frac{f(s, u(s))}{\left(\int_0^t f(x, u(x)) dx\right)^2} ds.$$

By using of (4), we get

$$\left| z(t) - u_0 - \frac{\lambda}{\Gamma(2\alpha)} \int_0^t (t-s)^{2\alpha-1} \frac{f(s, z(s))}{\left(\int_0^t f(x, z(x)) dx\right)^2} ds \right| \leq \frac{\epsilon h^{2\alpha}}{\Gamma(2\alpha+1)}$$

for all $t > 0$. From these relations, we have

$$\begin{aligned} |z(t) - u(t)| &\leq \left| z(t) - u_0 - \frac{\lambda}{\Gamma(2\alpha)} \int_0^t (t-s)^{2\alpha-1} \frac{f(s, z(s))}{(\int_0^t f(x, z(x)) dx)^2} ds \right| \\ &+ \frac{\lambda}{\Gamma(2\alpha)} \int_0^t (t-s)^{2\alpha-1} \left| \frac{f(s, z(s))}{(\int_0^t f(x, z(x)) dx)^2} - \frac{f(s, u(s))}{(\int_0^t f(x, u(x)) dx)^2} \right| ds \\ &\leq \frac{\epsilon h^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{\lambda}{\Gamma(2\alpha)} \int_0^t (t-s)^{2\alpha-1} \frac{1}{(\int_0^t f(x, z(x)) dx)^2} |f(s, z(s)) - f(s, u(s))| ds \\ &+ \frac{\lambda}{\Gamma(2\alpha)} \int_0^t (t-s)^{2\alpha-1} |f(s, u(s))| \left| \frac{1}{(\int_0^t f(x, z(x)) dx)^2} - \frac{1}{(\int_0^t f(x, u(x)) dx)^2} \right| ds. \end{aligned} \tag{15}$$

We get

$$
\begin{aligned}
|z(t)-u(t)| &\leq \frac{\epsilon h^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{\lambda L_f}{c_1^2\Gamma(2\alpha)}\int_0^t (t-s)^{2\alpha-1}|z(s)-u(s)|ds \\
&+ \frac{2\lambda c_2^2 L_f}{c_1^4\Gamma(2\alpha)}\int_0^t (t-s)^{2\alpha-1}|z(s)-u(s)|ds \\
&\leq \frac{\epsilon h^{2\alpha}}{\Gamma(2\alpha+1)} + \left(\frac{\lambda L_f}{c_1^2} + \frac{2\lambda c_2^2 L_f}{c_1^4}\right)\frac{1}{\Gamma(2\alpha)}\int_0^t (t-s)^{2\alpha-1}|z(s)-u(s)|ds,
\end{aligned}
$$

by using Remark 2 and Remark 3, we have

$$
|z(t)-u(t)| \leq \frac{\epsilon h^{2\alpha}}{\Gamma(2\alpha+1)}\left(E_{2\alpha}((\frac{\lambda L_f}{c_1^2} + \frac{2\lambda c_2^2 L_f}{c_1^4})t^{2\alpha})\right) := c\epsilon E_\alpha(t).
$$

Thus we obtain the Mittag-Leffler-Hyers-Ulam Stability for the problem (1). ■

Theorem 7 *With assumptions (H_1)–(H_4) and the inequality*

$$
\frac{\lambda h^{2\alpha} L_f}{c_1^2\Gamma(2\alpha+1)} + \frac{2\lambda c_2^2 h^{2\alpha} L_f}{c_1^4\Gamma(2\alpha+1)} < 1,
$$

problem (1) is Mittag-Leffler-Hyers-Ulam-Rassias stable.

