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On Dominated Graphs*

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Abstract

Let v be a vertex of the graph Γ and W a non-empty proper subset of the vertices of the graph Γ. The closed neighborhood $N[v]$ is a k-dominant neighborhood with respect to W, if the distance of every vertex of W to v is less or equal than k, where k is a non-negative integer. In this paper, we obtain a proper subset of vertices W of the graph Γ in such a way that every vertex of the graph is covered by k-dominant neighborhood $N[v]$, with respect to W. By this covering, we mean that $d(v,x) \leq l$, for all $x \in V(\Gamma) \setminus W$, where $N[v]$ is a k-dominant neighborhood with respect to W and l is a fixed non-negative integer. A graph which has a subset of vertices like W is called a (k,l)-dominated graph. The (k,l)-dominated graph is connected and $\mathsf{diam}(\Gamma) \leq 2k+2l$. Furthermore, $(2,1)$-dominated paths, cycles, trees and unicyclic graphs are specified and upper bounds for the number of vertices in a path and a cycle which is $(k,1)$-dominated are achieved.

1 Introduction

Suppose each of the squares of an $n \times n$ chessboard is equipped with an indicator light and a button. If the button of a square is pressed, the light of that square will change from off to on and vice versa; the same happens to the lights of all the edge-adjacent squares. Initially all lights are off. Is it possible to press a sequence of buttons in such a way that in the end all lights are on?

Let Γ be a locally finite graph, i.e., a graph such that every vertex v is adjacent to only finitely many vertices in Γ and $N[x]$ is the closed neighborhood of the vertex x. The subset of vertices of the graph is called a dominating set if $\cup_{d\in D} N[d] = V(\Gamma)$, where $N[d]$ is the closed neighborhood of d and $V(\Gamma)$ denotes the vertex set of the graph. The size of the smallest dominating set is called (closed) domination number and is denoted by $\gamma(\Gamma)$. The dominating set is generalized to k-dominating set, where k is a positive integer. A subset D of $V(\Gamma)$ is called a k-dominating set, if for every vertex $y \notin D$, there is at least one vertex $x \in D$, such that the distance between them $d(x,y) = k$. The k-domination number $\gamma_k(\Gamma)$ is the cardinality of the smallest k-dominating set.

In [4], for a given graph Γ, the k-dominating graph of Γ, $D_k(\Gamma)$, is defined to be the graph whose vertices correspond to the dominating sets of Γ that have cardinality at most k. Two vertices in $D_k(\Gamma)$ are adjacent if and only if the corresponding dominating sets of Γ differ by either adding or deleting a single vertex. The authors claimed that, the graph $D_k(\Gamma)$ aids in studying the reconfiguration problem for dominating sets.

There are some other types of dominating sets which are discussed such as even, odd, open dominating sets (see [3] for more details). We intend to investigate this notion from another point of view.

In this paper, we define the k-dominant neighborhood with respect to a proper subset of vertices of the graph W, where k is a non-negative integer. The closed neighborhood $N[v]$ is a k-dominant neighborhood with respect to W, if the distance of every vertex of W to v is less or equal than k. In other words, these neighborhoods are relatively dominant with respect to W. The 2-dominant neighborhood $N[v]$ with respect to W is a dominating set of the induced subgraph on the vertices $W \cup N[v]$. There is a large amount of results about the dominating set, domination number and notions related to it. We are going to obtain a proper subset of vertices of the graph in such a way that every vertex of a graph is covered by a k-dominant neighborhood with respect to that set. We introduce (k,l)-dominated set W, which is a non-empty proper

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subset of vertices of the graph such that, for all $v \in V(\Gamma) \setminus W$, there exists k-dominant neighborhood $N[x]$ with respect to W such that $d(v,x) \leq l$. The size of the biggest (k,l)-dominated set is called (k,l)-dominated number and is denoted by $\vartheta_{(k,l)}(\Gamma)$. If there exists (k,l)-dominated set for the graph Γ, then Γ is called (k,l)-dominated graph. For $l = 1$, we call W and the graph Γ, the k-dominated set and k-dominated graph for abbreviation, respectively. More precisely, the non-empty proper subset of vertices W is called a k-dominated set, if for every vertex $v \in V(\Gamma) \setminus W$, there exists a k-dominant neighborhood $N[x]$ such that $v \in N[x]$.

