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Characterization Of NH Distribution Through Generalized Record Values*

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Abstract

In this paper, we derive the exact expressions as well as recurrence relations for single and product moment of generalized record values from NH distribution. These relations generalize the results given by MirMostafee *et al.* (2016). Further, we characterize the given distribution through conditional expectation, recurrence relations and truncated moments.

1 Introduction

A generalization of the exponential distribution was introduced by Nadarajah and Haghighi (2011), which can be used as an alternative to the gamma, Weibull and exponentiated exponential distributions. The attractive feature of this distribution is always having the zero mode and yet allowing for increasing, decreasing and constant hazard rate functions. Lemonte (2013) called this distribution defined by Nadarajah and Haghighi (2011) also as the NH distribution.

A random variale X is said to have a NH distribution, if its probability density function (*pdf*) is of the form

$$f(x) = \theta(1+x)^{\theta-1} exp\{1-(1+x)^{\theta}\}, \quad x>0,\ \theta>0 \tag{1}$$

and the corresponding distribution function (*df*) is

$$F(x) = 1 - exp\{1-(1+x)^{\theta}\}, \quad x>0,\ \theta>0. \tag{2}$$

Note that for NH distribution defined in (1)

$$(1+x)f(x) = \theta[1-\ln \bar{F}(x)]\bar{F}(x), \tag{3}$$

where

$$\bar{F}(x) = 1 - F(x).$$

The exponential distribution arises when $\theta = 1$.

The concept of record values was introduced by Chandler (1952). An observation is called a record if its value is greater (or less) than all the previous observations. Record values are used in a wide variety of practical situations, such as industrial stress testing, meteorological analysis, hydrology, seismology, oil, mining surveys, sports and athletic events. For a survey on important results in this area one may refer to Ahsanullah (1995), Kamps (1995), Arnold *et al.* (1998) and Ahsanullah and Nevzorov (2015). Dziubdziela and Kopcoiński (1976) have generalized the concept of record values of Chandler (1952) by random variables of a more generalized nature and called them the k-th record values. Later, Minimol and Thomas (2013) called the record values defined by Dziubdziela and Kopcoiński (1976) also as the generalized record values,

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since the r-th member of the sequence of the ordinary record values is also known as the r-th record value. By setting $k = 1$, we obtain ordinary record statistics.

Kamps (1995) and Danielak and Raqab (2004) pointed out in reliability theory that in some situations record values themselves are viewed as outliers, then the second or third largest values are of special interest. For statistical inference based on ordinary records, serious difficulties arise if expected values of inter arrival time of records is infinite and occurrences of records are very rare in practice. This problem is avoided once we consider the model of generalized record values.

Let $\{X_n,\, n \geq 1\}$ be a sequence of independent and identical distributed (*iid*) continuous random variables with *df* $F(x)$ and *pdf* $f(x)$. Then for a fixed positive integer $k \geq 1$, the sequence of k-th upper record times $\{U_n^{(k)}, n \geq 1\}$ is defined as Nevzorov (2001):

$$U_1^{(k)} = k$$

and for $n \geq 1$,

$$U_{n+1}^{(k)} = \min\left\{ j : j > U_n^{(k)}, X_j > X_{U_n^{(k)}-k+1:U_n^{(k)}} \right\}.$$

The sequence $\{Y_n^{(k)}, n \geq 1\}$, where $Y_n^{(k)} = X_{U_n^{(k)}}$ is called the sequence of generalized upper record values (k-th upper record values) of $\{X_n, n \geq 1\}$. Note that for $k = 1$, we have $Y_n^{(1)} = X_{U_n}$, $n \geq 1$, which are the record values of $\{X_n, n \geq 1\}$ as defined in Ahsanullah (1995).