Proof. Let $z \in C[0,h]$ be a solution of inequality (5) and let $u \in C[0,h]$ be a unique solution of the following thermistor problem

$$
\begin{cases}
{}^C D_{0,t}^{2\alpha} u(t) = \dfrac{\lambda f(t,u(t))}{\left(\int_0^t f(x,u(x))dx\right)^2}, & t > 0, \\
u(t)|_{t=0} = z(t)|_{t=0} = u_0,
\end{cases}
$$

where $0 < \alpha < \frac{1}{2}$. From Lemma 3, we have

$$
u(t) = u_0 + \frac{\lambda}{\Gamma(2\alpha)}\int_0^t (t-s)^{2\alpha-1}\frac{f(s,u(s))}{(\int_0^t f(x,u(x))dx)^2}ds.
$$

By using of (5) and assumption (H_4), we get

$$
\left|z(t) - u_0 - \frac{\lambda}{\Gamma(2\alpha)}\int_0^t (t-s)^{2\alpha-1}\frac{f(s,z(s))}{(\int_0^t f(x,z(x))dx)^2}ds\right| \leq \epsilon\lambda_\varphi\varphi(t) \tag{16}
$$

for all $t > 0$. From above relations, it follows that:

$$
\begin{aligned}
|z(t)-u(t)| &\leq \left|z(t) - u_0 - \frac{\lambda}{\Gamma(2\alpha)}\int_0^t (t-s)^{2\alpha-1}\frac{f(s,u(s))}{(\int_0^t f(x,u(x))dx)^2}ds\right| \\
&\leq \left|z(t) - u_0 - \frac{\lambda}{\Gamma(2\alpha)}\int_0^t (t-s)^{2\alpha-1}\frac{f(s,z(s))}{(\int_0^t f(x,z(x))dx)^2}ds\right| \\
&+ \frac{\lambda}{\Gamma(2\alpha)}\int_0^t (t-s)^{2\alpha-1}\left|\frac{f(s,z(s))}{(\int_0^t f(x,z(x))dx)^2} - \frac{f(s,u(s))}{(\int_0^t f(x,u(x))dx)^2}\right|ds \\
&\leq \epsilon\lambda_\varphi\varphi(t) + \frac{\lambda}{\Gamma(2\alpha)}\int_0^t (t-s)^{2\alpha-1}\frac{1}{(\int_0^t f(x,z(x))dx)^2}|f(s,z(s)) - f(s,u(s))|ds \\
&+ \frac{\lambda}{\Gamma(2\alpha)}\int_0^t (t-s)^{2\alpha-1}|f(s,u(s))|\left|\frac{1}{(\int_0^t f(x,z(x))dx)^2} - \frac{1}{(\int_0^t f(x,u(x))dx)^2}\right|ds.
\end{aligned} \tag{17}
$$

Substituting (11) and (12) into (17), we get

$$\begin{aligned}|z(t)-u(t)| &\le \epsilon\lambda_\varphi\varphi(t)+\frac{\lambda L_f}{c_1^2\Gamma(2\alpha)}\int_0^t (t-s)^{2\alpha-1}|z(s)-u(s)|ds\\ &+\frac{2\lambda c_2^2 L_f}{c_1^4\Gamma(2\alpha)}\int_0^t (t-s)^{2\alpha-1}|z(s)-u(s)|ds\\ &\le \epsilon\lambda_\varphi\varphi(t)+(\frac{\lambda L_f}{c_1^2}+\frac{2\lambda c_2^2 L_f}{c_1^4})\frac{1}{\Gamma(2\alpha)}\int_0^t (t-s)^{2\alpha-1}|z(s)-u(s)|ds.\end{aligned}$$

By Remarks 2 and 3, we have

$$|z(t)-u(t)|\le \epsilon\lambda_\varphi\varphi(t)\left(E_{2\alpha}((\frac{\lambda L_f}{c_1^2}+\frac{2\lambda c_2^2 L_f}{c_1^4})t^{2\alpha})\right):=c_\varphi\epsilon\varphi(t)E_\alpha(t).$$

Thus the conclusion of our theorem holds. ■

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