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Error Bounds In Approximating The Exponential Beta Function With Ostrowski Type Quadrature Rules*

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Abstract

In this paper, by making use of Ostrowski type integral inequalities, we establish several error bounds in approximating the Exponential-Beta function

$$F(\alpha,\beta) := \int_0^1 \exp\left[x^\alpha (1-x)^\beta\right] dx,$$

where α, β are positive numbers, with some simple quadrature rules of Ostrowski and Trapezoid type.

1 Introduction

The ability to forecast project cash flow has profound impact on organizations ability to perform and secure its sustainability. The adverse impact of cash flow mismanagement can range from low financial performance to bankruptcy. The traditional and indeed dominant approach relies on significant knowledge about the project and the programme of work. The shortcomings of this approach have given rise to the development and use of mathematical approaches which tend to be easier, cheaper and faster.

One way by which the forecasting models are categorized is by examining the way the project dependant variables are related to the parameters of the mathematical expression. To this end, three groups of models are identified [8].

1) Inbuilt-Parameters: Typical of regression models and often referred to as black box model, here, the model is the product of the data and model parameters are determined by the data that generate the model.

2) No-Parameters: Here, the mathematical expression is simply a calculation of all cost elements by their relative quantity and rates as well as the identification of the time of their occurrence.

3) Independent-Parameters: Here, the mathematical expression is arbitrarily selected and attempts are made to establish link between its parameters and the domain data. These models tend have limited use, as they are not specifically designed to reflect the particular behavior of the domain data.

The proposed alternative model is a variation of the Independent-Parameters approach where the mathematical expression displays the general characteristics of the domain data and is capable of generating the specific characteristics of the domain data. However, the link between the data and the parameters of the mathematical expression is established through a common set of variables, namely the shape variables. In this approach, the model is independent of the data and new set of data can be processed without modifying the model. Here, the parameters have a real meaning which are then contextualised by the data. These meanings are defined in terms of two sets of characteristics: general and specific [9]. The former characteristics apply to all projects. Accordingly, the project starts and ends with zero values; there are no negative values and the periodic values are discrete values. The specific characteristics define the specific nature of each project pattern of expenditure. They are defined in terms of the coordinates of the main

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project peak on both time and cost axis (the curve monotonically ascends from zero to the peak point and then monotonically descends towards zero); the cumulative expenditure growth from the start to the peak; and distortion of the normal pattern resulting in the creation of additional peaks and troughs. Once these project variables are quantified, the mathematical model can generate an expenditure profile from which the expenditure values are extracted

Extensive analysis of project expenditure patterns has revealed that the main features of the shape of the project periodic expenditure pattern are defined in terms of a number of variables represented by the following expression (see, [10]):

$$Y_C := \exp\left[bx^a(1-x)^d\right] - 1$$

where

$$x_p := R = \frac{a}{a+d} \text{ and } y_p := Q = \exp\left[bR^a(1-R)^d\right] - 1,$$

where:

-Q, R , represent the positions of the project expenditure peak on both the cost and time access.

-a and b are parameterized in terms of x_p and y_p as follows

$$a = \frac{x_p d}{1-x_p}, \quad b = \frac{\ln(1+y_p)}{x_p^a (1-x_p)^d}.$$

-Parameter d is calculated through numerical method that is derived to rapidly converge towards a solution within desired error tolerance.

A relationship is established between the properties of the project and the physical shape of the project expenditure pattern. These are then related and reflected on the mathematical expression through its parameters.

Motivated by the above considerations, in the recent paper [4] we introduced the two-parameters family of functions

$$f_{\alpha,\beta}(x) := \exp\left[x^\alpha (1-x)^\beta\right], \; x \in [0,1], \; \alpha \,, \beta > 0$$

and the "*exponential beta function*" defined by the integral

$$F(\alpha,\beta) := \int_0^1 \exp\left[x^\alpha (1-x)^\beta\right] dx, \; \alpha, \; \beta > 0$$

and studied their mathematical properties.