All k-dominated graphs are connected. A bound is presented for the diameter of a (k,l)-dominated graph. It is also clear that, if $\mathrm{diam}(\Gamma) \leq k$, then Γ is k-dominated. In the third section, 2-dominated paths, cycles, trees and unicyclic graphs are characterized. Furthermore, an upper bound for the number of vertices in a path and a cycle which is k-dominated is computed.

Throughout the paper, all the notations and terminologies about the graphs follow the notations of [1, 2].

2 Preliminaries

We start with the following definition.

Definition 1 *Let Γ be a graph, $v \in V(\Gamma)$ and W a non-empty proper subset of the vertices of the graph Γ. The closed neighborhood $N[v]$ is a k-dominant neighborhood with respect to W, if the distance of every vertex of W to v is less or equal than k, where k is a non-negative integer.*

If $k = 2$ in Definition 1., then the notion of 2-dominant neighborhood follows. Since, its notion has a very close connection with the ordinary dominating set, let us present its definition. The closed neighborhood $N[v]$ is a 2-dominant neighborhood with respect to W, if every vertex of W which is not in $N[v]$ joins to at least a vertex of $N[v]$. The following lemma is direct result of the definition.

Lemma 1 *Let $N[v]$ be a 2-dominant neighborhood with respect to the proper subset of vertices of the graph W. Then the vertices of $N[v]$ form a dominating set of the induced subgraph on $W \cup N[v]$.*

Definition 2 *Suppose W is a non-empty proper subset of the vertices of the graph Γ. If for every vertex of the graph $x \in V(\Gamma) \setminus W$, there is a k-dominant neighborhood $N[v]$ with respect to W such that $d(x,v) \leq l$, then W is called (k,l)-dominated set. The size of the biggest (k,l)-dominated set of the graph Γ is (k,l)-dominated number and is denoted by $\vartheta_{(k,l)}(\Gamma)$. A graph which has a (k,l)-dominated set is called (k,l)-dominated graph.*

In other words, for every vertex outside of W say x, there exists the neighborhood $N[v]$ with $d(v,x) \leq l$ such that this neighborhood is chosen in such a way that $d(v,w) \leq k$ for every vertex inside W.
If $l = 1$, then we just call the graph k-dominated and k-dominated number and is denoted by $\vartheta_k(\Gamma)$ for abbreviation and the following more usable description is concluded.

Definition 3 *The non-empty proper subset of vertices W is called a k-dominated set, if for every vertex $x \in V(\Gamma) \setminus W$, there exists a k-dominant neighborhood $N[v]$ with respect to W such that $x \in N[v]$.*

In the above definitions, the existence of a neighborhood for every vertex outside of W, means it may exist several neighborhoods for one vertex or it may exist several vertices which covers by the same k-dominant neighborhood (see the following examples).

(i) If K_n is a complete graph with n vertices $\{v_1, \ldots, v_n\}$, then the set $V(K_n) \setminus \{v_i\}$ form a 1-dominated set. Clearly, every closed neighborhood covers whole n vertices and is 1-dominant with respect to $W = V(K_n) \setminus \{v_i\}$, $1 \leq i \leq n$. Here, for the vertex v_i outside W, there are n neighborhoods which are 1-dominant with respect to W and v_i belongs to it.

(ii) For the star graph S_n with n vertices $\{v_1, \ldots, v_n\}$, such that v_1 is the center vertex, $\vartheta_1(S_n) = n-1$. The closed neighborhood of the center vertex v_1 is 1-dominant with respect to the set $W = \{v_2, \ldots, v_n\}$. Here $N[v_1]$ is the unique 1-dominant neighborhood with respect to W for the vertex v_1 outside of W. Although, every neighborhood contains v_1, but $N[v_j]$ is not 1-dominant with respect to W, $j \neq 1$.

(iii) A wheel graph W_n with n vertices is 1-dominated, with the biggest 1-dominated, $\vartheta_1(W_n) = n - 1$.

(iv) Furthermore, if $K_{n_1,\ldots,n_k}$ is a complete k-partite graph and $n_i > 1$, then $W = V(K_{n_1,\ldots,n_k}) \setminus \{v_{1:i}\}$ is a 2-dominated set, where $v_{t:s}$ is the t-th vertex in s part $1 \leq s \leq k$. All the closed neighborhoods of the vertices in the j-th part are 2-dominant, $1 \leq i, j \leq k$, $i \neq j$. Clearly, $K_{n_1,\ldots,n_k}$ is not 1-dominated with respect to W, since there is not any 1-dominant neighborhood with respect to W.

