Applied Mathematics E-Notes, 20(2020), 398-405 © ISSN 1607-2510
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Note On Characterization Of Linear Hazard Rate Distribution By Generalized Record Values*

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Received 15 September 2019

Abstract

The aim of this short note is to provide the characterization of linear hazard rate distribution through the recurrence relation of product moments, conditional moments of generalized upper record values and truncated moment. Further, some important deductions are also discussed.

1 Introduction

Dziubdziela and Kopociński [1] have generalized the concept of record values of Chandler [2] by random variables of a more generalized nature and called them the *k-th* record values. Later Minimol and Thomas [3] called the record values defined by Dziubdziela and Kopociński [1] also as the generalized record values, since the *r-th* member of the sequence of the ordinary record values is also known as the *r-th* record value.

Let $\{X_n,\ n \geq 1\}$ be a sequence of independent and identically distributed (*iid*) random variables with distribution function(*df*) $F(x)$ and the probability density function (*pdf*) $f(x)$. For a fixed positive integer k, Dziubdziela and Kopociński [1] defined the sequence $\left\{U_n^{(k)}, n \geq 1\right\}$ of *k-th* upper record times for the sequence $\{X_n,\ n \geq 1\}$ as follows:

$$U_1^{(k)} = 1$$

$$U_{n+1}^{(k)} = \min\left\{j > U_n^{(k)} :\ X_{j:j+k-1} > X_{U_n^{(k)}:\ U_n^{(k)}+k-1}\right\},$$

where $X_{j:n}$ is the *j-th* order statistic of the sample $X_1,\ X_2, \ldots\ldots, X_n$. The sequence $\{Y_n^{(k)}, n \geq 1\}$, where $Y_n^{(k)} = X_{U_n^{(k)}}$ is called the sequence of *k-th* upper record values or generalized upper record values of $\{X_n,\ n \geq 1\}$. Note that for $k = 1$, we write $Y_n^{(1)} = X_{U_{(n)}},\quad n \geq 1$, which are the upper record values of $\{X_n,\ n \geq 1\}$ as defined (Ahsanullah, [4]) . Moreover, we see that $Y_0^{(k)} = 0$ and $Y_1^{(k)} = min(X_1,\ X_2, \ldots\ldots, X_n) = X_{1:k}$.

The *pdf* of $Y_n^{(k)}$ and the joint *pdf* of $Y_m^{(k)}$ and $Y_n^{(k)}$ are given by (Dziubdziela and Kopociński [1] and Grudzień [5])

$$f_{Y_n^{(k)}}(x) = \frac{k^n}{(n-1)!}[-ln\overline{F}(x)]^{n-1}[\overline{F}(x)]^{k-1}f(x), \quad n \geq 1 \tag{1}$$

and

$$\begin{aligned} f_{Y_m^{(k)}, Y_n^{(k)}} \quad (x,y) &= \frac{k^n}{(m-1)!\,(n-m-1)!}[-ln\overline{F}(x)]^{m-1}\frac{f(x)}{\overline{F}(x)} \\ &\quad \times [ln\overline{F}(x) - ln\overline{F}(y)]^{n-m-1}[\overline{F}(y)]^{k-1}f(y), \quad x < y,\ 1 \leq m < n,\ n \geq 2, \end{aligned} \tag{2}$$

where

$$\bar{F}(x) = 1 - F(x).$$

*Mathematics Subject Classifications: 62G30, 62E10, 60E05.

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The conditional *pdf* of $Y_n^{(k)}$ given $Y_m^{(k)} = x$ is

$$f_{Y_n^{(k)}|Y_m^{(k)}}(y|x) = \frac{k^{n-m}}{(n-m-1)!}[ln\overline{F}(x) - ln\overline{F}(y)]^{m-1}\left(\frac{\overline{F}(y)}{\overline{F}(x)}\right)^{k-1}\frac{f(x)}{\overline{F}(x)}, \quad x < y. \tag{3}$$

The statistical study of record values of a sequence of iid continuous random variables was carried out by Chandler [2]. For a survey on important results developed in this area one may refer to Glick [6] and Arnold et al. [7].