We obtained among others the following representation:

Theorem 1 *For any natural number $n \geq 1$ and any α, $\beta > 0$ we have*

$$F(\alpha,\beta) = 1 + \sum_{k=1}^{n} \frac{1}{k!} B(\alpha k+1, \beta k+1) + R_n(\alpha,\beta), \tag{1}$$

where the remainder $R_n(\alpha,\beta)$ is given by

$$R_n(\alpha,\beta) := \frac{1}{n!}\int_0^1 \left(\int_0^1 \left\{x^{\alpha(n+1)}(1-x)^{\beta(n+1)} \exp\left[sx^\alpha (1-x)^\beta\right]\right\} dx\right)(1-s)^n \, ds. \tag{2}$$

As a consequence, we derived the following convergence result:

Corollary 2 *We have the following beta series expansion*

$$F(\alpha,\beta) = 1 + \sum_{k=1}^{\infty} \frac{1}{k!} B(\alpha k+1, \beta k+1) \tag{3}$$

uniformly over α, $\beta > 0$.

We also obtained the following convexity results:

Theorem 3 *The function* $F-1$ *is logarithmically convex on* $(0,\infty)\times(0,\infty)$ *as a function of two variables, and, in particular,* F *is convex on* $(0,\infty)\times(0,\infty)$.

In this paper, by utilising various Ostrowski type inequalities, we first establish some error bounds in approximating the Exponential Beta function $F(\alpha,\beta)$ with the generating function $f_{\alpha,\beta}(x)$ in the case when $\alpha,\ \beta>1$. In the second part we consider the following Ostrowski type quadrature rule

$$\Omega_k\left(f_{\alpha,\beta},I_k,\boldsymbol{\alpha}\right):=\sum_{i=0}^{k}\left(\alpha_{i+1}-\alpha_i\right)\exp\left[x_i^{\alpha}\left(1-x_i\right)^{\beta}\right]$$

and the Trapezoid rule

$$T_k\left(f_{\alpha,\beta},I_k\right):=\frac{1}{2}\left[x_1+\sum_{i=1}^{k-1}\left(x_{i+1}-x_{i-1}\right)\exp\left[x_i^{\alpha}\left(1-x_i\right)^{\beta}\right]+1-x_{k-1}\right],$$

associated to the division of the interval $[0,1]$,

$$I_k:0=x_0<x_1<...<x_{k-1}<x_k=1$$

and the intermediate points $\alpha_0=0$, $\alpha_i\in[x_{i-1},x_i]$ $(i=1,...,k)$ and $\alpha_{k+1}=1$, and establish accurate error bounds in approximating the Exponential Beta function $F(\alpha,\beta)$ with these quadrature rules. The case of *equidistant Trapezoid rule* given by

$$T_k\left(f_{\alpha,\beta}\right):=\frac{1}{k}+\frac{1}{k}\sum_{i=1}^{k-1}\exp\left[\left(\frac{i}{k}\right)^{\alpha}\left(1-\frac{i}{k}\right)^{\beta}\right]$$

for $k\geq 2$ is also analyzed.

2 Bounds Via Ostrowski Type Inequalities

The following lemma provides an error estimate in approximating the integral mean by a value of the function in the case when the derivative is bounded. It was obtained in 1938 by Ostrowski, see [12].

Lemma 4 *Let* $f:[a,b]\longrightarrow\mathbb{R}$ *be continuous on* $[a,b]$ *and differentiable on* (a,b), *whose derivative is bounded on* (a,b) *and let* $\|f'\|_{\infty,(a,b)}:=\sup_{t\in(a,b)}|f'(t)|<\infty$. *Then*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\leq\left[\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a)\|f'\|_{\infty,(a,b)}\qquad(4)$$

for all $x\in[a,b]$. *The constant* $\frac{1}{4}$ *is sharp in the sense that it cannot be replaced by a smaller one.*

For a recent survey on this inequality, see [1] and the references therein.

We start with a simple fact incorporated in the following:

Lemma 5 *Let* $\alpha,\ \beta>0$. *The generating function* $f_{\alpha,\beta}$ *is increasing on* $\left[0,\frac{\alpha}{\alpha+\beta}\right]$, *decreasing on* $\left[\frac{\alpha}{\alpha+\beta},1\right]$ *and*

$$\max_{x\in[0,1]}f_{\alpha,\beta}(x)=f_{\alpha,\beta}\left(\frac{\alpha}{\alpha+\beta}\right)=\exp\left[\left(\frac{\alpha}{\alpha+\beta}\right)^{\alpha}\left(\frac{\beta}{\alpha+\beta}\right)^{\beta}\right].\qquad(5)$$

Proof. We have

$$f_{\alpha,\beta}(x) = \exp[g_{\alpha,\beta}(x)]$$

where $g_{\alpha,\beta}(x) = x^{\alpha}(1-x)^{\beta}$, $x \in [0,1]$ and

$$f'_{\alpha,\beta}(x) = g'_{\alpha,\beta}(x)\exp[g_{\alpha,\beta}(x)], \quad x \in [0,1] \tag{6}$$

showing that the sign of $f'_{\alpha,\beta}$ on $[0,1]$ is the same with the one of $g'_{\alpha,\beta}$.