Remark 1 *If Γ is a (k,l)-dominated graph, then Γ is a (h,m)-dominated graph for $h \geq k, m \geq l$. Suppose W is a (k,l)-dominated set of Γ. This means there exists a closed neighborhood $N[x]$ such that $d(x,w) \leq k$ and $d(x,v) \leq l$ for every vertex $w \in W$ and $v \in V(\Gamma) \setminus W$. Thus $d(x,w) \leq h$ and $d(x,v) \leq m$ which means k-dominant neighborhoods are h-neighborhoods and the assertion is obvious.*

According to the Remark 1., it is more preferable to find the minimal k and l. Therefore, in some of the results, we carry out the discussion on 1-dominated graph and so the results for k-dominated graph follows, where $k > 1$.

Lemma 2 *Let Γ be a graph with $n > 2$ vertices and $\gamma(\Gamma) = 1$. Then Γ is 1-dominated and $\vartheta_1(\Gamma) = n - 1$.*

Proof. Consider the set $V(\Gamma) \setminus \{y\}$ and the smallest dominating set of the graph $D = \{x\}$, $x \neq y$. By definition of the dominating set $N[x] = V(\Gamma)$, which implies $N[x]$ is the closed 1-dominant neighborhood with respect to $V(\Gamma) \setminus \{y\}$. ■

Lemma 3 *If W is a 1-dominated set of the graph Γ, then every subset of W is 1-dominated too.*

Proof. Assume $W' \subseteq W$ and by N, we denote an arbitrary 1-dominant neighborhood with respect to W. Clearly, every vertex $w' \in W'$ belongs to N, as $w' \in W$ and N is 1-dominant neighborhood with respect to W. This means N is 1-dominant neighborhood with respect to W' and for every vertex of the graph such a neighborhood exists, by the fact that W is 1-dominated set. If x is a vertex outside of W', then we have two cases. Firstly, suppose x does not belong to W, then clearly x covers by a 1-dominant neighborhood with respect to W, which is 1-dominant neighborhood with respect to W' by the above discussion. Secondly, if $x \in W$, then as W is a proper subset of $V(\Gamma)$, there is $y \in V(\Gamma)$ which is cover by 1-dominant neighborhood with respect to W. Thus the distance of y to all vertices of W and consequently to all vertices of W' is less or equal than 1. Hence the assertion follows. ■

Theorem 4 *Let Γ be a graph with an isolated vertex. Then Γ is 1-dominated if and only if $\Gamma \cong K_1$.*

Proof. If W is a 1-dominated set for the graph Γ with isolated vertex v, then $N[v]$ is the only neighborhood that includes v and is 1-dominant with respect to W. Let $w \in W$ be a vertex distinct from v. Therefore $w \notin N[v]$ and so by the definition of 1-dominant neighborhood, w and v are adjacent, but this is contradiction as v is isolated vertex. Thus $v = w$. We claim that Γ does not contain any other vertex than v, because otherwise if $v \neq u \in V(\Gamma)$, then all neighborhoods include u are not 1-dominant with respect to W as $v = w \in W$. ■

Theorem 4 implies that there is no 1-dominated graph with isolated vertex and number of vertices more than 1.

Theorem 5 *If Γ is a non-connected graph, then it is not 1-dominated graph.*

Proof. Consider v_1, v_2 two vertices of the graph for which there is no path between them. They belong to two disjoint components $\mathfrak{C}_1$ and $\mathfrak{C}_2$, respectively. Suppose W is a 1-dominated set for the graph such that its elements partitioned into two parts, belong to $\mathfrak{C}_1$ or $\mathfrak{C}_2$. Let $x \in W \cap \mathfrak{C}_1$ and $y \in W \cap \mathfrak{C}_2$. Because of the existence of $y \in W \cap \mathfrak{C}_2$, all the neighborhoods that x belongs to, are not 1-dominant with respect to W. Thus W contains the vertices of one component, say $\mathfrak{C}_1$. But this is again a contradiction. Since there is no 1-dominant neighborhood with respect to W that covers the vertices of $\mathfrak{C}_2$. Hence the non-connected graph Γ is not 1-dominated. ■

Lemmas 2, 3, Theorems 4 and 5 are true for k-dominated graphs, where $k \geq 1$ is arbitrary non-negative integer according to Remark 1. Moreover, by the hypothesis of Lemma 2., one can deduce that Γ is $(1, l)$-dominated and consequently (k, l)-dominated, where $l \geq 1$ is a non-negative integer. Similar to Lemma 3. it follows that every subset of a (k, l)-dominated set is also a (k, l)-dominated set. Theorems 4 and 5 can be generalized for (k, l)-dominated graph.