Applications of k-th record values can be found in the literature, for instance, see the examples cited in Kamps [8] and Danielak and Raqab [9] in reliability theory. Record values can be viewed as order statistics from a sample whose size is determined by the values and the order of occurrence of the observations.

Generalized upper record values have been considered among others, by Grudzień and Szynal [10] characterized the continuous distributions via moments of k-th record values with random indices. Pawlas and Szynal [11, 12] established relations for single and product moments of k-th record values from exponential, Gumbel , Pareto, generalized Pareto and Burr distributions. Minimol and Thomas [13] studied the properties of Makeham and Gompertz distribution and characterized using generalized upper record values. Kumar and Khan [14] obtained recurrence relations for moment$s$ of k-th record values from generalized beta II distribution and Khan, et al. [15] presented relations for moments of generalized record values from additive Weibull distribution and associated inference and Khan and Khan [16] characterized the distribution having power hazard function by properties of generalized upper record values.

In reliability theory, lifetime distributions are often specified by choosing a particular hazard rate function. One of these hazard rate functions is the linear hazard rate function. The linear hazard rate function has the form (Bain [17]):

$$h(x) = \alpha + \beta x, x > 0, \quad \alpha, \beta \geq 0 \tag{4}$$

where α, β are the parameters such that α+ $\beta > 0$.

The linear hazard rate function given in (4) induces the *pdf*

$$f(x) = (\alpha + \beta x)e^{-\left\{\alpha x + \frac{\beta}{2}x^2\right\}}, x > 0, \quad \alpha, \beta \geq 0. \tag{5}$$

which is known as linear hazard rate distribution.

The distribution function corresponding to (5) is

$$F(x) = 1 - e^{-\left\{\alpha x + \frac{\beta}{2}x^2\right\}}, \quad x > 0, \quad \alpha, \beta \geq 0. \tag{6}$$

Henceforth, linear hazard rate distribution will be denoted by LHR distribution.

The LHR distribution contains the following well known life time distributions:

(i) When $\beta = 0$, LHR distribution reduces to the exponential distribution.

(ii) When $\alpha = 0$, LHR distribution reduces to the Rayleigh distribution.

From equation (5) and (6), we note that the characterizing differential equation for LHR distribution is given by,

$$f(x) = (\alpha + \beta x)\overline{F}(x) \tag{7}$$

This relation will be used to characterize the LHR distribution. For more details, properties and application of LHR distribution, see (Bain [17]). Neither the characterization results through the recurrence relations of product moments and conditional moments of usual upper record values nor those of generalized upper record values arising from the LHR distribution are known to have been obtained in the available literature.

2 Characterizations

Characterization of probability distributions play an important role in statistical theory. Different methods are used for characterization of continuous distributions. Characterization based on recurrence relations, conditional moments and truncation moment are amongst them.

In the following theorems, we introduce the characterization of linear hazard rate distribution using the recurrence relations for product moments and conditional moments based on generalized upper record values.

Theorem 1 *For a positive integer k, let i and j be non-negative integers. A necessary and sufficient condition for a random variable X to be distributed with pdf given by (5) is that*

$$\begin{aligned} E\left(Y_m^{(k)}\right)^i\left(Y_n^{(k)}\right)^j &= \frac{\alpha k}{(j+1)}\left\{\left[E\left(Y_m^{(k)}\right)^i\left(Y_n^{(k)}\right)^{j+1}\right]-\left[E\left(Y_m^{(k)}\right)^i\left(Y_{n-1}^{(k)}\right)^{j+1}\right]\right\} \\ &+\frac{\beta k}{(j+2)}\left\{\left[E\left(Y_m^{(k)}\right)^i\left(Y_n^{(k)}\right)^{j+2}\right]-\left[E\left(Y_m^{(k)}\right)^i\left(Y_{n-1}^{(k)}\right)^{j+2}\right]\right\} \end{aligned} \tag{8}$$