The *pdf* of $Y_n^{(k)}$ and the joint *pdf* of $Y_m^{(k)}$ and $Y_n^{(k)}$ are given by (Dziubdziela and Kopcoiński (1976), Grudziaeń (1982))

$$f_{Y_n^{(k)}}(x) = \frac{k^n}{(n-1)!}[-\ln \bar{F}(x)]^{n-1}[\bar{F}(x)]^{k-1} f(x), \quad n \geq 1, \tag{4}$$

$$\begin{aligned} f_{Y_m^{(k)}, Y_n^{(k)}}(x, y) \quad = \quad & \frac{k^n}{(m-1)!\,(n-m-1)!}[-\ln \bar{F}(x)]^{m-1} \frac{f(x)}{\bar{F}(x)} \\ & \times [\ln \bar{F}(x) - \ln \bar{F}(y)]^{n-m-1}[\bar{F}(y)]^{k-1} f(y), \quad x < y,\, 1 \leq m < n, \end{aligned} \tag{5}$$

and the conditional *pdf* of $Y_n^{(k)}$ given $Y_m^{(k)} = x$, is

$$f_{Y_n^{(k)}|Y_m^{(k)}}(y|x) = \frac{k^{n-m}}{(n-m-1)!}[\ln \bar{F}(x) - \ln \bar{F}(y)]^{n-m-1} \Big[\frac{\bar{F}(y)}{\bar{F}(x)}\Big]^{k-1} \frac{f(y)}{\bar{F}(x)}, \qquad x < y. \tag{6}$$

Properties of the k-th upper record values of $\{X_n, n \geq 1\}$ have been extensively studied in the literature, see for instance Dziubdziela and Kopcoiński (1976), Deheuvels (1982, 1988), Grudzień (1982), Grudzień and Szynal (1997). For some recent developments on generalized record values with special reference to those arising from exponential, Gumble, Pareto, generalized Pareto, Burr, Weibull, Makeham, Gompertz, modified-Weibull, exponential-Weibull, additive Weibull and Kumaraswamy log-logistic distributions carried out by Grudzień and Szynal (1983, 1997), Pawlas and Szynal (1998, 1999, 2000), Paul and Thomas (2015, 2016), Minimol and Thomas (2013, 2014), Khan and Khan (2016), Khan *et al*. (2015, 2017) and Singh *et al*. (2019a,b) respectively.

In this paper we mainly focus on the study of generalized upper record values arising from NH distribution and discussed exact explicit expressions as well as several recurrence relations satisfied by single and product moments. In addition, conditional expectations and recurrence relations for single moments of generalized record values and truncated moment are used to characterize this distribution.

2 Relations for Single Moments

In this section, we derive the exact expressions for single moments of generalized upper record values and recurrence relations in the following theorems.

Theorem 1 *For the distribution given in (2), fix a positive integer $k \geq 1$, for $n \geq 1$ and $j = 0, 1, \dots$*

$$E\big(Y_n^{(k)}\big)^j = \frac{e^k\, k^n}{(n-1)!} \sum_{u=0}^{n-1} \sum_{v=0}^{j} (-1)^{n+j-u-v-1} \binom{n-1}{u} \binom{j}{v} \frac{\Gamma[u+(\frac{v}{\theta})+1, k]}{k^{u+(\frac{v}{\theta})+1}}. \tag{7}$$

Proof. In view of (4), we have

$$E\big(Y_n^{(k)}\big)^j = \frac{k^n}{(n-1)!} \int_0^\infty x^j [-\ln \bar{F}(x)]^{n-1} [\bar{F}(x)]^{k-1} f(x) dx, \ \ n \geq 1\,.$$

Now using (2), we get

$$E\big(Y_n^{(k)}\big)^j = \frac{k^n}{(n-1)!} \int_0^\infty [\{1-\ln \bar{F}(x)\}^{\frac{1}{\theta}} - 1]^j \,[-\ln \bar{F}(x)]^{n-1} [\bar{F}(x)]^{k-1} f(x) dx. \tag{8}$$