Proposition 6 *Let Γ be a 2-dominated graph. Then* $\mathrm{diam}(\Gamma) \leq 6$.

Proof. Suppose W a 2-dominated set and $v_1, v_2 \in V(\Gamma) \setminus W$ are two non-adjacent vertices. By the definition of a 2-dominated graph, all vertices which does not in W belongs to a 2-dominant neighborhoods with respect to W. If $N[x_i]$ are 2-dominant neighborhoods with respect to W which contain v_i, then $d(x_i, w) \leq 2$, where $w \in W$, $i = 1, 2$. Hence the result is clear. ■

Theorem 7 *If Γ is a (k, l)-dominated graph, then* $\mathrm{diam}(\Gamma) \leq 2k + 2l$.

The proof of Theorem 7. is similar to reasoning of Proposition 6. so we omit it.

The diameter of 1-dominated graphs is at most 4.

Theorem 8 *If* $\mathrm{diam}(\Gamma) \leq k$, *then Γ is k-dominated graph.*

Proof. Suppose V is the set of all vertices and $N[x]$ an arbitrary closed neighborhood. Since $\mathrm{diam}(\Gamma) \leq k$, we conclude $d(v, x) \leq k$, for vertex v. Hence all neighborhoods are k-dominant with respect to every proper subset of V. We claim that every proper subset of V is the k-dominated set. For every vertex of the graph v, v belongs to the k-dominant neighborhood $N[v]$ which is dominant with respect to every proper subset of V. ■

Note that the non-negative integer k in Theorem 8., is not the least number. The following proposition is direct result of Theorem 8.

Proposition 9 *If Γ is a graph with diameter at most 2, then*

(i) $\nu_2(\Gamma) = n - 1$, *where n is the number of vertices,*

(ii) *the least number of the 2-dominant neighborhoods is equal to $\gamma(\Gamma)$.*

The Peterson graph is 2-dominated, since the diameter of this graph is 2. Moreover, the least number of its 2-dominant neighborhoods is 3, since the domination number of this graph is 3.

Lemma 10 *Let Γ be a 1-dominated graph. Then the least number of 1-dominant neighborhoods of Γ with respect to the 1-dominated set of the graph is $\gamma(\Gamma)$.*

Proof. Suppose W is the 1-dominated set of the graph Γ. By definition, there are 1-dominant neighborhoods $N[i_1], N[i_2], \ldots, N[i_t]$ which covers all the vertices. It is clear that $\{i_1, i_2, \ldots, i_t\}$ form a dominating set for the graph and so $\gamma(\Gamma) \leq t$. ■

Let Γ be any graph of order n and maximum degree Δ, then $n/(1 + \Delta) \leq \gamma(\Gamma)$. Therefore by Lemma 10. least number of 2-dominant neighborhoods of Γ with respect to the 2-dominated set of the graph is $n/(1 + \Delta)$.

Note that Lemma 10. is correct for k-dominated graphs, $k \geq 1$ (see Remark 1.). Proposition 9. shows the the lower bound of the Lemma 10. is sharp for all the graphs with diameter at most 2. Lemma 10. guarantee the existence of a dominating set for k-dominated graphs, $k \geq 1$. But the converse is not true, for instance we can present a dominating set for the path P_8 which is not 2-dominated (see Theorem 11.).

3 Some k-dominated Graphs

In this section, we prove an equivalence condition in which a path and a cycle are k-dominated. It is clear that $P_2,\ P_3$ are 1-dominated and $\vartheta_1(P_n) = n-1$, $n = 2, 3$.