Proof. In view of Saran and Singh [18] necessary part can be proved. Alternatively, if the relation in (8) is satisfied, then by applying (2) and (7), we have:

$$\begin{aligned} E\left(Y_m^{(k)}\right)^i\left(Y_n^{(k)}\right)^j &= \frac{\alpha k}{(j+1)}\frac{k^n}{(m-1)!\,(n-m-1)!}\int_0^\infty\int_x^\infty x^i y^{j+1}\left[-\ln\overline{F}(x)\right]^{m-1} \\ &\times\frac{f(x)}{\overline{F}(x)}\left[\overline{F}(y)\right]^{k-1}\left\{\left[-ln\overline{F}(y)+ln\overline{F}(x)\right]-\frac{(n-m-1)}{k}\right\}f(y)\,dydx \\ &+\frac{\beta k}{(j+2)}\frac{k^n}{(m-1)!\,(n-m-1)!}\int_0^\infty\int_x^\infty x^i y^{j+2}\left[-\ln\overline{F}(x)\right]^{m-1}\frac{f(x)}{\overline{F}(x)}\left[\overline{F}(y)\right]^{k-1} \\ &\times\left\{\left[-ln\overline{F}(y)+ln\overline{F}(x)\right]-\frac{(n-m-1)}{k}\right\}f(y)\,dydx. \end{aligned} \tag{9}$$

Let

$$G(y)=-\frac{1}{k}\left[-ln\overline{F}(y)+ln\overline{F}(x)\right]^{n-m-1}\left[\overline{F}(y)\right]^k. \tag{10}$$

Differentiating both sides of (10), we get

$$\begin{aligned} G^{'}(y) &= \left[-ln\overline{F}(y)+ln\overline{F}(x)\right]^{n-m-2}\left[\overline{F}(y)\right]^{k-1}f(y) \\ &\times\left\{\left(-ln\overline{F}(y)+ln\overline{F}(x)\right)-\frac{(n-m-1)}{k}\right\} \end{aligned} \tag{11}$$

Thus,

$$\begin{aligned} &E\left(Y_m^{(k)}\right)^i\left(Y_n^{(k)}\right)^j \\ =\ &\frac{\alpha k}{(j+1)}\frac{k^n}{(m-1)!\,(n-m-1)!}\int_0^\infty\int_x^\infty x^i y^{j+1}\left[-\ln\overline{F}(x)\right]^{m-1}\frac{f(x)}{\overline{F}(x)}G^{'}(y)dydx \\ &+\frac{\beta k}{(j+2)}\frac{k^n}{(m-1)!\,(n-m-1)!}\int_0^\infty\int_x^\infty x^i y^{j+2}\left[-\ln\overline{F}(x)\right]^{m-1}\frac{f(x)}{\overline{F}(x)}G^{'}(y)dydx. \end{aligned} \tag{12}$$

Now consider

$$I(X)=\int_x^\infty y^{j+1}G^{'}(y)\,dy+\int_x^\infty y^{j+2}G^{'}(y)\,dy. \tag{13}$$

Integrating (13) by parts and using the value of $G(y)$ from (10), we have:

$$\begin{aligned}E\left(Y_m^{(k)}\right)^i\left(Y_n^{(k)}\right)^j &= \frac{\alpha k^n}{(m-1)!\,(n-m-1)!}\int_0^\infty\int_x^\infty x^i y^{j+1}\left[-ln\overline{F}\left(x\right)\right]^{m-1}\frac{f\left(x\right)}{\overline{F}\left(x\right)}\\ &\times\left[-ln\overline{F}\left(y\right)+ln\overline{F}\left(x\right)\right]^{n-m-1}\left[\overline{F}\left(y\right)\right]^k dydx\\ &+\frac{\beta k^n}{(m-1)!\,(n-m-1)!}\int_0^\infty\int_x^\infty x^i y^{j+1}\left[-ln\overline{F}\left(x\right)\right]^{m-1}\frac{f\left(x\right)}{\overline{F}\left(x\right)}\\ &\times\left[-ln\overline{F}\left(y\right)+ln\overline{F}\left(x\right)\right]^{n-m-1}\left[\overline{F}\left(y\right)\right]^k dydx. \end{aligned} \tag{14}$$

