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The Odd Log-Logistic Logarithmic Lindley Distribution: Properties And Applications*

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Abstract

In this article, we introduce a new three-parameter odd log-logistic logarithmic Lindley distribution and discuss some of its properties. These include the shapes of the density and hazard rate functions, mixture representation, the moments, the quantile function, and order statistics. Maximum likelihood estimation of the parameters and their estimated asymptotic standard errors are derived. Three algorithms are proposed for generating random data from the proposed distribution. A simulation study is carried out to examine the bias and mean square error of the maximum likelihood estimators of the parameters. An application of the model to two real data sets is presented finally and compared with the fit attained by some other well-known two and three-parameter distributions for illustrative purposes. It is observed that the proposed model has some advantages in analyzing lifetime data as compared to other popular models in the sense that it exhibits varying shapes and shows more flexibility than many currently available distributions.

1 Introduction

Statistical distributions are very useful in describing and predicting real world phenomena. Numerous extended distributions have been extensively used over the last decades for modeling data in several areas. Recent developments focus on defining new families that extend well-known distributions and at the same time provide greater flexibility in modelling data in practice. Hence, several classes to generate new distributions by adding one or more parameters have been proposed in the statistical literature. Some well-known generators are the Marshall-Olkin generated (MO-G) by Marshall and Olkin [21], beta-G by Eugene et al. [8], Kumaraswamy-G (Kw-G) by Cordeiro and de Castro [5],Weibull-G by Bourguignon et. al.[4], exponentiated half-logistic-G by Cordeiro et al. [6], among others.

Let $G(x;\xi)$ be a baseline cumulative distribution function (cdf) and ξ be the vector of associated parameters. Gleaton and Lynch ([11, 12, 13]) defined the cdf of the odd log-logistic family with one extra shape parameter $\alpha > 0$ by

$$H(x) = \frac{G(x;\xi)^\alpha}{G(x;\xi)^\alpha + \bar{G}(x;\xi)^\alpha},$$

where $\bar{G}(x;\xi) = 1 - G(x;\xi)$. In addition, Gleaton and Rahman ([14, 15]) obtained asymptotic results for the maximum likelihood estimates (MLEs) of the parameters of these two distributions. They proved that for distributions generated from either a two-parameter Weibull or a two-parameter inverse Gaussian distributions by a GLL transformation, the joint MLEs of the parameters are asymptotically normal and efficient, provided the GLL transformation parameter exceeds three.

Recently, Alizadeh et al. [2] defined the cdf of the odd log-logistic logarithmic-G (OLLL-G) family by

$$H(x,\alpha,\beta,\xi) = \frac{\log\left(1 - \frac{\beta\, G(x;\xi)^\alpha}{G(x;\xi)^\alpha + \bar{G}(x;\xi)^\alpha}\right)}{\log(1-\beta)}, \tag{1}$$

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where $G(x;\xi)$ is the baseline cdf depending on a parameter vector ξ and $\alpha > 0$ and $\beta < 1$ are two additional shape parameters. It includes the odd log-logistic-G (OLL-G) family (Gleaton and Lynch 2004,2006) and the logarithmic-G family. In addition, the pdf of the family is

$$h(x,\alpha,\beta,\xi) = \frac{\alpha\,\beta\,g(x,\xi)G(x,\xi)^{\alpha-1}\bar{G}(x,\xi)^{\alpha-1}}{-[G(x;\xi)^{\alpha}+\bar{G}(x;\xi)^{\alpha}][(1-\beta)G(x;\xi)^{\alpha}+\bar{G}(x;\xi)^{\alpha}]\log(1-\beta)}$$

A motivation of this family can be explained as follows. Suppose that a parallel system is made up of N components and the lifetimes of the components are independent and identically distributed (iid) random variables, denoted by Z_1, Z_N, with common cdf (1). Then, the system fails as soon as the last component fails, namely the lifetime of the whole system is represented by $X = \max\{Z_1, Z_N\}$. In many survival parallel systems, it is almost impossible to have a fixed number of components because some of them get lost or censored for various reasons. Therefore we may assume that N is a discrete random variable. Suppose that N has the logarithmic distribution with probability mass function given by

$$P(N=n) = \frac{-1}{\log(1-\beta)}.\frac{\beta^n}{n}, \qquad n=1,2,...; \quad 0<\beta<1.$$

Then, the cdf of the life length of the whole system, X, is obtained as

$$\begin{aligned} F(x) &= \sum_{n=1}^{\infty} P(X \le x | N=n) P(N=n) \\ &= \sum_{n=1}^{\infty} \left[\frac{G(x,\xi)^{\alpha}}{G(x,\xi)^{\alpha}+(1-G(x,\xi))^{\alpha}}\right]^n \frac{-1}{\log(1-\beta)}.\frac{\beta^n}{n} \\ &= \frac{-\log\left[1-\frac{\beta\,G(x,\xi)^{\alpha}}{G(x,\xi)^{\alpha}+(1-G(x,\xi))^{\alpha}}\right]}{-\log(1-\beta)} \end{aligned}$$

which is identical to (1).

On the other hand, the Lindley distribution specified by the probability density function (pdf)

$$f(x) = \frac{\lambda^2}{\lambda+1}(1+x)\exp(-\lambda x), \qquad x>0,\ \lambda>0, \tag{2}$$

was introduced by Lindley [20] in the context of Bayesian statistics. Ghitany et al. [10] investigated properties of the Lindley distribution with application and outlined that the Lindley distribution is a better model than one based on the exponential distribution, in other words many mathematical properties of the Lindley distribution are more flexible than those of the exponential distribution. Ghitany et al. [10] showed that the Lindley distribution can be written as a mixture of an exponential distribution and a gamma distribution with shape parameter 2.

Now, by letting $G(x;\xi)$ in (1) to be the cdf of the Lindley distribution, where $\xi = \lambda$, we can obtain a new extension of the Lindley distribution, called the odd log-logistic logarithmic Lindley (henceforth, OLLL-L) distribution. The new distribution is very flexible in the sense that it can be skewed and symmetric depending upon the specific choices of the parameters. Furthermore, the associated cdf is in closed form. Consequently, this distribution can be applied to modelling censored data too.

The article is outlined as follows: In section 2, we introduce the OLLL-L distribution and provide plots of the density and hazard rate functions. Shapes, quantile function, moments, and moment generating function are also obtained. Moreover, mean deviation, Lorenz and Bonferroni curves, order statistics. In section 3, the asymptotic properties and extreme values are obtained. Estimation by the method of maximum likelihood and an explicit expression for the observed information matrix are presented in Section 4. The simulation study is presented in section 5. The applications to real data sets are considered in section 6. Finally, section 7 offers some concluding remarks.

2 Main Properties

2.1 Probability Density and Cumulative Density Functions

Inserting (2) in (1), the cdf of the OLLL-L with three parameters α, β, λ is defined as

$$F(x, \alpha, \beta, \lambda) = \frac{\log\left[1 - \frac{\beta\,[1-(1+\frac{\lambda x}{1+\lambda})e^{-\lambda x}]^{\alpha}}{[1-(1+\frac{\lambda x}{1+\lambda})e^{-\lambda x}]^{\alpha}+[(1+\frac{\lambda x}{1+\lambda})e^{-\lambda x}]^{\alpha}}\right]}{\log(1-\beta)}, \tag{3}$$

$$f(x, \alpha, \beta, \lambda) = \frac{\alpha\,\beta\,\lambda^2(1+x)\,e^{-\lambda\,x}k(x)^{\alpha-1}(1-k(x))^{\alpha-1}}{-(\lambda+1)\log(1-\beta)\,[k(x)^{\alpha}+(1-k(x))^{\alpha}]\,[k(x)^{\alpha}+(1-\beta)(1-k(x))^{\alpha}]}, \tag{4}$$

respectively, where

$$k(x) = (1 + \frac{\lambda\,x}{1+\lambda})e^{-\lambda x}$$

and $\alpha, \lambda > 0$ and $\beta < 1$. We write $X \sim OLLLL(\alpha, \beta, \lambda,)$ if the pdf of X can be written as (4).

It is easy to see that:

(i) When $\beta \uparrow 1$, we obtain OLL-L.

(ii) When $\alpha = 1$, we obtain the Logarithmic-Lindley distribution.