Further, we have

$$\begin{aligned} g'_{\alpha,\beta}(x) &= \alpha x^{\alpha-1}(1-x)^{\beta} - \beta x^{\alpha}(1-x)^{\beta-1} = x^{\alpha-1}(1-x)^{\beta-1}[\alpha(1-x)-\beta x] \\ &= x^{\alpha-1}(1-x)^{\beta-1}[\alpha-(\alpha+\beta)x], \quad x \in (0,1). \end{aligned} \tag{7}$$

This shows that $g'_{\alpha,\beta}(x) > 0$ for $x \in \left(0, \frac{\alpha}{\alpha+\beta}\right)$ and $g'_{\alpha,\beta}(x) < 0$ for $\left(\frac{\alpha}{\alpha+\beta}, \infty\right)$, which proves the statement. ■

Lemma 6 *For α, $\beta > 1$ we have*

$$\max_{x\in[0,1]} \left|g'_{\alpha,\beta}(x)\right| \le \max\{\alpha,\beta\}\left(\frac{\alpha-1}{\alpha+\beta-2}\right)^{\alpha-1}\left(\frac{\beta-1}{\alpha+\beta-2}\right)^{\beta-1}. \tag{8}$$

Proof. From (7), we have

$$g'_{\alpha,\beta}(x) = g_{\alpha-1,\beta-1}(x)[\alpha-(\alpha+\beta)x], \quad x \in [0,1],$$

which implies that for α, $\beta > 1$ we have

$$\max_{x\in[0,1]} \left|g'_{\alpha,\beta}(x)\right| \le \max_{x\in[0,1]} g_{\alpha-1,\beta-1}(x) \max_{x\in[0,1]} |\alpha-(\alpha+\beta)x| = \max\{\alpha,\beta\} \max_{x\in[0,1]} g_{\alpha-1,\beta-1}(x). \tag{9}$$

From (7) we get

$$g'_{\alpha-1,\beta-1}(x) = g_{\alpha-2,\beta-2}(x)[\alpha-1-(\alpha+\beta-2)x], x \in (0,1).$$

This shows that $g'_{\alpha-1,\beta-1}(x) > 0$ for $x \in \left(0, \frac{\alpha-1}{\alpha+\beta-2}\right)$ and $g'_{\alpha-1,\beta-1}(x) < 0$ for $\left(\frac{\alpha-1}{\alpha+\beta-2}, \infty\right)$, which gives that

$$\max_{x\in[0,1]} g_{\alpha-1,\beta-1}(x) = g_{\alpha-1,\beta-1}\left(\frac{\alpha-1}{\alpha+\beta-2}\right) = \left(\frac{\alpha-1}{\alpha+\beta-2}\right)^{\alpha-1}\left(\frac{\beta-1}{\alpha+\beta-2}\right)^{\beta-1}. \tag{10}$$

By (9) and (10) we get the desired inequality (8). ■

We have the following result via Ostrowski's inequality:

Theorem 7 *For α, $\beta > 1$ we have*

$$\begin{aligned} |F(\alpha,\beta) - f_{\alpha,\beta}(x)| \le &\left[\frac{1}{4} + \left(x-\frac{1}{2}\right)^2\right] \times \max\{\alpha,\beta\}\left(\frac{\alpha-1}{\alpha+\beta-2}\right)^{\alpha-1}\left(\frac{\beta-1}{\alpha+\beta-2}\right)^{\beta-1} \\ &\times \exp\left[\left(\frac{\alpha}{\alpha+\beta}\right)^{\alpha}\left(\frac{\beta}{\alpha+\beta}\right)^{\beta}\right] \end{aligned} \tag{11}$$

for all $x \in [0,1]$. In particular,

$$\begin{aligned} &\left|F(\alpha,\beta) - \exp\left(\frac{1}{2^{\alpha+\beta}}\right)\right| \\ &\le \frac{1}{4}\max\{\alpha,\beta\}\left(\frac{\alpha-1}{\alpha+\beta-2}\right)^{\alpha-1}\left(\frac{\beta-1}{\alpha+\beta-2}\right)^{\beta-1}\exp\left[\left(\frac{\alpha}{\alpha+\beta}\right)^{\alpha}\left(\frac{\beta}{\alpha+\beta}\right)^{\beta}\right]. \end{aligned} \tag{12}$$

