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A Non-Algebraic Limit Cycle For A Class Of Quintic Differential Systems With Non-Elementary Singular Point*

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Abstract

In this paper, we consider a class of planar quintic differential systems, for which a non-algebraic limit cycle around a non-elementary critical point is given and it is the unique limit cycle. The non-algebraic limit cycle is constructed explicitly by using polar coordinates.

1 Introduction

In the study of planar differential systems, it is not always possible to find explicit solutions for such systems, we resort to qualitative theory to seek information about solutions for non-linear systems to investigate their behavior. In the qualitative theory, limit cycles, or isolated periodic solutions, were and still remain the most sought solutions when modeling physical systems in the plane. Most of the early examples in the theory of limit cycles in planar differential systems were commonly related to practical problems with mechanical and electronic systems, but periodic behavior appears in all branches of science, both the technological and natural sciences. Existence of limit cycles is one of the most difficult subjects in the qualitative theory of planar differential equations. A large amount of references deals with the subject of limit cycle, for instance, the famous Hilbert's 16th problem [11] motivated researchers to enter this domain of research. In particular, to deal with autonomous ordinary differential systems in two real variables, which have the following form

$$\begin{cases} \dot{x} = \dfrac{dx}{dt} = P_N(x,y), \\ \\ \dot{y} = \dfrac{dx}{dt} = Q_N(x,y), \end{cases} \tag{1}$$

where P and Q are real polynomials in the variables x and y of degree $N = \max\{\deg P, \deg Q\}$. The dot denotes derivative with respect to the independent variable t. Recall that, limit cycle of (1) is an isolated periodic orbit in the set of all its periodic orbits, and the algebraic curve $U(x,y) = 0$ is called an invariant curve for (1) if and only if there exists a cofactor $\kappa(x,y)$ which is a polynomial satisfying

$$P_N(x,y)\frac{\partial U}{\partial x} + Q_N(x,y)\frac{\partial U}{\partial y} = \kappa(x,y)U(x,y). \tag{2}$$

A limit cycle of (1) is said to be algebraic if it is contained in the zero set of an invariant algebraic curve of system.

Nowadays, most limit cycles known in an explicit way are algebraic, see for instance [4, 5, 9]. In 1998, M. Abdelkadder [1] presented for the first time an example of Liénard equations with an exact algebraic limit cycle. This example was obtained as a particular case by Bendjeddou and Cheurfa [4] by considering a more general class of planar systems. Limit cycles of planar polynomial differential systems are not in

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general algebraic. For instance, the limit cycle for the Van der Pol equation is non-algebraic as shown by Odani [12]. the first examples of explicit non-algebraic limit cycles were given by Gasull [8], Al-Dossary [2] for $n = 5$ and by Llibre [6] for $n = 3$. The first result about coexistence of algebraic and non-algebraic limit cycles goes back to Giné and Grau [9] with $n = 9$.

In this work, we are mainly interested in the study of the existence of one and only one limit cycle which is non-algebraic for a class of quintic systems around a non-elementary critical point.

Our main result is the following theorem.

Theorem 1 *Consider the following quintic system*

$$\begin{cases} \dot{x} = P_5(x,y) = bx^3 + dmx^5 - nx^4y + cxy^2 + dnx^3y^2 \\ \qquad\qquad - (2a+n)x^2y^3 + adxy^4 - 2ay^5, \\ \dot{y} = Q_5(x,y) = 2mx^5 + bx^2y + dmx^4y + (2m+n)x^3y^2 \\ \qquad\qquad + cy^3 + dnx^2y^3 + nxy^4 + ady^5. \end{cases} \tag{3}$$

Then, for $bc > 0$, $ab > 0$ and $d < 0$, system (3) has one and only one limit cycle which is non-algebraic, given in polar coordinates by the formula

$$r(\theta; r_*) = \left(\int_0^\theta \frac{f_2(u)}{g(u)} \exp\left(-\int_0^u \frac{f_1(s)}{g(s)} ds \right) du + r_*^2 \right)^{\frac{1}{2}} \exp\left(\frac{1}{2} \int_0^\theta \frac{f_1(u)}{g(u)} du \right), \tag{4}$$

where

$$\begin{aligned} f_1(\theta) &= ad\sin^4\theta + dm\cos^4\theta + dn\sin^2\theta\cos^2\theta \\ &\quad + (n-2a)\sin^3\theta\cos\theta + (2m-n)\sin\theta\cos^3\theta, \end{aligned}$$

$$f_2(\theta) = b\cos^2\theta + c\sin^2\theta,$$

$$g(\theta) = a\sin^4\theta + m\cos^4\theta + n\sin^2\theta\cos^2\theta,$$

and

$$r_* = \left(\frac{\int_0^{2\pi} \frac{f_2(u)}{g(u)} \exp\left(-\int_0^u \frac{f_1(s)}{g(s)} ds \right) du}{\exp\left(-\int_0^{2\pi} \frac{f_1(u)}{g(u)} du \right) - 1} \right)^{\frac{1}{2}}, \tag{5}$$

provided that

$$n^2 - 4am < 0. \tag{6}$$

Proof. Since

$$yP_5(x,y) - xQ_5(x,y) = 2(x^2+y^2)(mx^4 + nx^2y^2 + ay^4),$$

we see that from (6), the unique critical point is the origin $(0,0)$. In order to search for the limit cycle, we use polar coordinates. System (3) becomes

$$\begin{cases} \dot{r} = f_2\theta)r^3 + f_1(\theta)r^5, \\ \dot{\theta} = 2g(\theta)r^4. \end{cases} \tag{7}$$

We can rewrite system (7) as the first-order Bernoulli differential equation as follows

$$\frac{dr}{d\theta} = \frac{1}{2r}\frac{f_2(\theta)}{g(\theta)} + \frac{r}{2}\frac{f_1(\theta)}{g(\theta)}, \tag{8}$$