(iii) When $\beta \uparrow 1$ and $\alpha = 1$, we obtain the Lindley distribution.

Some of the possible shapes of the density function (4) for the selected parameter values are illustrated in Figure 1. As seen in Figure 1, the density function can take various forms depending on the parameter values. It is evident that the OLLL-L distribution is much more flexible than the Lindley distribution, i.e. the additional shape parameter allows for a high degree of flexibility of the OLLL-L distribution. Both unimodal and monotonically decreasing and increasing shapes appear to be possible.

2.2 Survival and Hazard Rate Functions

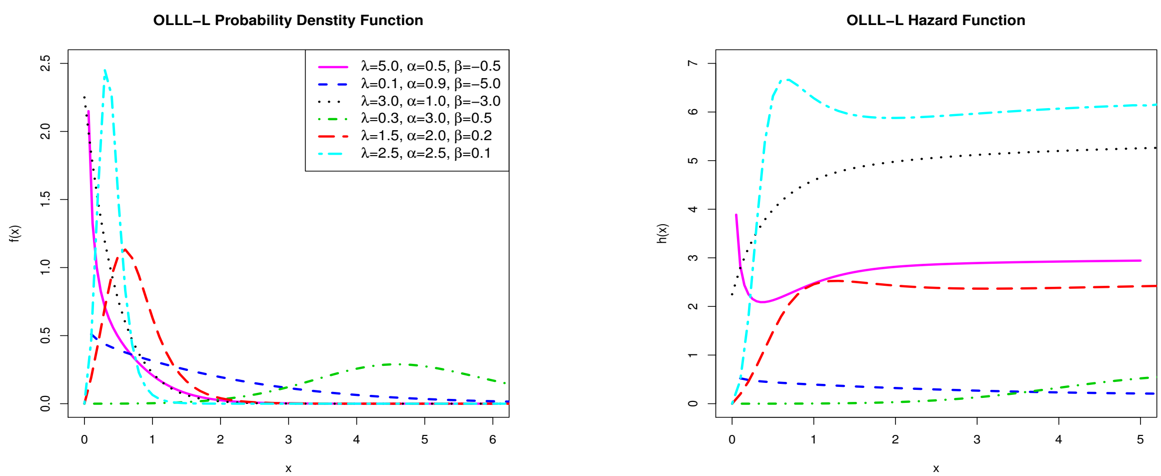
Central role is played in the reliability theory by the quotient of the pdf and survival function. We obtain the survival function corresponding to (3) as

$$S(x;\lambda,\alpha,\gamma) = \frac{1}{\log(1-\beta)}\log\left[\frac{1-\beta}{1 - \frac{\beta\,[1-(1+\frac{\lambda x}{1+\lambda})e^{-\lambda x}]^{\alpha}}{[1-(1+\frac{\lambda x}{1+\lambda})e^{-\lambda x}]^{\alpha}+[(1+\frac{\lambda x}{1+\lambda})e^{-\lambda x}]^{\alpha}}}\right].$$

In reliability studies, the hrf is an important characteristic and fundamental to the design of safe systems in a wide variety of applications. Therefore, we discuss these properties of the OLLL-L distribution. The hrf of X takes the form

$$h(x;\lambda,\alpha,\gamma) = \frac{\frac{\alpha\,\beta\,\lambda^2(1+x)\,e^{-\lambda\,x}k(x)^{\alpha-1}(1-k(x))^{\alpha-1}}{-(\lambda+1)(\log(1-\beta))^2[k(x)^{\alpha}+(1-k(x))^{\alpha}][k(x)^{\alpha}+(1-\beta)(1-k(x))^{\alpha}]}}{\log(\frac{1-\beta}{1-\frac{\beta\,[1-(1+\frac{\lambda x}{1+\lambda})e^{-\lambda x}]^{\alpha}}{[1-(1+\frac{\lambda x}{1+\lambda})e^{-\lambda x}]^{\alpha}+[(1+\frac{\lambda x}{1+\lambda})e^{-\lambda x}]^{\alpha}}})}.$$

Plots of the hrf of the OLLL-L distribution for several parameter values are displayed in Figure 1. Figure 1 shows that the hrf of theOLL-L distribution can have very flexible shapes, such as increasing, decreasing, upside-down bathtub, and bathtub. This attractive flexibility makes the hrf of the OLLL-L useful and suitable for non-monotone empirical hazard behaviors which are more likely to be encountered or observed in real-life situations



(a) (b)

Figure 1: Plots of pdfs and hazard functions of the OLLL-L model for selected α, β and λ.

2.3 Mixture Representations for the Pdf and Cdf

In this subsection, we provide alternative mixture representations for the pdf and cdf of X. Some useful expansions for (3) can be derived by using the concept of power series. We know $-\log(1-u) = \sum_{i=1}^{\infty} \frac{u^i}{i}$ (which converges for $|u| < 1$), the OLLL-G family cdf can be expanded as

$$F(x) = \frac{-1}{\log(1-\beta)} \sum_{i=1}^{\infty} \frac{\beta^i}{i} \left[\frac{G(x)^{\alpha}}{G(x)^{\alpha} + (1-G(x))^{\alpha}} \right]^i . \tag{5}$$

On the other hand, for $|u| < 1$, we have, (see Gradshteyn and Ryzhik [16], Section 0.313)

$$\frac{u^a}{u^a + (1-u)^a} = \frac{\sum_{k=1}^{\infty} a_k(\alpha)\, u^k}{\sum_{k=1}^{\infty} b_k(\alpha)\, u^k} = \sum_{k=1}^{\infty} c_k(\alpha)\, u^k, \tag{6}$$

where

$$a_k(\alpha) = \sum_{j=k}^{\infty} (-1)^{k+j} \binom{j}{\alpha} \binom{k}{j},$$

$$b_k(\alpha) = a_k(\alpha) + (-1)^k \binom{k}{\alpha},$$

and

$$c_k(\alpha) = \frac{1}{b_0(\alpha)} [a_k(\alpha) - \frac{1}{b_0(\alpha)} \sum_{r=1}^{k} b_r\, C_{h-r}(\alpha)].$$

Based on (6) and using a result of Gradshteyn and Ryzhik ([16] Section 0.314) for a power series raised to a positive integer number, we obtain

$$\left[\frac{u^a}{u^a + (1-u)^a} \right]^m = \left[\sum_{i=1}^{\infty} c_k(\alpha)\, u^k \right]^m = \sum_{k=1}^{\infty} h_k(\alpha, m)\, u^k, \tag{7}$$

the coefficients $h_k(\alpha, i)$ can be determined from the recursive formula as bellow:

$$h_k(\alpha, i) = \frac{1}{k\, c_0(\alpha)} \sum_{q=1}^{k} [(m+1)q - k] c_q(\alpha) h_{k-q}(\alpha, i).$$

From (5) and (7), we obtain

$$F(x) = \sum_{k=0}^{\infty} d_k H_k(x), \tag{8}$$

where $H_k(x)$ is the Exp-G cdf with power parameter k ($H_k(x) = G(x)^k$) and

$$d_k = \sum_{i=0}^{\infty} \frac{-\beta^i\, h_k(\alpha, i)}{i\, \log(1-\beta)}. \tag{9}$$

By differentiating (8), the pdf of X follows as

$$f(x) = \sum_{k=1}^{\infty} d_k h_k(x), \tag{10}$$

where $h_k(x) = k\, G(x)^{k-1} g(x)$ is the Exp-G density function with power parameter k.

Equation (10) reveals that the OLLL-G density function is a linear combination of Exp-G densities. Some structural properties of the new family such as the ordinary and incomplete moments and generating function can be determined from well-established properties of the Exp-G distribution.