On applying binomial expansion in (8), we find that

$$E\big(Y_n^{(k)}\big)^j = \frac{k^n}{(n-1)!} \sum_{v=0}^{j} (-1)^{j-v} \binom{j}{v} \int_0^\infty [1-\ln \bar{F}(x)]^{\frac{v}{\theta}} [-\ln \bar{F}(x)]^{n-1} [\bar{F}(x)]^{k-1} f(x) dx. \tag{9}$$

After substituting $t = [1-\ln \bar{F}(x)]$ in (9), we get

$$\begin{aligned} E\big(Y_n^{(k)}\big)^j &= \frac{k^n}{(n-1)!} \sum_{v=0}^{j} (-1)^{j-v} \binom{j}{v} \int_1^\infty t^{\frac{v}{\theta}} (t-1)^{n-1} e^{k(1-t)} dt \\ &= \frac{e^k\, k^n}{(n-1)!} \sum_{u=0}^{n-1} \sum_{v=0}^{j} (-1)^{n+j-u-v-1} \binom{n-1}{u} \binom{j}{v} \int_1^\infty t^{u+\frac{v}{\theta}} e^{-kt} dt. \end{aligned}$$

In view of result on generalized incomplete gamma function obtained by Chaudhry and Zubair (1994), we may obtain the result given in (7). ■

Remark 1 *At $\theta = 1$ in (8) and then substituting $t = -\ln \bar{F}(x)$ in the resulting expression, we have*

$$E\big(Y_n^{(k)}\big)^j = \frac{k^n}{(n-1)!} \int_0^\infty t^{j+n-1}\, e^{-kt} dt = \frac{\Gamma(j+n)}{(n-1)!\, k^j},$$

which verifies the result obtained by Kamps (1995) for exponential distribution.

Remark 2 *Setting $k = 1$ in (7), we get the exact moments of upper records from NH distribution as obtained by MirMostafaee et al. (2016), for $n = s$ and $j = r$.*

Theorem 2 *For the distribution given in (2), fix a positive integer $k \geq 1$, for $n \geq 1$, $n \geq k$ and $j = 0, 1, \dots$*

$$E(Y_{n+1}^{(k)})^{j+1} = E(Y_n^{(k)})^{j+1} + \frac{j+1}{n\theta}\Big[E(Y_n^{(k)})^j + E(Y_n^{(k)})^{j+1}\Big] - \frac{k}{n}\Big[E(Y_n^{(k)})^{j+1} - E(Y_{n-1}^{(k)})^{j+1}\Big]. \tag{10}$$

Proof. From (4) and (3), for $n \geq 1$ and $j = 0, 1, \dots,$ we have

$$\begin{aligned} & [E(Y_n^{(k)})^j + E(Y_n^{(k)})^{j+1}] \\ = \;& \frac{\theta\, k^n}{(n-1)!} \int_0^\infty x^j [-\ln \bar{F}(x)]^{n-1} [\bar{F}(x)]^k [1-\ln \bar{F}(x)] dx \\ = \;& \frac{\theta k^n}{(n-1)!} \Big\{ \int_0^\infty x^j [-\ln \bar{F}(x)]^{n-1} [\bar{F}(x)]^k dx + \int_0^\infty x^j [-\ln \bar{F}(x)]^n [\bar{F}(x)]^k dx \Big\}. \end{aligned} \tag{11}$$

Now, (10) can be seen by noting that in view of Khan *et al.* (2017)

$$[E(Y_n^{(k)})^j - E(Y_{n-1}^{(k)})^j] = \frac{j\,k^{n-1}}{(n-1)!}\int_\alpha^\beta x^{j-1}[-\ln \bar{F}(x)]^{n-1}[\bar{F}(x)]^k dx.$$

■

Remark 3 **(i)** *Setting $k = 1$ and $\theta = 1$ in (10), we deduce the recurrence relations for single moments of upper records from exponential distribution as established by Pawlas and Szynal* (1998)*.*

(ii) *Putting $k = 1$ in (10), we get the recurrence relations for single moments from standard NH distribution as obtained by MirMostafaee et al.* (2016)*.*

3 Relations for Product Moments

In this section, we derive the recurrence relations for the product moments of generalized upper record values in the following theorem.