Theorem 11 *Let P_n be a path with n vertices. Then P_n is 2-dominated if and only if $n \leq 7$. Moreover, $\vartheta_2(P_n) = n-1$ for $2 \leq n \leq 4$, $\vartheta_2(P_5) = 3$, $\vartheta_2(P_6) = 2$ and $\vartheta_2(P_7) = 1$.*

Proof. Easily $\vartheta_2(P_n) = n-1$, for $2 \leq n \leq 4$. Suppose $n \geq 5$. If P_n is 2-dominated, then there exists a proper subset of vertices W such that every vertex of the path P_n outside of W belongs to a 2-dominant neighborhood with respect to W. Let us labeling the vertices of the graph by $1, 2, \ldots, n$. Since the vertex 1, just belongs to the neighborhoods $N[1]$ and $N[2]$, one of the neighborhoods $N[1]$ or $N[2]$ is 2-dominant with respect to W. The set W is the largest 2-dominated set so it is appropriate to consider the closed neighborhood $N[2]$ which covers more vertices than $N[1]$. Similar situation happened for the vertex n. According to the definition of the 2-dominated set the closed bound for the vertices which belong to W are vertices 4 and $n-3$ and the assertion follows. ■

In Theorem 12, we present the maximum number of the vertices in a k-dominated path.

Theorem 12 *The path P_n is a k-dominated graph if and only if $n \leq 5 + 2(k-1)$.*

Proof. Suppose P_n is a k-dominated graph with a k-dominated set W such that P_n vertices were labeled by numbers. Similar to the proof of Theorem 11. we can consider $N[2]$ and $N[n-1]$ as k-dominant neighborhoods. Thus the greatest vertex belongs to W is $3 + (k-1)$ as $d(2, 3 + (k-1)) = k$. Moreover, another bound for the integers that belongs to W is $(n-2) - (k-1)$, $d((n-2) - (k-1), n-1) = k$. Since $N[2]$ is k-dominat neighborhood so $3 + (k-1) \geq (n-2) - (k-1)$. Hence the assertion follows clearly. ■

One can deduce that the cycles $C_3,\ C_4$ and C_5 are 1-dominated, with 2, 2 and 1, 1-dominated number, respectively.

Theorem 13 *Let C_n be a cycle with n vertices. Then C_n is 2-dominated if and only if $n \leq 7$. Moreover, $\vartheta_2(C_n) = n-1$ for $3 \leq n \leq 5$, $\vartheta_2(C_6) = 4$ and $\vartheta_2(C_7) = 1$.*

Proof. Assume C_n is 2-dominated and W is its maximal 2-dominated set. If we label the vertices by numbers $1, 2, \ldots, n$, then for every vertex $1 \leq i \leq n$ we have three neighborhoods $N[i-1] = \{i-2, i-1, i\}$, $N[i] = \{i-1, i, i+1\}$, and $N[i+1] = \{i, i+1, i+2\}$ that it belongs. Since W is maximal 2-dominated set so at least one of the above closed neighborhoods is 2-dominant. Therefore, W is the subset of one of the sets $\{i-3, i-2, i-1, i, i+1\}$, $\{i-2, i-1, i, i+1, i+2\}$ or $\{i-1, i, i+1, i+2, i+3\}$. For $i+1, i+2$ and $i+3$, if they are strictly greater than n, then consider the integer modulo n. By computing $\{2, 3, 5, 6\}$ and $\{1\}$ are one of the maximal 2-dominated sets for C_6 and C_7, respectively. Finally, if $n \geq 8$, then there is no 2-dominated set for C_n, because there exists a vertex in C_n which is not in a 2-dominated neighborhood with respect to any proper subset of vertices. ■

Proposition 14 *The cycle C_n is a k-dominated graph if and only if $n \leq 2k + 3$.*

Proof. Similar to the proof of Theorem 13., with out loss of generality, if $N[1]$ is k-dominant neighborhood, then the greatest vertex in W is $k+1$, whenever the distance of another path from 1 to $k+1$ is more than $k+1$. Thus C_n has at most $2k+3$ vertices. ■

A Hamiltonian cycle is a cycle that visits each vertex exactly once. A Hamiltonian graph is a graph possessing a Hamiltonian cycle.

Proposition 15 *Let Γ be a Hamiltonian graph with Hamiltonian cycle C_n, $n \leq 2k+3$. Then Γ is k-dominated .*

Proof. From Proposition 14 follows C_n has the largest k-dominated set W. Thus every vertex of C_n belongs to a closed k-dominant neighborhood of C_n with respect to W. Since C_n pass through all the vertices of the graph Γ, the subset W is the k-dominated set for Γ. ■

A tree is an undirected graph in which any two vertices are connected by exactly one path. In other words, any acyclic connected graph is a tree.