The EL distribution has the pdf

$$f(x, \alpha, \lambda) = \frac{\alpha\, \lambda^2}{\lambda + 1} (1+x) e^{-\lambda\, x} \left[1 - (1 + \frac{\lambda\, x}{1+\lambda}) e^{-\lambda\, x} \right]^{\alpha - 1}, \quad x > 0;\ \alpha, \lambda > 0. \tag{11}$$

We write $X \sim EL(x; \alpha, \lambda)$ if the pdf of X can be expressed as (11). In addition, the cdf of the EL model is

$$F(x, \alpha, \lambda) = \left[1 - (1 + \frac{\lambda\, x}{1+\lambda}) e^{-\lambda\, x} \right]^{\alpha}, \quad x > 0;\ \alpha, \lambda > 0.$$

Now, using (8), we show that the OLLL-L distribution can be viewed as a mixture of EL distributions as bellow:

$$F(x) = \sum_{k=0}^{\infty} d_k F_{EL}(x; k, \lambda) \quad \text{and} \quad f(x) = \sum_{k=0}^{\infty} d_k f_{EL}(x; k, \lambda).$$

2.4 Moments and Moment Generating Function

Some of the most important features and characteristics of a distribution can be studied through moments (e.g. tendency, dispersion, skewness and kurtosis). Now we obtain ordinary moments and the moment generating function of the OLLL-L distribution. We define and compute

$$A(a_1, a_2, a_3; \lambda) = \int_0^{\infty} x^{a_1}\, (1+x)\, \mathrm{e}^{-a_2\, x} \left[1 - (1 + \frac{\lambda}{1+\lambda}\, x) \mathrm{e}^{-\lambda x} \right]^{a_3} \mathrm{d}x.$$

Using generalized binomial expansion, one can obtain

$$A(a_1, a_2, a_3; \lambda) = \sum_{i=0}^{\infty} \sum_{j=0}^{l} \sum_{l=0}^{1} (-1)^l \binom{a_3}{i} \binom{j}{i} (\frac{\lambda}{1+\lambda})^j \times \frac{\Gamma(a_1 + j + l + 1)}{(\lambda\, i + a_2)^{a_1 + j + l + 1}}.$$

Next, the nth moment of the OLLL-L distribution is given by

$$E\left[X^m\right] = \frac{\lambda^2}{1+\lambda} \sum_{k=0}^{\infty} (k+1)\, A(m, \lambda, k; \lambda). \tag{12}$$

For integer values of k, let $\mu'_k = E(X^k)$ and $\mu = \mu'_1 = E(X)$, then one can also find the kth central moment of the EGL distribution through the following well-known equation

$$\mu_k = E(X-\mu)^k = \sum_{r=0}^{k} \binom{k}{r} \mu'_r (-\mu)^{k-r}. \tag{13}$$

The moment generating function of a random variable provides the basis of an alternative route to analytical results compared with working directly with its pdf and cdf. Using (12) and (13), we obtain

$$M_X(t) = E\left[\mathrm{e}^{\mathrm{t\,X}}\right] = \frac{\lambda^2}{1+\lambda} \sum_{k=0}^{\infty} (k+1)\, A(0, \lambda - t, k; \lambda).$$

Using (13), the variance, skewness and kurtosis measures can be obtained. Skewness measures the degree of the long tail and kurtosis is a measure of the degree of tail heaviness. For the OLLL-L distribution, the skewness can be computed as

$$S = \frac{\mu_3}{\mu_2^{3/2}} = \frac{\mu'_3 - 3\mu'_2\mu'_1 + 2\mu'^3_1}{(\mu'_2 - \mu'^2_1)^{3/2}}$$

and the kurtosis is based on octiles as

$$K = \frac{\mu_4}{\mu_2^2} = \frac{\mu'_4 - 4\mu'_1\mu'_3 + 6\mu'^2_1\mu'_2 - 3\mu'^4_1}{\mu'_2 - \mu'^2_1}.$$

When the distribution is symmetric $S = 0$, and when the distribution is right (or left) skewed $S > 0 (or\ S < 0)$. As K increases, the tail of the distribution becomes heavier. These measures are less sensitive to outliers and they exist even for distributions without moments. Plots for skewness and kurtosis are presented in Figure 3.

Next, we define and compute

$$B(a_1, a_2, a_3; y, \lambda) = \int_0^y x^{a_1}\,(1+x)\,\mathrm{e}^{-a_2\,x} \left[1 - (1 + \frac{\lambda}{1+\lambda}\,x)\mathrm{e}^{-\lambda x}\right]^{a_3} \mathrm{d}x.$$

From the generalized binomial expansion, we have

$$B(a_1, a_2, a_3; \lambda, y) = \sum_{i=0}^{\infty} \sum_{j=0}^{l} \sum_{l=0}^{1} (-1)^l \binom{a_3}{i} \binom{j}{i} (\frac{\lambda}{1+\lambda})^j \times \frac{\gamma(a_1 + j + l + 1, y(a_2 + i\lambda))}{(\lambda\, i + a_2)^{a_1+j+l+1}},$$

where $\gamma(\lambda, z) = \int_0^z t^{\lambda-1}\, e^{-t}\, dt$ denotes the incomplete gamma function. Now, the nth incomplete moment of the EGL distribution is found to be

$$m_n(y) = E\left[X^n \,|X < y\right] = \frac{\lambda^2}{1+\lambda} \sum_{k=0}^{\infty} (k+1)\, B(n, \lambda, k; \lambda, y).$$

2.5 Stochastic Ordering

Let X and Y be two random variables. X is said to be stochastically less than or equal to Y, denoted by $X \leq_{st} Y$ if $P(X > x) = P(Y > x)$ for all x in the support set of X.

Theorem 1 *Suppose $X \sim OLLL - L(\alpha_1, \beta_1, \lambda)$ and $Y \sim OLLL - L(\alpha_2, \beta_2, \lambda)$. If $\alpha_1 < \alpha_2$ and $\beta_1 < \beta_2$, then $X \leq_{st} Y$.*

Proof. Straightforward and hence omitted. ■

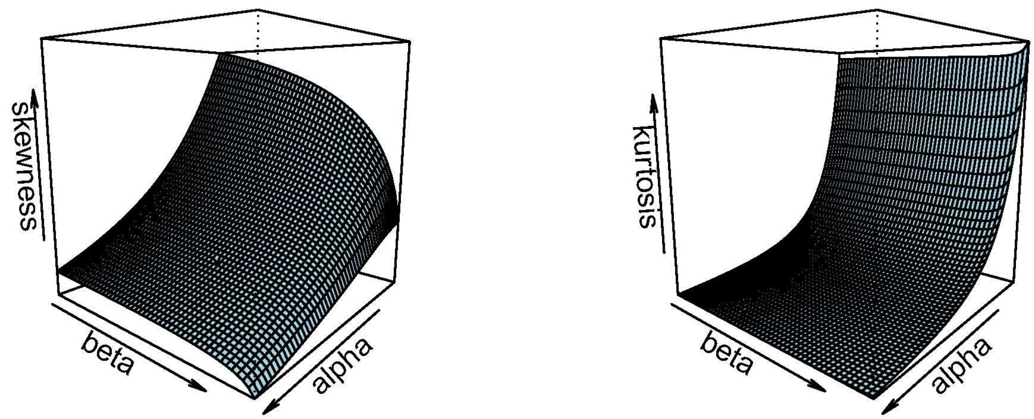


Figure 2: Plots of the skewness (left) and kurtosis (right) of the OLLL-L distribution for $\lambda = 2$.

2.6 Quantile Function

Quantile functions are in widespread use in statistics and often find representations in terms of lookup tables for key percentiles. Let $X \sim OLLL - L(\alpha, \beta, \lambda)$. The quantile function of X, say $Q(p)$, is defined by $F(Q(p)) = p$ and is the root of the following equation

$$(1+\lambda+\lambda\, Q(p))e^{-\lambda\, Q(p)} = \frac{(1+\lambda)(\beta-1+(1-\beta)^p)^{1/\alpha}}{(1-(1-\beta)^p)^{1/\alpha}+(\beta-1+(1-\beta)^p)^{1/\alpha}}. \tag{14}$$

Inserting $Z(p) = -1-\lambda-\lambda\, Q(p)$ into (14), we have

$$Z(p)e^{Z(p)} = \frac{-(1+\lambda)(\beta-1+(1-\beta)^p)^{1/\alpha}e^{-1-\lambda}}{(1-(1-\beta)^p)^{1/\alpha}+(\beta-1+(1-\beta)^p)^{1/\alpha}}.$$