Theorem 3 *For the distribution given in (2) and $m \geq 1$, $m \geq k$ and $i, j = 0, 1, \ldots,$*

$$\begin{aligned} & E[(Y_m^{(k)})^i (Y_{m+1}^{(k)})^j] \\ = & \frac{\theta}{i+1}\Big\{m\,E(Y_{m+1}^{(k)})^{i+j+1} + k\,E(Y_m^{(k)})^{i+j+1}\Big\} \\ & -\frac{\theta}{i+1}\Big\{k\,E[(Y_{m-1}^{(k)})^{i+1}(Y_m^{(k)})^j] + m\,E[(Y_m^{(k)})^{i+1}(Y_{m+1}^{(k)})^j]\Big\} - E[(Y_m^{(k)})^{i+1}(Y_{m+1}^{(k)})^j]. \end{aligned} \tag{12}$$

and for $1 \leq m \leq n-2$, and $i, j = 0, 1, \ldots,$

$$\begin{aligned} & E[(Y_m^{(k)})^i (Y_n^{(k)})^j] \\ = & \frac{\theta}{i+1}\Big\{m\,E[(Y_{m+1}^{(k)})^{i+1}(Y_n^{(k)})^j] + k\,E[(Y_m^{(k)})^{i+1}(Y_{n-1}^{(k)})^j]\Big\} \\ & -\frac{\theta}{i+1}\Big\{k\,E[(Y_{m-1}^{(k)})^{i+1}(Y_{n-1}^{(k)})^j] + m\,E[(Y_m^{(k)})^{i+1}(Y_n^{(k)})^j]\Big\} - E[(Y_m^{(k)})^{i+1}(Y_n^{(k)})^j]. \end{aligned} \tag{13}$$

Proof. From (5) and (3), for $m \leq n-1$, we have

$$\begin{aligned} & E[(Y_m^{(k)})^i (Y_n^{(k)})^j] + E[(Y_m^{(k)})^{i+1}(Y_n^{(k)})^j] \\ = & \frac{\theta\,k^n}{(m-1)!\,(n-m-1)!}\int_0^\infty\int_0^y x^i\,y^j[-\ln\bar{F}(x)]^{m-1} \\ & \times[1-\ln\bar{F}(x)][\ln\bar{F}(x)-\ln\bar{F}(y)]^{n-m-1}[\bar{F}(y)]^{k-1}f(y)dxdy \\ = & \frac{\theta\,k^n}{(m-1)!\,(n-m-1)!}\int_0^\infty\int_0^y x^i\,y^j[-\ln\bar{F}(x)]^{m-1}[\ln\bar{F}(x)-\ln\bar{F}(y)]^{n-m-1} \\ & \times[\bar{F}(y)]^{k-1}f(y)dxdy + \frac{\theta\,k^n}{(m-1)!\,(n-m-1)!}\int_0^\infty\int_0^y x^i\,y^j[-\ln\bar{F}(x)]^m \\ & \times[\ln\bar{F}(x)-\ln\bar{F}(y)]^{n-m-1}[\bar{F}(y)]^{k-1}f(y)dxdy \\ = & \frac{\theta\,k^n}{(m-1)!\,(n-m-1)!}\int_0^\infty y^j[\bar{F}(y)]^{k-1}f(y)I_1(y)dy \\ & +\frac{\theta\,k^n}{(m-1)!\,(n-m-1)!}\int_0^\infty y^j[\bar{F}(y)]^{k-1}f(y)I_2(y)dy, \end{aligned} \tag{14}$$

where

$$I_1(y) = \int_0^y x^i[-\ln\bar{F}(x)]^{m-1}\,[\ln\bar{F}(x)-\ln\bar{F}(y)]^{n-m-1}dx.$$