Upon simplification (14), we have:

$$\begin{aligned}&\frac{k^n}{(m-1)!\,(n-m-1)!}\int_0^\infty\int_x^\infty x^i y^j\left[-\ln\overline{F}\left(x\right)\right]^{m-1}\frac{f\left(x\right)}{\overline{F}\left(x\right)}\left[-ln\overline{F}\left(y\right)+\quad ln\overline{F}\left(x\right)\right]^{n-m-1}\\ &\times\left[\overline{F}\left(y\right)\right]^{k-1}\left\{f\left(y\right)-(\alpha+\beta y)\overline{F}\left(y\right)\right\}dydx=0.\end{aligned} \tag{15}$$

Applying the extension of Müntz-Szász Theorem, (see, for example, Hwang and Lin, [19]) to (15), we get

$$f\left(y\right)=(\alpha+\beta y)\overline{F}\left(y\right).$$

This proves that $f\left(y\right)$ is the *pdf* of the linear hazard rate distribution. ■

Remark 1 *Putting $i=0$ in Theorem 1, we deduce the characterizing result for single moments of k-th upper record values from the linear hazard rate distribution as obtained by Saran and Singh [18].*

Remark 2 *Putting $k=1$ and $i=0$ in Theorem 1, we deduce the characterizing result for single moments of upper record values from the linear hazard rate distribution as obtained by Saran and Pushkarna [20].*

Remark 3 *Putting $k=1$ and $\beta=0,\ \alpha=1$ in Theorem 1, we deduce the characterizing result for product moments of upper record values from the standard exponential distribution*

$$E\left[Y_{U(m)}^i\ Y_{U(n)}^{j+1}\right]=E\left[Y_{U(m)}^i\ Y_{U(n-1)}^{j+1}\right]+(j+1)E\left[Y_{U(m)}^i\ Y_{U(n)}^j\right]$$

as obtained by Ahsanullah et al. [21].

Remark 4 *Putting $k=1$ and $\alpha=0,\quad\beta=2\theta$ in Theorem 1, we deduce the characterizing result for product moments of upper record values from Rayleigh distribution as follows*

$$E\left[Y_{U(m)}^i\ Y_{U(n)}^j\right]=E\left[Y_{U(m)}^i\ Y_{U(n-1)}^{j+2}\right]+\left(\frac{j+2}{2\theta}\right)E\left[Y_{U(m)}^i\ Y_{U(n)}^{j+2}\right]$$

obtained by Mohsin et al. [22].

Theorem 2 *Let X be a non-negative random variable having an absolutely continuous df $F(x)$ with $F\left(0\right)=0$ and $0\le F(x)\le 1$ for all $x>0$. Then*

$$E[\xi(Y_n^{(k)})|(Y_l^{(k)})=x]=e^{-(\alpha x+\frac{\beta}{2}x^2)}\left(\frac{k}{k+1}\right)^{n-l},\quad l=m,\ m+1 \tag{16}$$

if and only if

$$F\left(x\right)=1-e^{-\left(\alpha x+\frac{\beta}{2}x^2\right)},\ x>0,\quad \alpha,\beta\ge 0 \tag{17}$$

where

$$\xi\left(y\right)=e^{-\left(\alpha y+\frac{\beta}{2}y^2\right)}$$

Proof. From (3), we have:

$$E[\xi(Y_n^{(k)})|(Y_m^{(k)})=x]=\frac{k^{n-m}}{(n-m-1)!}\int_x^\infty e^{-(\alpha y+\frac{\beta}{2}y^2)}[\ln\overline{F}\,(x)-\ln\overline{F}\,(y)]^{n-m-1}\left(\frac{\overline{F}\,(y)}{\overline{F}\,(x)}\right)^{k-1}\frac{f(y)}{\overline{F}(x)}dy. \quad (18)$$