Hence, the solution of $Z(p)$ is

$$Z(p) = W_{-1}\left[\frac{-(1+\lambda)(\beta-1+(1-\beta)^p)^{1/\alpha}e^{-1-\lambda}}{(1-(1-\beta)^p)^{1/\alpha}+(\beta-1+(1-\beta)^p)^{1/\alpha}}\right],$$

where $W_{-1}[.]$ is the negative branch of the Lambert W function (Corless et al. 1996). Thus, we obtain the quantile function as

$$Q(p) = \left\{-1-\frac{1}{\lambda}-\frac{1}{\lambda}W_{-1}\left[\frac{-(1+\lambda)(\beta-1+(1-\beta)^p)^{1/\alpha}e^{-1-\lambda}}{(1-(1-\beta)^p)^{1/\alpha}+(\beta-1-(1-\beta)^p)^{1/\alpha}}\right]\right\}.$$

Here, we also propose three different algorithms for generating random data from the OLLL-L distribution. The first algorithm is based on generating random data from the Lindley distribution using the exponential-gamma mixture (see Ghitany et al., [16])

- Algorithm 1 (Mixture Form of the Lindley Distribution)

  1. Generate $U_i \sim Uniform(0,1), i = 1,\ldots,n$;
  2. Generate $V_i \sim Exponential(\lambda), i = 1,\ldots,n$;
  3. Generate $W_i \sim Gamma(2,\lambda), i = 1,\ldots,n$;

4. If $\frac{(1-(1-\beta)^{u_i})^{1/\alpha}}{(1-(1-\beta)^{u_i})^{1/\alpha}+(\beta-1+(1-\beta)^{u_i})^{1/\alpha}} \leq \frac{\lambda}{1+\lambda}$ set $X_i = V_i$, otherwise, set $X_i = W_i, i = 1, \ldots, n$.

- Algorithm 2 (Inverse cdf)
  1. Generate $U_i \sim Uniform(0,1), i = 1, \ldots, n$;
  2. Set

$$X_i = \left\{-1 - \frac{1}{\lambda} - \frac{1}{\lambda} W_{-1}\left[\frac{-(1+\lambda)(\beta-1+(1-\beta)^{U_i})^{1/\alpha} e^{-1-\lambda}}{(1-(1-\beta)^{U_i})^{1/\alpha}+(\beta-1-(1-\beta)^{U_i})^{1/\alpha}}\right]\right\},\ i = 1, \ldots, n.$$

Algorithm 1 is the simplest data generation algorithm and therefore is preferable. Data generation from the classical distributions like uniform, exponential and gamma distributions is included normally in many statistical software. We have used Algorithm 1 in our simulation study. Algorithm 2 involves the Lambert W function and therefore is somehow complicated, see Corless et al. [7] for more details regarding the Lambert W function.

2.7 Mean Deviations

The amount of scatter in a population may be measured to some extent by deviations from the mean and median. These are known as the mean deviation about the mean and the mean deviation about the median, defined by

$$\delta_1(X) = \int_0^\infty |x-\mu|\, f(x)\, \mathrm{d}x,$$

and

$$\delta_2(X) = \int_0^\infty |x-M|\, f(x)\, \mathrm{d}x$$

respectively, where $\mu = E(X)$ and $M = Median(X) = Q(0.5)$ denotes the median and $Q(p)$ is the quantile function. The measures $\delta_1(X)$ and $\delta_2(X)$ can be calculated using the relationships

$$\delta_1(X) = 2\mu\, F(\mu) - 2\int_0^\mu x\, f(x)\, \mathrm{d}x \quad \text{and} \quad \delta_2(X) = \mu - 2\int_0^M x\, f(x)\, \mathrm{d}x.$$

Finally using (14), we have

$$\delta_1(X) = 2\mu\, F(\mu) - \frac{\lambda^2}{1+\lambda}\sum_{k=0}^{\infty}(k+1)\, B(1,\lambda,k;\lambda,\mu)$$

and

$$\delta_2(X) = \mu - \frac{2\,\lambda^2}{1+\lambda}\sum_{k=0}^{\infty}(k+1)\, B(1,\lambda,k;\lambda,M).$$

2.8 Bonferroni and Lorenz Curves

The Bonferroni and Lorenz curves have applications in economics as well as other fields like reliability, medicine and insurance. Let $X \sim OLLL-L(\lambda,\alpha,\beta)$ and $F(x)$ be the cdf of X, then the Bonferroni curve of the OLLL-L distribution is given by

$$B(F(x)) = \frac{1}{\mu\, F(x)}\int_0^x t f(t)\mathrm{d}t,$$

where $\mu = E(Y)$. Therefore, from (14), we have

$$B(F(x)) = \frac{1}{\mu\, F(x)} \times \frac{\lambda^2}{1+\lambda}\sum_{k=0}^{\infty}(k+1)\, B(1,\lambda,k;\lambda,x).$$

The Lorenz curve of the EGL distribution can be obtained using the relation

$$L(F(x)) = F(x)B(F(x)) = \frac{1}{\mu} \times \frac{\lambda^2}{1+\lambda} \sum_{k=0}^{\infty}(k+1)\, B(1,\lambda,k;\lambda,x).$$

2.9 Order Statistics

Order statistics make their appearance in many areas of statistical theory and practice. Suppose $X_1, \ldots, X_n$ is a random sample from any EGL distribution. Let $X_{i:n}$ denote the ith order statistic. The pdf of $X_{i:n}$ can be expressed as

$$f_{i:n}(x) = K\, f(x)\, F^{i-1}(x)\, \{1-F(x)\}^{n-i} = K \sum_{j=0}^{n-i}(-1)^j \binom{n-i}{j} f(x)\, F(x)^{j+i-1},$$

where $K = 1/B(i, n-i+1)$. We use the result of Gradshteyn and Ryzhik [16] for a power series raised to a positive integer n (for $n \geq 1$)

$$\left(\sum_{i=0}^{\infty} a_i\, u^i\right)^n = \sum_{i=0}^{\infty} d_{n,i}\, u^i,$$

where the coefficients $d_{n,i}$ (for $i = 1, 2, \ldots$) are determined from the recurrence equation (with $d_{n,0} = a_0^n$)

$$d_{n,i} = (i\, a_0)^{-1} \sum_{m=1}^{i} [m(n+1) - i]\, a_m\, d_{n,i-m}. \tag{15}$$

We can show that the density function of the ith order statistic of any OLLL-L distribution can be expressed as

$$f_{i:n}(x) = \sum_{r,k=0}^{\infty} m_{r,k}\, f_{EL}(x;\lambda, r+k), \tag{16}$$

where $f_{GL}(x;\lambda, r+k)$ denotes the density function of exponentiated Lindley distribution with parameters λ and $r+k$,

$$m_{r,k} = \frac{n!\, r\, d_r}{(i-1)!\,(r+k+1)} \sum_{j=0}^{n-i} \frac{(-1)^j\, c_{j+i-1,k}}{(n-i-j)!\, j!}.$$

Here, d_r is given by (9) and the quantities $c_{j+i-1,k}$ can be determined given that $c_{j+i-1,0} = d_0^{j+i-1}$ and recursively we have:

$$c_{j+i-1,k} = (k\, d_0)^{-1} \sum_{m=1}^{k} [m\,(j+i) - k]\, c_m\, d_{j+i-1,k-m}, k \geq 1.$$

Equation (16) is the main result of this section. It reveals that the pdf of the ith order statistic is a linear combination of exponentiated Lindley distributions. Therefore, several mathematical quantities of these order statistics like ordinary and incomplete moments, factorial moments, mgf, mean deviations and others can be derived using this result.