Integrating $I_1(y)$ by parts, we get

$$\begin{aligned} I_1(y) &= \frac{(n-m-1)}{(i+1)} \int_0^y x^{i+1}\,[-\ln \bar{F}(x)]^{m-1}[\ln \bar{F}(x) - \ln \bar{F}(y)]^{n-m-2}\frac{f(x)}{[\bar{F}(x)]}d(x) \\ &\quad -\frac{(m-1)}{(i+1)} \int_0^y x^{i+1}\,[-\ln \bar{F}(x)]^{m-2}[\ln \bar{F}(x) - \ln \bar{F}(y)]^{n-m-1}\frac{f(x)}{[\bar{F}(x)]}d(x). \end{aligned} \tag{15}$$

Similarly

$$\begin{aligned} I_2(y) &= \frac{(n-m-1)}{(i+1)} \int_0^y x^{i+1}\,[-\ln \bar{F}(x)]^{m}[\ln \bar{F}(x) - \ln \bar{F}(y)]^{n-m-2}\frac{f(x)}{[\bar{F}(x)]}d(x) \\ &\quad -\frac{m}{(i+1)} \int_0^y x^{i+1}\,[-\ln \bar{F}(x)]^{m-1}[\ln \bar{F}(x) - \ln \bar{F}(y)]^{n-m-1}\frac{f(x)}{[\bar{F}(x)]}d(x). \end{aligned} \tag{16}$$

Substituting $I_1(y)$ and $I_2(y)$ in (14) and simplifying the resulting expression, yields the result given in (13).

Now putting $n = m+1$ in (13) and noting that $E[(Y_m^{(k)})^i (Y_m^{(k)})^j] = E[(Y_m^{(k)})^{i+j}]$, the recurrence relations given in (12) can be easily be established. ■

Remark 4 **(i)** *Setting $j = 0$ in (13), we deduce the recurrence relations for single moments from NH distribution as obtained in (10).*

(ii) *Setting $k = 1$ and $\theta = 1$ in (13), we deduce the recurrence relations for product moments of upper record values from exponential distribution as obtained by Pawlas and Szynal (1998).*

(iii) *Setting $k = 1$ in (13), we deduce the recurrence relations for product moments of upper record values from NH distribution as obtained by MirMostafaee et al. (2016).*

(iv) *Setting $k = 1$ and $j = 0$ in (13), we get the recurrence relations for single moments of upper record values from NH distribution as established by MirMostafaee et al. (2016).*

4 Characterization

This section contains the characterizations of NH distribution, we start with the following result of Lin (1986).

Proposition 4 *Let n_0 be any fixed non-negative integer, $-\infty < a < b < \infty$ and $g(x) \geq 0$ an absolutely continuous function with $g^{'}(x) \neq 0$ a.e. on (a, b). Then the sequence of functions $\{(g(x))^n\, e^{-g(x)}, n \geq n_0\}$ is complete in $L(a, b)$ iff $g(x)$ is strictly monotone on (a, b).*

Theorem 5 *Fix a positive integer $k \geq 1$ and let j be a nonnegative integer. A necessary and sufficient condition for a random variable X to be distributed with pdf given by (1) is that*

$$E(Y_{n+1}^{(k)})^{j+1} = E(Y_n^{(k)})^{j+1} + \frac{j+1}{n\theta}\Big[E(Y_n^{(k)})^{j} + E(Y_n^{(k)})^{j+1}\Big] - \frac{k}{n}\Big[E(Y_n^{(k)})^{j+1} - E(Y_{n-1}^{(k)})^{j+1}\Big], \tag{17}$$

for $n = 1, 2, \ldots$ and $n \geq k$.