By setting

$$u=\frac{\overline{F}\,(y)}{\overline{F}\,(x)}=\frac{e^{-(\alpha y+\frac{\beta}{2}y^2)}}{e^{-\left(\alpha x+\frac{\beta}{2}x^2\right)}}$$

from (6) in (18). Then (18) reduces to

$$E[\xi(Y_n^{(k)})|(Y_m^{(k)})=x]=\frac{k^{n-m}}{(n-m-1)!}e^{-\left(\alpha x+\frac{\beta}{2}x^2\right)}\int_0^1 u^k(-lnu\ )^{n-m-1}du. \quad (19)$$

Since

$$\int_0^1(-lnx\ )^{\mu-1}x^{\upsilon-1}dx=\frac{\Gamma\mu}{\upsilon^\mu},\ \mu>0,\quad \upsilon>0. \quad (20)$$

(Gradshteyn and Ryzhik [23], p-551). On using (20) in (19), the necessary part is proved.

For the sufficiency part, we consider

$$\frac{k^{n-m}}{(n-m-1)!}\int_x^\infty e^{-\left(\alpha y+\frac{\beta}{2}y^2\right)}\left[\ln\overline{F}\,(x)\ -\ln\overline{F}\,(y)\ \right]^{n-m-1}\left[\overline{F}\,(y)\right]^{k-1}f\,(y)\,dy=\left[\overline{F}\,(x)\right]^k g_{n|m}\,(x) \quad (21)$$

where

$$g_{n|m}\,(x)=e^{-(\alpha x+\frac{\beta}{2}x^2)}\left(\frac{k}{k+1}\right)^{n-m}.$$

Differentiating (21) both sides with respect to x, we get

$$\begin{aligned}&-\frac{k^{n-m}}{(n-m-2)!}\int_x^\infty e^{-\left(\alpha y+\frac{\beta}{2}y^2\right)}\left[\ln\overline{F}\,(x)\ -\ln\overline{F}\,(y)\ \right]^{n-m-2}\left[\overline{F}\,(y)\right]^{k-1}\frac{f(x)}{\overline{F}\,(x)}f\,(y)\,dy\\ =\ & g_{n|m}^{'}\,(x)\left[\overline{F}\,(x)\right]^k-kg_{n|m}\,(x)\left[\overline{F}\,(x)\right]^{k-1}f(x).\end{aligned}$$

Or:

$$-kg_{n|m+1}\,(x)\left[\overline{F}\,(x)\right]^{k-1}f(x)=g_{n|m}^{'}\,(x)\left[\overline{F}\,(x)\right]^k-kg_{n|m}\,(x)\left[\overline{F}\,(x)\right]^{k-1}f(x).$$

Therefore:

$$\frac{f(x)}{\overline{F}\,(x)}=-\frac{g_{n|m}'\,(x)}{k[g_{n|m+1}(x)\ \ -g_{n|m}(x)]}=(\alpha+\beta x).$$

Integrating both sides of the above equation with respect to x between $(0,\ y)$, the sufficiency part is proved. ■

The following theorem contains the characterization result based on truncated moment.