3 Asymptotic Properties and Extreme Value

One of the main usage of the idea of an asymptotic distribution is in providing approximations to the cumulative distribution functions of the statistical estimators. Moreover, the extreme value theory is a branch of statistics dealing with the extreme deviations from the median of probability distributions. It seeks to assess, from a given ordered sample of a given random variable, the probability of events that are more extreme than any previously observed. Extreme value analysis is widely used in many disciplines,

3.1 Asymptotic Properties

The asymptotic of cdf, pdf and hrf of the OLLL-L distribution as $x \to 0$ are, respectively, given by

$$F(x) \sim \frac{\beta\,\lambda^{\alpha}\,x^{\alpha}}{-\log(1-\beta)} \quad \text{as } x \to 0,$$

$$f(x) \sim \frac{\alpha\,\beta\,\lambda^{\alpha}\,x^{\alpha-1}}{-\log(1-\beta)} \quad \text{as } x \to 0,$$

$$h(x) \sim \frac{\alpha\,\beta\,\lambda^{\alpha}\,x^{\alpha-1}}{-\log(1-\beta)} \quad \text{as } x \to 0.$$

The asymptotic of cdf, pdf and hrf of the OLLL-L distribution as $x \to \infty$ are, respectively, given by

$$1 - F(x) \sim 1 - \frac{\log\left\{1-\beta\left[1-\frac{\lambda}{1+\lambda}\,x\,\mathrm{e}^{-\lambda\,\mathrm{x}}\right]^{\alpha}\right\}}{\log(1-\beta)} \qquad \text{as } x \to \infty,$$

$$f(x) \sim \frac{\alpha\,\beta\,\lambda^{\alpha}\,x^{\alpha}\,\mathrm{e}^{-\alpha\lambda\,\mathrm{x}}}{-(1+\lambda)^{\alpha}\,(1-\beta)\log(1-\beta)} \qquad \text{as } x \to \infty,$$

$$h(x) \sim \frac{\alpha\,\beta\,\lambda^{\alpha}\,x^{\alpha}\,\mathrm{e}^{-\alpha\lambda\,\mathrm{x}}}{-(1+\lambda)^{\alpha}\,(1-\beta)\log\left\{\frac{1-\beta}{1-\beta\left[1-\frac{\lambda}{1+\lambda}\,x\,\mathrm{e}^{-\lambda\,\mathrm{x}}\right]^{\alpha}}\right\}} \quad \text{as } x \to \infty.$$

These equations show the effect of parameters on the tails of the OLLL-L distribution.

3.2 Extreme Value

If $X_1, ..., X_n$ is a random sample from (3) and if $\bar{X} = (X_1 + ... + X_n)/n$ denotes the sample mean, then by the usual central limit theorem, $\sqrt{n}(\bar{X} - E(X))/\sqrt{\mathrm{Var}(X)}$ approaches the standard normal distribution as $n \to \infty$. One may be interested in the asymptotic of the extreme values $M_n = \max(X_1, ..., X_n)$ and $m_n = \min(X_1, ..., X_n)$. Let $\tau(x) = \frac{1}{\lambda}$, we obtain following equations for the cdf in (3) as

$$\lim_{t\to 0} \frac{F(t\,x)}{F(t)} = \lim_{t\to 0} \frac{x\,f(t\,x)}{f(t)} = x^{\alpha}$$

and

$$\lim_{t\to\infty} \frac{1-F(t+x\,\tau(t))}{1-F(t)} = \lim_{t\to\infty} \frac{f(t+x\,\tau(t))}{f(t)} = \mathrm{e}^{-\alpha\,x}.$$

Thus, from Leadbetter et al. [18], there must be norming constants $a_n > 0$, b_n, $c_n > 0$ and d_n such that

$$Pr\left[a_n(M_n - b_n) \le x\right] \to \mathrm{e}^{-\mathrm{e}^{-\alpha \mathrm{x}}}$$

and

$$Pr\left[c_n(m_n - d_n) \le x\right] \to 1 - \mathrm{e}^{-\mathrm{x}^{\alpha}}$$

as $n \to \infty$. The form of the norming constants can also be determined. For instance, using Corollary 1.6.3 in Leadbetter et al. [18], one can see that $b_n = F^{-1}(1 - \frac{1}{n})$ and $a_n = \lambda$, where $F^{-1}(.)$ denotes the inverse function of $F(.)$.

4 Estimation

Several approaches for parameter estimation have been proposed in the literature but the maximum likelihood method is the most commonly employed. In this section, we discuss maximum likelihood estimation (MLE) and inference for the OLLL-L distribution. Let $x_1, x_2, ..., x_n$ be observed values from the OLLL-L distribution with parameters α, β and λ. The log-likelihood function for $(\alpha; \beta; \lambda\ )$ is given by

$$
\begin{aligned}
\ell_n &= n\log(\frac{\lambda^2}{1+\lambda}) + n\log(\alpha) + n\log(\beta) - n\log(-\log(1-\beta)) \\
&+ \sum_{i=1}^{n}\log(1+x_i) - \lambda\sum_{i=1}^{n}x_i + \alpha\sum_{i=1}^{n}\log\ k_i - (\alpha-1)\sum_{i=1}^{n}\log\left(1-k_i\right) \\
&- \sum_{i=1}^{n}\log(k_i^{\alpha} + (1-k_i)^{\alpha}) - \sum_{i=1}^{n}\log\left((1-\beta)k_i^{\alpha} + (1-k_i)^{\alpha}\right),
\end{aligned}
$$

where

$$
k_i = 1 - (1 + \frac{\lambda}{1+\lambda}x_i)e^{-\lambda x_i}.
$$

The derivatives of the log-likelihood function with respect to the parameters α, γ and λ are given respectively, by

$$
\begin{aligned}
\frac{\partial \ell_n}{\partial \alpha} &= \frac{n}{\alpha} + \sum_{i=1}^{n}\log\ k_i - \sum_{i=1}^{n}\log\left(1-k_i\right) - \sum_{i=1}^{n}\frac{\log(k_i)k_i^{\alpha} + \log(1-k_i)(1-k_i)^{\alpha}}{k_i^{\alpha} - (1-k_i)^{\alpha}} \\
&- \sum_{i=1}^{n}\frac{(1-\beta)\log(k_i)k_i^{\alpha} + \log(1-k_i)(1-k_i)^{\alpha}}{(1-\beta)k_i^{\alpha} - (1-k_i)^{\alpha}},
\end{aligned}
$$

$$
\frac{\partial \ell_n}{\partial \beta} = \frac{n}{\beta} - \frac{n}{(1-\beta)\log(1-\beta)} + \sum_{i=1}^{n}\frac{k_i^{\alpha}}{(1-\beta)k_i^{\alpha} - (1-k_i)^{\alpha}},
$$

$$
\begin{aligned}
\frac{\partial \ell_n}{\partial \lambda} &= \frac{2n}{\lambda} - \frac{n}{1+\lambda} - \sum_{i=1}^{n}x_i + \alpha\sum_{i=1}^{n}\frac{k_i'}{k_i} + (\alpha-1)\sum_{i=1}^{n}\frac{k_i'}{1-k_i} \\
&- \sum_{i=1}^{n}\frac{\alpha\, k_i'\left(k_i^{\alpha-1} - (1-k_i)^{\alpha-1}\right)}{k_i^{\alpha} - (1-k_i)^{\alpha}} - \sum_{i=1}^{n}\frac{\alpha\, k_i'\left((1-\beta)k_i^{\alpha-1} - (1-k_i)^{\alpha-1}\right)}{(1-\beta)k_i^{\alpha} - (1-k_i)^{\alpha}},
\end{aligned}
$$

where

$$
k_i' = \frac{\partial k_i}{\partial \lambda} = x_i e^{-\lambda x_i}\left[1 + \frac{\lambda}{\lambda+1}x_i + \frac{1}{(1+\lambda)^2}\right].
$$

The maximum likelihood estimates (MLEs) of $(\alpha; \beta; \lambda)$, say $(\widehat{\alpha}; \widehat{\beta}; \widehat{\lambda})$, are the simultaneous solution of the equations $\frac{\partial \ell_n}{\partial \alpha} = 0$; $\frac{\partial \ell_n}{\partial \beta} = 0$; $\frac{\partial \ell_n}{\partial \lambda} = 0$.