Using the change of variable $\rho = r^2$, equation (8) becomes the following linear first order differential equation

$$\frac{d\rho}{d\theta} = \frac{f_1(\theta)}{g(\theta)}\rho + \frac{f_2(\theta)}{g(\theta)}. \tag{9}$$

Now immediately from (9) it follows that

$$r(\theta; k) = \left(\int_0^{\theta} \frac{f_2(u)}{g(u)} \exp\left(-\int_0^{u} \frac{f_1(s)}{g(s)} ds \right) du + k \right)^{\frac{1}{2}} \exp\left(\frac{1}{2} \int_0^{\theta} \frac{f_1(u)}{g(u)}\, du \right) \tag{10}$$

solution for (8), where k is a constant. The Cartesian coordinate form of (10) proves that it is a non-algebraic curve. It is clear that $r(0, k) = r_0 > 0$, corresponds to $k = r_0^2$, so (10) becomes

$$r(\theta; r_0) = \left(\int_0^{\theta} \frac{f_2(u)}{g(u)} \exp\left(-\int_0^{u} \frac{f_1(s)}{g(s)} ds \right) du + r_0^2 \right)^{\frac{1}{2}} \exp\left(\frac{1}{2} \int_0^{\theta} \frac{f_1(u)}{g(u)}\, du \right). \tag{11}$$

Periodic solutions must verify the following condition

$$r(2\pi; r_0) = r_0, \tag{12}$$

solving (12) with respect to r_0 gives

$$r_*^2 = \frac{\int_0^{2\pi} \frac{f_2(u)}{g(u)} \exp\left(-\int_0^{u} \frac{f_1(s)}{g(s)} ds \right) du}{\exp\left(-\int_0^{2\pi} \frac{f_1(u)}{g(u)} du \right) - 1}. \tag{13}$$

In order to show that the right hand side into (13) is strictly positive, we can first easily show that

$$\exp\left(-\int_0^{2\pi} \frac{f_1(u)}{g(u)} du \right) - 1 = e^{-2\pi d} - 1 > 0,$$

when $d < 0$, and because the denominator into (13) is positive, the sign of the numerator is the same as the sign of $f_2(u)/g(u)$, but we have $n^2 - 4am < 0$ this means that $am > 0$, which makes $f_2(u)/g(u)$ always positive if and only if $bc > 0$ and $ab > 0$.

Let's now consider from (11)

$$\tilde{g}(\theta) = r(\theta; r_*), \tag{14}$$

r_* is given by (13). From the previous considerations of parameters, we must have $\tilde{g} > 0$ by construction. Knowing that

$$e^{-\int_0^{2\pi} \frac{f_1(u)}{g(u)} du} = e^{-2\pi d},$$

replacing it into (14), with simple calculations, we can easily show that the function $\tilde{g}$ is periodic, i.e.,

$$\tilde{g}(\theta + 2\pi) = \tilde{g}(\theta).$$

Now we turn to the final step, i.e., the question whether the graph of the function $\tilde{g}$ is indeed a limit cycle. We consider the Poincaré return map, from (11), we calculate the derivative of $r(2\pi, r_0)$ with respect to r_0 at the point r_*, thus

$$\left. \frac{dr}{dr_0}(2\pi, r_0) \right|_{r_0 = r_*} = \frac{e^{\pi d} r_*}{(G(2\pi) + r_*^2)^{\frac{1}{2}}}, \tag{15}$$

where

$$G(2\pi) = \int_0^{2\pi} \frac{f_2(u)}{g(u)} \exp\left(-\int_0^{u} \frac{f_1(s)}{g(s)} ds \right) du.$$

From (15),

$$\frac{r_*}{(G(2\pi) + r_*^2)^{\frac{1}{2}}} < 1,$$

and

$$e^{\pi d} < 1,$$

therefore

$$\left.\frac{dr}{dr_0}(2\pi, r_0)\right|_{r_0=r_*} < 1.$$

For that reason, limit cycle for the ordinary differential equation (8) is stable. Finally, system (3) has exactly one non-algebraic limit cycle which is the only existing limit cycle. ■

2 Example

Let the parameters in system (3) be $a = 1$, $m = 2$, $n = 1$, $b = 2$, $c = 1$ and $d = -1$. Then system (3) becomes

$$\begin{cases} \dot{x} = 2x^3 - 2x^5 - x^4y + xy^2 - x^3y^2 - 3x^2y^3 - xy^4 - 2y^5, \\ \dot{y} = 4x^5 + 2x^2y - 2x^4y + 5x^3y^2 + y^3 - x^2y^3 + xy^4 - y^5. \end{cases} \tag{16}$$

Clearly, conditions of theorem 1 can be easily verified, system (16) has one limit cycle as shown in figure 1.

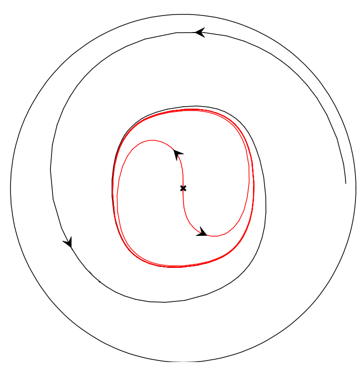

Figure 1: The phase portrait in the Poincaré disc for system (16), with limit cycle included.

3 Conclusion

In this work, we determine the conditions for which a class of planar quintic systems, have a unique non-algebraic limit cycle that is explicitly constructed. The method adopted is simple and gives interesting results of this kind of systems. For polynomials of lower degrees, explicit results are difficult, for instance, an explicitly given non-algebraic limit cycle with a polynomial of second degree still remains an open problem to this day [7].

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