Lemma 16 *T is a 2-dominated tree if and only if T is an induced subgraph of Δ in Figure (1). Moreover* $1 \leq \vartheta_2(T) \leq 3$.

Proof. Proposition 6., if T is a 2-dominated graph, then $diam(T) \leq 6$. Therefore, T includes the paths with 7 vertices. Hence T is an induced subgraph of the tree Δ in the Figure (1). The rest follows by Theorem 11. ■

By Lemma 16, it follows that a 2-dominated tree contains paths with at most 7 vertices.

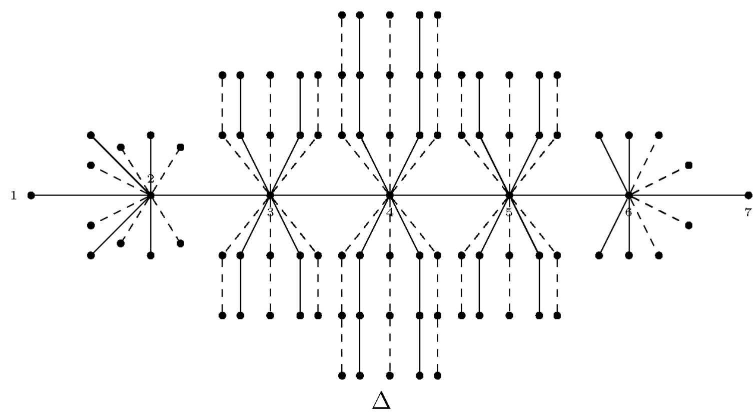

Figure 1: Every 2-dominated tree is an induced subgraph of Δ.

If Γ is a 2-dominated graph, then every induced subgraph of Γ is not 2-dominated, in general. For instance, its induced subgraph may be non-connected. Moreover, this is not true for all connected induced subgraph of Γ. Consider the graph in Figure (2) which is 2-dominated with dominated set $\{1\}$, but P_8 is connected induced subgraph of it, which is not 2-dominated.

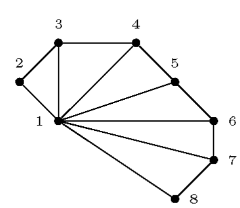

Figure 2: 2-dominated graph with non-2-dominated connected induced subgraph.

A spanning tree T of an undirected graph Γ is a subgraph that is a tree which includes all of the vertices of Γ, with minimum possible number of edges. The graph in Figure 2 shows that if the graph is 2-dominated, then its spanning tree may not be 2-dominated.

Let C_n be a an induced cycle for the unicyclic graph Γ. If the vertex x of Γ, which is not in C_n, is a pendant vertex (vertex of degree 1) of Γ, that belongs to a path $P = u, v, w, \ldots, y, x$ of length s from C_n, where u is the only vertex in C_n, we say that P is (with respect to C_n) a s-pendant path of Γ. In Figure 3 some 2-dominated unicyclic graphs are presented.

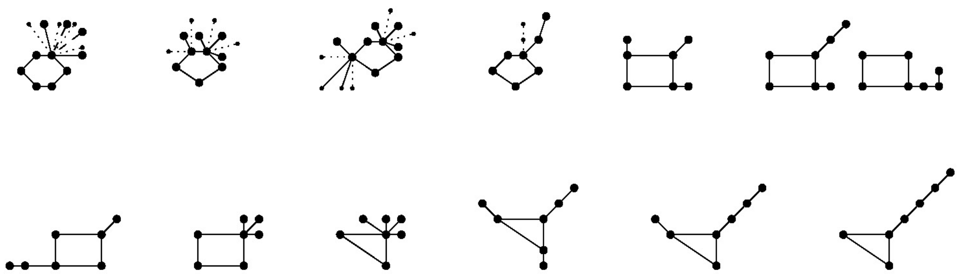

Figure 3: Some 2-dominated unicyclic graphs.

Theorem 17 *Γ is a 2-dominated unicyclic graph if and only if Γ is isomorphic to C_3, C_4, C_5, C_6, C_7 or induced subgraph of a graph with the cycle C_n, $n \leq 7$ and s-pendant path whose distance of its pendant vertex x to all other vertices of Γ is less or equal than 6.*

Proof. Suppose Γ is connected 2-dominated unicyclic graph which is not a cycle. Theorems 11 and 13 deduce the result. ■

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