For estimating the model parameters, numerical iterative techniques should be used to solve these equations.We can investigate the global maxima of the log-likelihood by setting different starting values for the parameters. The information matrix will be required for interval estimation. Let $\theta = (\alpha, \beta, \lambda)^T$, then the asymptotic distribution of $\sqrt{n}(\theta - \hat{\theta})$ is $N_4(0, K(\theta)^{-1})$, under standard regularity conditions (see Lehmann and Casella, [19], pp. 461-463), where $K(\theta)$ is the expected information matrix. The asymptotic behavior remains valid if $K(\theta))$ is superseded by the observed information matrix multiplied by $1/n$, say $I(\theta)/n$, approximated by $\hat{\theta}$, i.e. $I(\hat{\theta})/n$. We have

$$
I(\theta) = -\begin{bmatrix} I_{\alpha\alpha} & I_{\alpha\beta} & I_{\alpha\lambda} \\ I_{\beta\alpha} & I_{\gamma\beta} & I_{\beta\lambda} \\ I_{\lambda\alpha} & I_{\lambda\beta} & I_{\lambda\lambda} \end{bmatrix},
$$

where

$$I_{\alpha\alpha} = \frac{\partial^2 \ell_n}{\partial \alpha^2}; \qquad I_{\alpha\beta} = I_{\beta\alpha} = \frac{\partial^2 \ell_n}{\partial \alpha \partial \beta},$$

$$I_{\beta\lambda} = I_{\lambda\beta} = \frac{\partial^2 \ell_n}{\partial \beta \partial \lambda}; \quad I_{\alpha\lambda} = I_{\lambda\alpha} = \frac{\partial^2 \ell_n}{\partial \alpha \partial \lambda}.$$

5 Simulation Study

The performance of the maximum likelihood method is evaluated for estimating the OLLL-L parameters using a Monte Carlo simulation study. The mean square error (MSEs) and the bias of the parameter estimates are calculated. We generate $N = 10,000$ samples of sizes $n = 50, 55, ..., 100$ from the OLLL-L distribution with $\alpha = 1.5$, $\beta = 0.3$, $\lambda = 2.0$. Let $(\widehat{\alpha}, \widehat{\gamma}, \widehat{\lambda})$ be the MLEs of the new model parameters and $(s_{\widehat{\alpha}}, s_{\widehat{\gamma}}, s_{\widehat{\lambda}})$ be the standard errors of the MLEs. The estimated biases and MSEs are given by

$$\widehat{Bias}_{\epsilon}(n) = \frac{1}{N}\sum_{i=1}^{N}(\hat{\epsilon}_i - \epsilon) \;\; \text{and} \;\; \widehat{MSE}_{\epsilon}(n) = \frac{1}{N}\sum_{i=1}^{N}(\hat{\epsilon}_i - \epsilon)^2,$$

for $\epsilon = \alpha, \gamma, \lambda$. Figure 3 displays the numerical results for the above measures. We conclude below results from these plots:

(i) The estimated biases decrease when the sample size n increases.

(ii) The estimated MSEs decay toward zero as n increases.

These results reveal the consistency property of the MLEs.

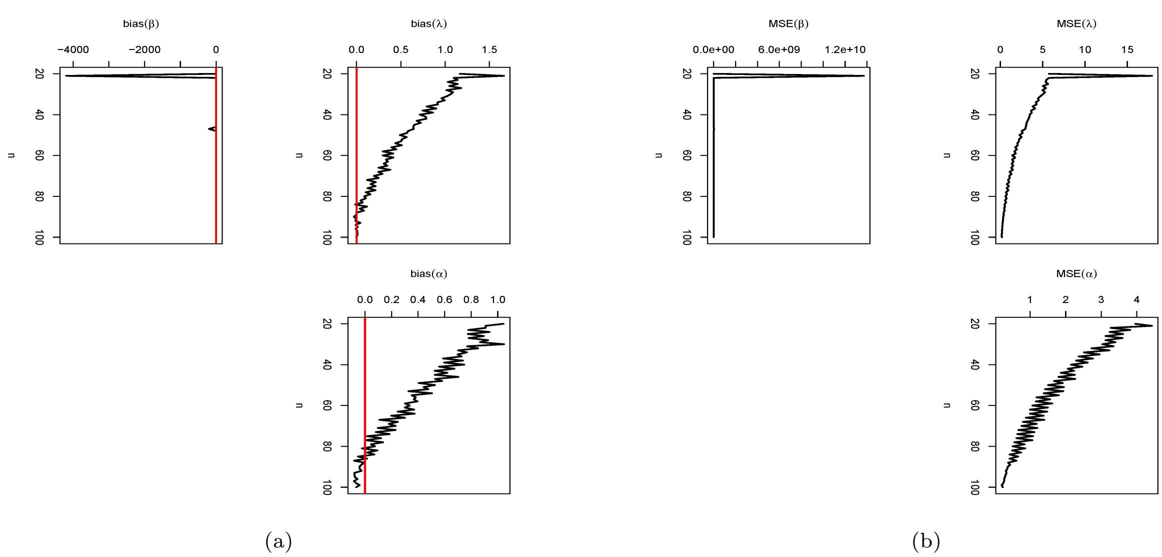


Figure 3: Estimated biases and MSEs for the selected parameter values.

6 Application

In this section, we illustrate the fitting performance of the OLLL-L distribution using two real data sets. For the purpose of comparison, we fitted the following models to show the fitting performance of OLLL-L distribution by means of real data set:

- Lindley Distribution, $L(\lambda)$.
- Power Lindley distribution, $PL(\beta, \lambda)$.
- Generalized Lindley, $GL(\alpha, \lambda)$, (Nadarajah et al. [22], with distribution function given by

$$F(x) = L(\lambda)^{\alpha} = \left(1 - (1 + \frac{\lambda\, x}{1+\lambda})\mathrm{e}^{-\lambda\,\mathrm{x}}\right)^{\alpha}.$$

- Beta Lindley, $BL(\alpha, \beta, \lambda)$, with distribution function given by

$$F(x) = \int_{0}^{L(\lambda)} t^{\alpha-1}(1-t)^{\beta-1}dt.$$

- Exponentiated power Lindley distribution, $EPL(\alpha, \beta, \lambda)$, with distribution function given by

$$F(x) = PL(\beta, \lambda)^{\alpha} = \left(1 - (1 + \frac{\lambda\, x^{\beta}}{1+\lambda})\mathrm{e}^{-\lambda\,\mathrm{x}^{\beta}}\right)^{\alpha.}$$

- Odd log-logistic power Lindley distribution $OLL - PL(\alpha, \beta, \lambda)$, (Alizadeh et al. [1]), with distribution function given by

$$F(x) = \frac{PL(x, \beta, \lambda)^{\alpha}}{PL(x, \beta, \lambda)^{\alpha} + (1 - PL(x, \beta, \lambda))^{\alpha}}.$$

- Kumaraswamy Power Lindley, $KPL(\alpha, \beta, \gamma, \lambda)$ (Broderick et al. [23])

$$F(x) = 1 - (1 - PL(x, \beta, \lambda)^{\alpha})^{\gamma}.$$

- Log-Lindley, $LL(\beta, \lambda)$

$$F(x) = \frac{\log(1 - \beta\, L(\lambda))}{\log(1-\beta)}.$$

Estimates of the parameters of OLLL-L distribution, Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), Cramer Von Mises and Anderson-Darling statistics (W^* and A^*) are presented for each dataset. We have also considered the Kolmogorov-Smirnov (K-S) statistic and its corresponding p-value and the minimum value of the minus log-likelihood function (-Log(L)) for the sake of comparison. Generally speaking, the smaller values of AIC, BIC, W^* and A^*, the better fit to a data set. All the computations were carried out using the software R.

Note that initial values of model parameters are quite important to obtain the correct MLEs of parameters. To avoid local minima problem, we first obtain the parameter estimate of the Lindley distribution. Then, the estimated parameter of the Lindley distribution is used as the initial value of the parameter of the PL and GL distributions. Then, the estimated parameters of PL distribution, λ and β, are used as the initial values of the OLLL-L distribution. This approach is quite useful to obtain correct parameter estimates of extended models.

6.1 First Application

Fonseca and Franca [9] studied the soil fertility in influence and the characterization of the biologic fixation of N2 for the Dimorphandra wilsonii rizz growth. For 128 plants, they made measures of the phosphorus concentration in the leaves. The data, which have also been analyzed by Bidram and Nekoukhou [3], are listed in Table 1.

Table 1: First data set.

0.22	0.17	0.11	0.10	0.15	0.06	0.05	0.07	0.12	0.09	0.23	0.25	0.23	0.24	0.20
0.08	0.11	0.12	0.10	0.06	0.20	0.17	0.20	0.11	0.16	0.09	0.10	0.12	0.12	0.10
0.09	0.17	0.19	0.21	0.18	0.26	0.19	0.17	0.18	0.20	0.24	0.19	0.21	0.22	0.17
0.08	0.08	0.06	0.09	0.22	0.23	0.22	0.19	0.27	0.16	0.28	0.11	0.10	0.20	0.12
0.15	0.08	0.12	0.09	0.14	0.07	0.09	0.05	0.06	0.11	0.16	0.20	0.25	0.16	0.13
0.11	0.11	0.11	0.08	0.22	0.11	0.13	0.12	0.15	0.12	0.11	0.11	0.15	0.10	0.15
0.17	0.14	0.12	0.18	0.14	0.18	0.13	0.12	0.14	0.09	0.10	0.13	0.09	0.11	0.11
0.14	0.07	0.07	0.19	0.17	0.18	0.16	0.19	0.15	0.07	0.09	0.17	0.10	0.08	0.15
0.21	0.16	0.08	0.10	0.06	0.08	0.12	0.13							

The ML estimates of the parameters and the goodness-of-fit test statistics for the real data set is presented in Table 2 and 3 respectively. As we can see, the smallest values of AIC, BIC, A^*, W^* and $-l$ statistics and the largest p-values belong to the OLLL-L distribution. Therefore the OLLL-L distribution outperforms the other competitive considered distribution in the sense of this criteria.