Proof. The necessary part follows from (10). On the other hand if the recuerrence relations (17) is satisfied, then on rearranging the terms in (17) and using Khan et al. (2017), we have

$$\begin{aligned} &\frac{k^n}{(n-1)!}\int_0^\infty x^j (1+x)[-\ln \bar{F}(x)]^{n-1}[\bar{F}(x)]^{k-1} f(x)dx \\ = \;&\frac{\theta\, k^n}{(n-1)!}\int_0^\infty x^j [-\ln \bar{F}(x)]^{n}[\bar{F}(x)]^{k} dx + \frac{\theta\, k^n}{(n-1)!}\int_0^\infty x^j [-\ln \bar{F}(x)]^{n-1}[\bar{F}(x)]^{k} dx \end{aligned}$$

which implies

$$\frac{\theta\, k^n}{(n-1)!}\int_0^\infty x^j[-\ln\bar F(x)]^{n-1}[\bar F(x)]^{k-1}\Big\{(1+x)f(x)-\theta[-\ln\bar F(x)][\bar F(x)]-\theta[\bar F(x)]\Big\}dx=0. \tag{18}$$

It now follow from the above proposition with

$$g(x)=-\ln\bar F(x)$$

that

$$(1+x)f(x)=\theta\,[1-\ln\bar F(x)]\bar F(x)$$

which proves that $f(x)$ has the form as given in (1). ■

Theorem 6 *Let X be a non-negative random variable having an absolutely continuous df $F(x)$ and $F(0)=0$ and $0\le F(x)\le 1$ for all $x>0$. Then*

$$E[\xi(Y_n^{(k)})|(Y_l^{(k)})=x]=exp\{1-(1+x)^\theta\}\Big(\frac{k}{k+1}\Big)^{n-l},\ \ l=m,\,m+1,\,m\ge k \tag{19}$$

if and only if

$$\bar F(x)=exp\{1-(1+x)^\theta\},\ x>0,\ \theta>0,$$

where

$$\xi(y)=exp\{1-(1+y)^\theta\}.$$

Proof. From (6), we have

$$\begin{aligned}&E[\xi(Y_n^{(k)})|(Y_m^{(k)})=x]\\ =\ &\frac{k^{n-m}}{(n-m-1)!}\int_x^\infty exp\{1-(1+y)^\theta\}[\ln\bar F(x)-\ln\bar F(y)]^{n-m-1}\Big[\frac{\bar F(y)}{\bar F(x)}\Big]^{k-1}\frac{f(y)}{\bar F(x)}dy.\end{aligned} \tag{20}$$

By setting $u=\frac{\bar F(y)}{\bar F(x)}=\frac{exp\{1-(1+y)^\theta\}}{exp\{1-(1+x)^\theta\}}$ from (2) in (20), we have

$$E[\xi(Y_n^{(k)})|(Y_m^{(k)})=x]=\frac{k^{n-m}}{(n-m-1)!}exp\{1-(1+x)^\theta\}\int_0^1 u^k\,[-\ln u]^{n-m-1}du. \tag{21}$$

We have Gradshteyn and Ryzhik ((2007), $p-551$)

$$\int_0^1[-\ln x]^{\mu-1}\,x^{\nu-1}dx=\frac{\Gamma(\mu)}{\nu^\mu},\quad \mu>0,\,\nu>0. \tag{22}$$

On using (22) in (21), we have the result given in (19).

To prove sufficient part, we have

$$\frac{k^{n-m}}{(n-m-1)!}\int_x^\infty exp\{1-(1+y)^\theta\}[-\ln\bar F(y)+\ln\bar F(x)]^{n-m-1}[\bar F(y)]^{k-1}f(y)dy=[\bar F(x)]^k\,g_{n|m}(x), \tag{23}$$

where

$$g_{n|m}(x)=[exp\{1-(1+x)^\theta\}]\Big(\frac{k}{k+1}\Big)^{n-m}.$$

Differentiating (23) both sides with respect to x, we get

$$\begin{aligned}&-\frac{k^{n-m}}{(n-m-2)!}\frac{f(x)}{\bar F(x)}\int_x^\infty exp\{1-(1+y)^\theta\}[-\ln\bar F(y)+\ln\bar F(x)]^{n-m-2}[\bar F(y)]^{k-1}f(y)dy\\ =\ &g'_{n|m}(x)[\bar F(x)]^k-k\,g_{n|m}(x)\,[\bar F(x)]^{k-1}f(x)\end{aligned}$$

or

$$-k\, g_{n|m+1}(x)[\bar{F}(x)]^{k-1} = g^{'}_{n|m}(x)[\bar{F}(x)]^{k} - k\, g_{n|m}(x)\, [\bar{F}(x)]^{k-1} f(x).$$