Theorem 3 *Suppose an absolutely continuous (with respect to Lebesgue measure) random variable X has the df $F(x)$ and pdf $f(x)$ for $x\geq 0$ such that $f'(x)$ and $E(X|X\leq x)$ exist then*

$$E\,(X\mid X\leq x)=g(x)\eta(x) \quad (22)$$

where

$$g\,(x)=\frac{1}{(\alpha+\beta x)}\left[-x+e^{\left\{\alpha x+\frac{\beta}{2}x^2\right\}}\int_0^x e^{-\left\{\alpha u+\frac{\beta}{2}u^2\right\}}du\right]$$

and $\eta\,(x)=\frac{f(x)}{F(x)}$ if and only if

$$f\,(x)=(\alpha+\beta x)e^{-\left\{\alpha x+\frac{\beta}{2}x^2\right\}},x>0\quad \alpha,\beta\ \geq 0\ .$$

Proof. We have

$$E\left(X \mid X \leq x\right) = \frac{1}{F(x)} \int_0^x uf\left(u\right) du = \frac{1}{F(x)} \int_0^x u(\alpha + \beta u)e^{-\left\{\alpha u + \frac{\beta}{2} u^2\right\}} du. \tag{23}$$

Integrating (23) by parts, treating$(\alpha + \beta u)e^{-\left\{\alpha u + \frac{\beta}{2} u^2\right\}}$ for integration and the remaining part for the integrand for differentiation, we get

$$E\left(X \mid X \leq x\right) = \frac{1}{F(x)} \left\{ -xe^{-\left\{\alpha x + \frac{\beta}{2} x^2\right\}} + \int_0^x e^{-\left\{\alpha u + \frac{\beta}{2} u^2\right\}} du \right\}. \tag{24}$$

After multiplying and dividing by $f(x)$ in (24), we have

$$E\left(X \mid X \leq x\right) = \frac{1}{(\alpha + \beta x)} \left[-x + e^{\left\{\alpha x + \frac{\beta}{2} x^2\right\}} \int_0^x e^{-\left\{\alpha u + \frac{\beta}{2} u^2\right\}} du \right] \frac{f(x)}{F(x)}.$$

This proves the necessary part.

To prove the sufficiency part, we have from (23):

$$\frac{1}{F(x)} \int_0^x uf\left(u\right) du = \frac{g\left(x\right) f(x)}{F(x)} \quad \text{or} \quad \int_0^x uf\left(u\right) du = g\left(x\right) f(x). \tag{25}$$

Differentiating (25) on both the sides with respect to x, we get:

$$xf\left(x\right) = g^{'}\left(x\right) f\left(x\right) + g(x) f^{'}\left(x\right).$$

Therefore

$$\frac{f'\left(x\right)}{f(x)} = \frac{x - g'\left(x\right)}{g(x)} \text{ Ahsanullah et al. [24]},$$

$$\frac{f'\left(x\right)}{f(x)} = \frac{x - g'\left(x\right)}{g(x)} = \left[\frac{\beta}{\alpha + \beta x} - (\alpha + \beta x)\right], \tag{26}$$

where

$$g'\left(x\right) = x - g\left(x\right) \left[\frac{\beta}{\alpha + \beta x} - (\alpha + \beta x)\right].$$

Integrating both sides in (26) with respect to x, we get

$$f\left(x\right) = c(\alpha + \beta x)e^{-\left(\alpha x + \frac{\beta}{2} x^2\right)}$$

Now, using the condition: $\int_{-\infty}^{\infty} f(x)dx = 1$,

$$\int_0^\infty c(\alpha + \beta x)e^{-\left(\alpha x + \frac{\beta}{2} x^2\right)} dx = 1,$$

$$\frac{1}{c} = \int_0^\infty (\alpha + \beta x)e^{-\left(\alpha x + \frac{\beta}{2} x^2\right)} dx,$$

$$c = 1.$$

This proves that,

$$f\left(x\right) = (\alpha + \beta x)e^{-\left(\alpha x + \frac{\beta}{2} x^2\right)}, x > 0 \quad \alpha, \beta \ \geq 0 \,.$$

This completes the proof. ■

Acknowledgment. The author is grateful to the referees for their valuable comments and suggestions that have led to the improvement and presentation of this paper.

References

[1] W. Dziubdziela, and B. Kopociński, Limiting properties of the *k-th* record value, Appl. Math., 15(1976), 187–190.