Table 2: Parameter ML estimates and theirs standard errors (in parentheses) for first data set.

$Model$	α	β	γ	λ
$Lindley(\lambda)$	–	–	–	7.901
				(0.6343)
$GL(\alpha,\lambda)$	10.5303	–	–	21.9739
	(2.0203)			(1.6518)
$PL(\beta,\lambda)$	–	2.8183	–	180.7664
		(0.1919)		(60.2260)
$BL(\alpha,\beta,\lambda)$	6.3358	45.4292	–	1.5111
	(0.8070	(64.7580)		(1.5113)
$EPL(\alpha,\beta,\lambda)$	1.5150	3.5946	–	37.6386
	(0.5103)	(2.8255)		(22.0225)
$OLLL(\alpha,\lambda)$	2.9955	–	–	6.0085
	(0.2183)			(0.1971)
$OLLPL(\alpha,\beta,\lambda)$	1.1537	2.4982	–	99.0435
	(0.3756)	(0.7218)		(134.6303)
$KPL(\alpha,\beta,\gamma,\lambda)$	6.6698	0.9258	3.0713	10.7987
	(3.2657)	(0.0598)	(1.2352)	(14.3619)
$EGL(\alpha,\gamma,\lambda)$	19.3599	–	69.6518	30.4042
	(6.1287)		(53.5897)	(3.6772)
$OLLL-L(\alpha,\beta,\lambda)$	6.5780	-13939.221	—	3.7721
	(0.6889)	1704.24		(0.1382)

Table 3: Goodness-of-fit test statistics for first data set.

$Model$	W^*	A^*	AIC	BIC	$CAIC$	$-l$
$Lindley(\lambda)$	0.1332	0.7556	-245.3218	-242.4697	-245.2900	123.6609
$GL(\alpha,\lambda)$	0.1323	0.8146	-388.0868	-382.3827	-387.9908	196.0434
$PL(\beta,\lambda)$	0.2071	1.1569	-385.6296	-379.9255	-385.5336	194.8148
$BL(\alpha,\beta,\lambda)$	0.1348	0.7607	-387.9407	-379.3846	-387.7471	196.9703
$EPL(\alpha,\beta,\lambda)$	0.1378	0.7824	-387.4424	-378.8863	-387.2489	196.7212
$OLLL(\alpha,\lambda)$	0.1883	1.0973	-383.0465	-377.3424	-382.9505	193.5232
$OLLPL(\alpha,\beta,\lambda)$	0.1986	1.1065	-383.8543	-375.2982	-383.6607	194.9271
$KPL(\alpha,\beta,\gamma,\lambda)$	0.1406	0.8002	-385.0695	-373.6613	-384.7443	196.5347
$EGL(\alpha,\gamma,\lambda)$	0.0818	0.4857	-391.0308	-382.4747	-390.8373	198.5154
$OLLLL(\alpha,\gamma,\lambda)$	0.0623	0.3879	-393.4425	-384.8864	-393.2489	199.7212

Here, we also applied likelihood ratio (LR) tests. The LR tests can be used for comparing the EGL distribution with its sub-models. For example, the test of $H_0 : \alpha = 1$ against $H_1 : \alpha \neq 1$ is equivalent to comparing the OLLL-L and Log-Lindley distributions with each other. For this test, the LR statistic can be calculated by the following relation

$$LR = 2\left[l(\widehat{\alpha},\widehat{\beta},\widehat{\lambda}) - l(1,\widehat{\beta^*},\widehat{\lambda^*})\right],$$

where $\widehat{\alpha^*}, \widehat{\beta^*}$ and $\widehat{\lambda^*}$ are the ML estimators of α, β and λ, respectively, obtained under H_0. Under the regularity conditions and if H_0 is assumed to be true, the LR test statistic converges in distribution to a chi square with r degrees of freedom, where r equals the difference between the number of parameters estimated under H_0 and the number of parameters estimated in general, (for $H_0 : \beta = 1$, we have $r = 1$). Table 4 gives the LR statistics and the corresponding p-values.

Table 4: The LR test results for first data set.

	Hypotheses	LR	p-value
OLLL-L versus OLL-L	$H_0 : \beta = 1$	12.396	0.0004
OLLL-L versus Log-Lindley	$H_0 : \alpha = 1$	9.5236	0.0012
OLLL-L versus Lindley	$H_0 : \alpha = \beta = 1$	152.1206	0

From Table 4, we observe that the computed p-values are too small so we reject all the null hypotheses and conclude that the OLLL-L fits the first data better than the considered sub-models according to the LR criterion.

We also plotted the fitted pdfs and cdfs of the considered models for the sake of visual comparison, in Figure 4. Figure 4 suggests that the OLLL-L fits the skewed data very well.

Table 5: Second data set.

0.014	0.034	0.059	0.061	0.069	0.080	0.123	0.142	0.165	0.212
0.381	0.464	0.479	0.556	0.574	0.839	0.917	0.969	0.991	1.064
1.088	1.091	1.174	1.271	1.275	1.355	1.397	1.477	1.578	1.649
1.702	1.893	1.932	2.001	2.161	2.292	2.326	2.337	2.628	2.785
2.811	2.886	2.993	3.122	3.248	3.715	3.791	3.857	3.912	4.121
4.106	4.116	4.315	4.510	4.580	5.267	5.299	5.583	6.065	9.701

6.2 Second Application

In this example, we analyze a real data set from Lawless [17]. It shows the number of 1000s of cycles to failure for electrical appliances in a life test. Table 6 and 7 present the parameter ML estimates and the goodness-of- fit test statistics for the second data, respectively. We see that the OLLL-L distribution outperforms the considered models according to the considered goodness-of-fit criteria.

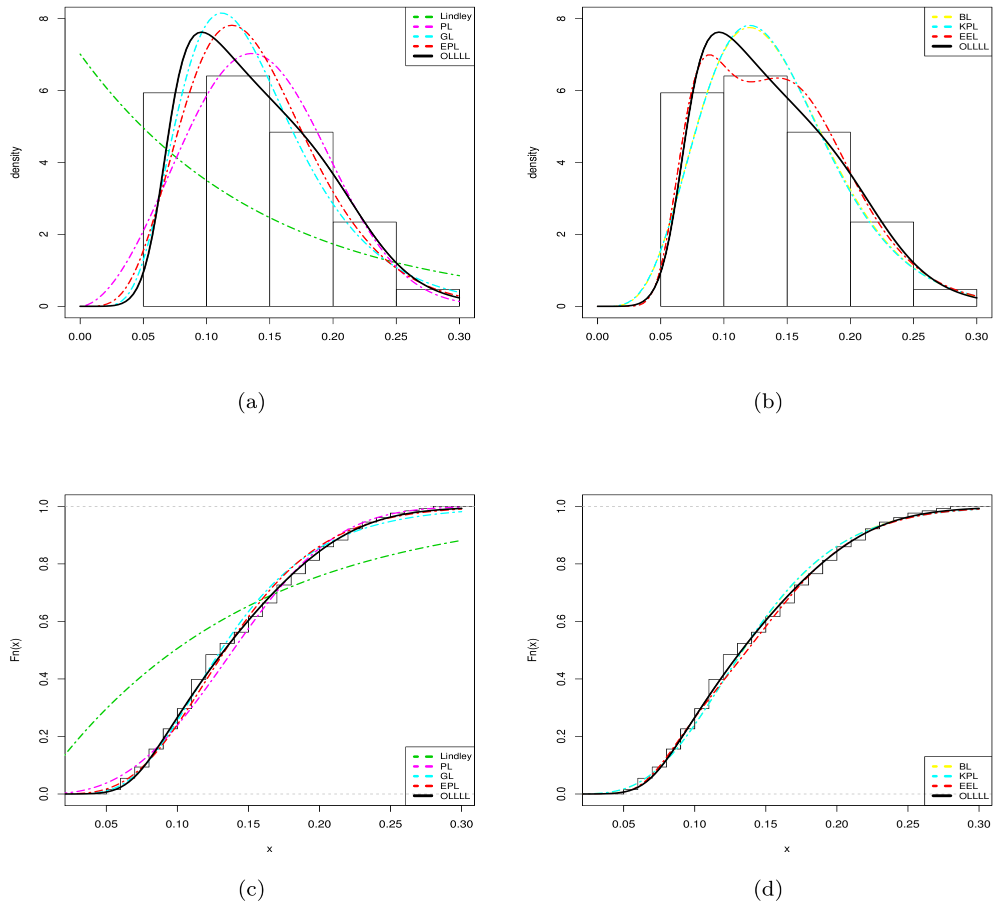


Figure 4: Fitted densities of distributions for first data set.