Therefore,

$$\frac{f(x)}{\bar{F}(x)} = -\frac{g^{'}_{n|m}(x)}{k[g_{n|m+1}(x) - g_{n|m}(x)]} = \theta(1+x)^{\theta-1}, \tag{24}$$

where

$$g^{'}_{n|m}(x) = -\theta(1+x)^{\theta-1}\, exp\{1-(1+x)^{\theta}\}\Big(\frac{k}{k+1}\Big)^{n-m},$$

$$g_{n|m+1}(x) - g_{n|m}(x) = \frac{1}{k} exp\{1-(1+x)^{\theta}\}\Big(\frac{k}{k+1}\Big)^{n-m}.$$

Integrating both the sides (24) with respect to x over $(0, y)$, the sufficiency part is proved. ∎

Theorem 7 *Suppose an absolutely continuous (with respect to Lebesque measure) random variable X has the df $F(x)$ and the pdf $f(x)$ for $0 < x < \infty$, such that $f'(x)$ and $E(X|X \leq x)$ exist for all x. Then*

$$E(X|X \leq x) = g(x)\eta(x), \tag{25}$$

where

$$\eta(x) = \frac{f(x)}{F(x)}$$

and

$$g(x) = -\frac{x}{\theta(1+x)^{\theta-1}} + \frac{\int_0^x\ exp\{1-(1+t)^{\theta}\}dt}{\theta(1+x)^{\theta-1}\, exp\{1-(1+x)^{\theta}\}}$$

if and only if

$$f(x) = \theta(1+x)^{\theta-1} exp\{1-(1+x)^{\theta}\}, \quad x > 0,\, \theta > 0.$$

Proof. From (1), we have

$$E(X|X \leq x) = \frac{\theta}{F(x)} \int_0^x t\,(1+t)^{\theta-1} exp\{1-(1+t)^{\theta}\}dt. \tag{26}$$

Integrating (26) by parts treating $'^{\theta-1} exp\{1-(1+t)^{\theta}\}'$ for integration and rest of the integrand for differentiation, we get

$$E(X|X \leq x) = \frac{1}{F(x)}\Big[x\,\{1 - exp(1-(1+x)^{\theta})\} - \int_0^x \{1 - exp(1-(1+t)^{\theta})\}dt\Big]. \tag{27}$$

After multiplying and dividing by $f(x)$ in (27), we have the result given in (25).

To prove the sufficient part, we have from (25)

$$\int_0^x t f(t)dt = g(x)f(x). \tag{28}$$

Differentiating (28) on both the sides with respect to x, we find that

$$xf(x) = g'(x)f(x) + g(x)f'(x).$$

Therefore,

$$\begin{aligned} \frac{f'(x)}{f(x)} &= \frac{x - g'(x)}{g(x)} \qquad\qquad [Ahsanullah, et\ al.\ (2016)] \\ &= (\theta-1)(1+x)^{-1} - \theta(1+x)^{\theta-1}, \end{aligned} \tag{29}$$

where

$$g'(x) = x - g(x)\big[(\theta-1)(1+x)^{-1} - \theta(1+x)^{\theta-1}\big].$$

Integrating both the sides in (29) with respect to x, we get

$$f(x) = c\,(1+x)^{\theta-1} exp\{1-(1+x)^{\theta}\}.$$

Now, using the condition $\int_0^\infty f(x)dx = 1$, we obtains

$$f(x) = \theta(1+x)^{\theta-1} exp\{1-(1+x)^{\theta}\}, \quad x>0, \quad \theta>0.$$

■

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