[2] K. N. Chandler, The distribution and frequency of record values, J. Roy. Statist. Soc. Ser. B, 14(1952), 220–228.

[3] S. Minimol and P. Yageen Thomos, On some properties of Makeham distribution using generalized record values and its characterization, Braz. J. Probab. Stat., 27(2013), 487–501.

[4] M. Ahsanullah, Record Statistics, Nova Science Publishers, New York, 1995.

[5] Z. Grudzień, Characterization of Distribution of Time Limits in Record Statistics as well as Distributions and Moments of Linear Record Statistics From the Samples of Random Numbers [Praca Doktorska], UMCS, Lublin, Poland, 1982.

[6] N. Glick, Breaking record and breaking boards, Amer. Math. Monthly, 85(1978), 2–26.

[7] B. C. Arnold, N. Balakrishnan and H. N. Nagaraja, Record, Johan Wiley and Sons, New York, 1998.

[8] U. Kamps, A Concept of Generalized Order Statistics, B.G. Teubner Stuttgart, Germany, 1995.

[9] K. Danielak and M. Z. Raqab, Sharp bounds for expectations of *k-th* record increments, Aust. N. Z. J. Stat., 46(2004), 665–673.

[10] Z. Grudzień and D. Szynal, Characterization of continuous distributions via moments of *k-th* record values with random indices, J. Appl. Statist. Sci., 5(1997), 259–266.

[11] P. Pawlas and D. Szynal, Relations for single and product moments of *k-th* record values from exponential and Gumbel distributions, J. Appl. Statist. Sci., 7(1998), 53–62.

[12] P. Pawlas and D. Szynal, Recurrence relations for single and product moments of *k-th* record values from Pareto, generalized Pareto and Burr distributions, Comm. Statist. Theory Methods, 28(1999), 1699–1709.

[13] S. Minimol and P. Yageen Thomos, On characterization of Gompertz distribution by properties of generalized record values, J. Stat. Theory Appl., 13(2014), 38–45.

[14] D. Kumar and M. I. Khan, Recurrence relations for moment$s$ of *k-th* record values from generalized beta II distribution and a characterization, Seluck J. Appl. Math., 13(2012), 75–82.

[15] R. U. Khan, M. A. Khan and M. A. R. Khan, Relations for moments of generalized record values from additive Weibull distribution and associated inference, Stat., Optim. Inf. Comput., 5(2017), 127–136.

[16] M. I. Khan and M. A. R. Khan, Generalized record values from distributions having power hazard function and characterization, J. Stat. Appl. Prob., 8(2019), 103–111.

[17] L. J. Bain, Analysis for the linear failure-rate life testing distribution, Technometrics, 15(1974), 551–559.

[18] J. Saran and S. K. Singh, Recurrence relations for single and product moments of *k-th* record values from linear- exponential distribution and a characterization, Asian J. Math. Statist., 1(2008), 159–164.

[19] J. S. Hwang and G. D. Lin, On a generalized moments problem II, Proc. Amer. Math. Soc., 91(1984), 577–580.

[20] J. Saran and N. Pushkarna, Relationships for moments of record values from linear- exponential distribution, J. Appl. Statist. Sci., 10(2000), 69–76.

[21] M. Ahsanullah, G. Hamedani and M. Shakil, On record values of univariate exponential distribution, J. Sat. Res., 2(2010), 267–288.

[22] M. Mohsin, M. Q. Shahbaz and B. M. Golam Kibria, Recurrence relations for single and product moments of generalized order statistics for Rayleigh distribution, Appl. Math. Inf. Sci., 4(2016), 273–279.

[23] I. S. Gradshteyn. and I. M. Ryzhik, Table of Integrals, Series and Products, Academic Press, New York, 2007.

[24] M. Ahsanullah, M. Shakil and B. M. Golam Kibria, Characterization of continuous distributions by truncated moment, J. Modern Appl. Satist. Methods, 15(2016), 316–331.