Table 6: Parameter ML estimates and theirs standard errors (in parentheses) for second data set.

$Model$	α	β	γ	λ
$Lindley(\lambda)$	–	–	–	0.7210
	–	–	–	(0.068)
$GL(\alpha,\lambda)$	0.6202	–	–	0.7522
	(0.0873)	–	–	(0.1274)
$PL(\beta,\lambda)$	–	0.8039	–	0.8882
	–	(0.1030)	–	(0.0891)
$BL(\alpha,\beta,\lambda)$	0.0589	0.6247	–	34.2119
	(0.0598)	(0.1029)	–	(61.4033)
$EPL(\alpha,\beta,\lambda)$	0.2803	1.4252	–	0.4353
	(0.2012)	(0.3684)	–	(0.1866)
$OLLL(\alpha,\lambda)$	0.7857	–	–	0.8144
	(0.0976)	–	–	(0.1003)
$OLLPL(\alpha,\beta,\lambda)$	0.5668	1.3608	–	0.6220
	(0.1547)	(0.2771)	–	(0.1668)
$KPL(\alpha,\beta,\gamma,\lambda)$	0.3946	2.9333	1.6113	0.0626
	(0.1825)	(2.9166)	(0.5376)	(0.0992)
$LL(\beta,\lambda)$	0.6426	-0.8572	–	–
	(0.1674)	(2.3433)	–	–
$OLLLL(\alpha,\beta,\lambda)$	0.6739	0.8329	–	1.2644
	(0.1283)	(0.1907)	–	(0.4047)

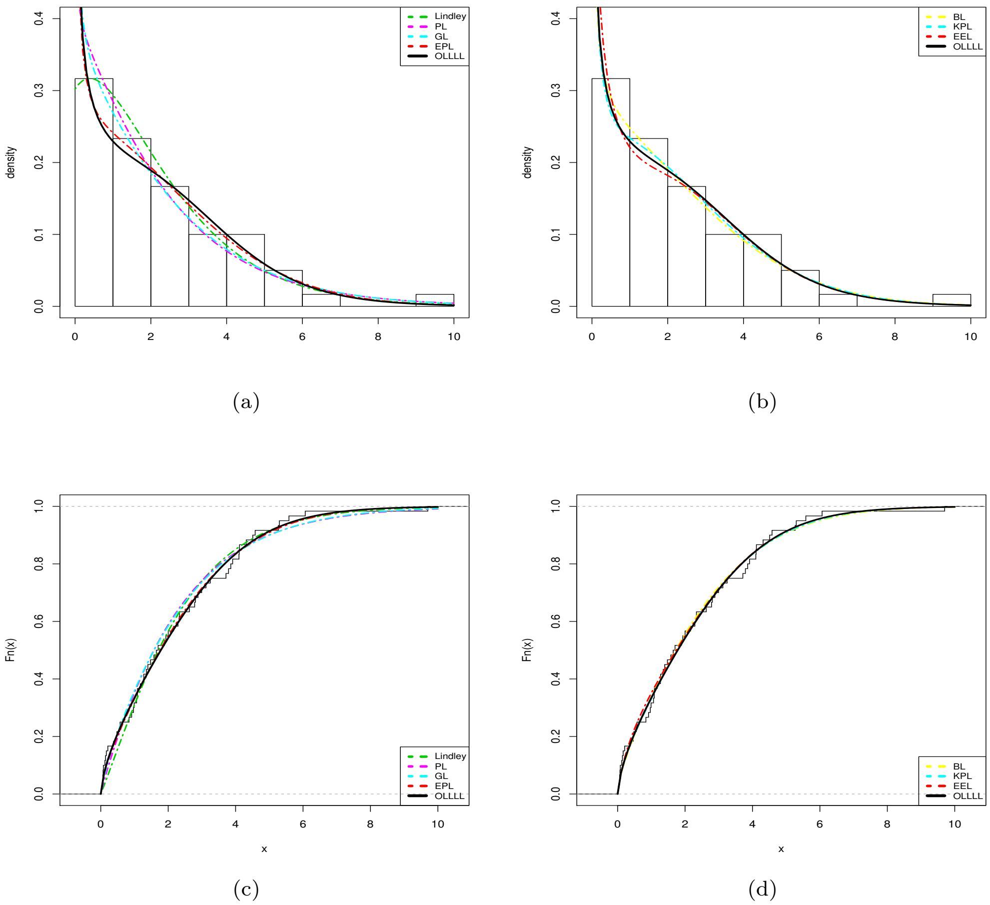


Figure 5: Fitted densities of distributions for first data set.

Table 7: Goodness-of-fit test statistics for second data set.

$Model$	W^*	A^*	AIC	BIC	$CAIC$	$-l$
$Lindley(\lambda)$	0.0635	0.4756	216.5948	218.6892	216.6638	107.7974
$GL(\alpha,\lambda)$	0.0591	0.4478	215.6075	219.7962	215.8180	105.8038
$PL(\beta,\lambda)$	0.0965	0.6768	217.0997	221.2884	217.3102	106.5499
$BL(\alpha,\beta,\lambda)$	1.5545	8.4762	216.1607	222.4437	216.5892	105.0803
$EPL(\alpha,\beta,\lambda)$	0.2886	1.7720	215.9393	222.2223	216.3679	104.9696
$OLLL(\alpha,\lambda)$	0.0610	0.4525	215.4906	219.6793	215.7011	106.7453
$OLLPL(\alpha,\beta,\lambda)$	0.0301	0.2511	215.7051	221.9881	216.1337	104.8525
$KPL(\alpha,\beta,\gamma,\lambda)$	0.0240	0.2194	217.7420	226.1194	218.4693	104.8710
$LL(\beta,\lambda)$	0.0820	0.5881	218.3651	222.5538	218.5756	107.1825
$OLLLL(\alpha,\beta,\lambda)$	0.0240	0.2070	215.3536	221.6366	215.7821	104.6768

Here, we also applied likelihood ratio (LR) tests for the second data set. The LR test results for the second data set are given in Table 8. The null hypotheses are all rejected in favor of the OLLL-L distribution since the p-values are less than 0.01.

Table 8: The LR test results for second data set.

	Hypotheses	LR	p-value
OLLL-L versus OLL-L	$H_0 : \beta = 1$	4.137	0.0419
OLLL-L versus Log-Lindley	$H_0 : \alpha = 1$	5.0114	0.0251
OLLL-L versus Lindley	$H_0 : \alpha = \beta = 1$	6.2412	0.0441

We also plotted the fitted pdfs and cdfs of the considered models for the sake of visual comparison, in Figure 5. Figure 5 suggests that the OLLL-L fits the skewed data very well.

7 Conclusion

In this paper, a new distribution called odd log-logistic logarithmic Lindley (OLLL-L) distribution was introduced. The statistical properties of the OLLL-L distribution including the hazard and reverse hazard functions, quantile function, moments, incomplete moments, generating functions, mean deviations, Bonferroni and Lorenz curves, order statistics and maximum likelihood estimation for the model parameters are given. Simulation studies was conducted to examine the performance of the new OLLL-L distribution. We also presented applications of this new model to a real life data set in order to illustrate the usefulness of the distribution.

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