Proof. If we write Ostrowski's inequality for the function $f_{\alpha,\beta}$ on the interval $[0,1]$, then we have

$$\left|f_{\alpha,\beta}(x)-\int_0^1 f_{\alpha,\beta}(t)\,dt\right|\le\left[\frac{1}{4}+\left(x-\frac{1}{2}\right)^2\right]\left\|f'_{\alpha,\beta}\right\|_{\infty,[0,1]} \tag{13}$$

for all $x\in[0,1]$. From (6) we have

$$f'_{\alpha,\beta}(x)=g'_{\alpha,\beta}(x)\,f_{\alpha,\beta}(x)\,,\ x\in[0,1]\,,$$

which shows that

$$\begin{aligned}&\max_{x\in[0,1]}\left|f'_{\alpha,\beta}(x)\right|\\&\le\max_{x\in[0,1]}\left|g'_{\alpha,\beta}(x)\right|\max_{x\in[0,1]}f_{\alpha,\beta}(x)\\&\le\max\{\alpha,\beta\}\left(\frac{\alpha-1}{\alpha+\beta-2}\right)^{\alpha-1}\left(\frac{\beta-1}{\alpha+\beta-2}\right)^{\beta-1}\exp\left[\left(\frac{\alpha}{\alpha+\beta}\right)^{\alpha}\left(\frac{\beta}{\alpha+\beta}\right)^{\beta}\right],\end{aligned}$$

where for the last inequality we used Lemmas 4 and 5. By employing (13) we obtain the desired result (11). ■

In 1997, Dragomir and Wang proved the following Ostrowski type inequality [5], see also [1, p. 26]:

Lemma 8 *Let $f:[a,b]\to\mathbb{R}$ be an absolutely continuous function on $[a,b]$. Then we have the inequality*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\le\left[\frac{1}{2}+\frac{\left|x-\frac{a+b}{2}\right|}{b-a}\right]\|f'\|_{[a,b],1}\,, \tag{14}$$

for all $x\in[a,b]$, where $\|\cdot\|_1$ is the Lebesgue norm on $L_1[a,b]$, i.e., we recall it

$$\|g\|_{[a,b],1}:=\int_a^b|g(t)|\,dt.$$

The constant $\frac{1}{2}$ is best possible.

Note the fact that $\frac{1}{2}$ is the best constant for differentiable functions was proved in [13].

Theorem 9 *For $\alpha,\ \beta>1$ we have*

$$\left|\frac{f_{\alpha,\beta}(x)}{F(\alpha,\beta)}-1\right|\le\left[\frac{1}{2}+\left|x-\frac{1}{2}\right|\right]\times\max\{\alpha,\beta\}\left(\frac{\alpha-1}{\alpha+\beta-2}\right)^{\alpha-1}\left(\frac{\beta-1}{\alpha+\beta-2}\right)^{\beta-1} \tag{15}$$

for all $x\in[0,1]$ and, in particular,

$$\left|\frac{\exp\left(\frac{1}{2^{\alpha+\beta}}\right)}{F(\alpha,\beta)}-1\right|\le\frac{1}{2}\max\{\alpha,\beta\}\left(\frac{\alpha-1}{\alpha+\beta-2}\right)^{\alpha-1}\left(\frac{\beta-1}{\alpha+\beta-2}\right)^{\beta-1}. \tag{16}$$

For $\alpha,\ \beta>0$, we also have

$$\left|F(\alpha,\beta)-f_{\alpha,\beta}(x)\right|\le\left[\frac{1}{2}+\left|x-\frac{1}{2}\right|\right]\times\max\{\alpha,\beta\}\,B(\alpha,\beta)\exp\left[\left(\frac{\alpha}{\alpha+\beta}\right)^{\alpha}\left(\frac{\beta}{\alpha+\beta}\right)^{\beta}\right] \tag{17}$$

for all $x\in[0,1]$ and, in particular,

$$\left|F(\alpha,\beta)-\exp\left(\frac{1}{2^{\alpha+\beta}}\right)\right|\le\frac{1}{2}\max\{\alpha,\beta\}\,B(\alpha,\beta)\exp\left[\left(\frac{\alpha}{\alpha+\beta}\right)^{\alpha}\left(\frac{\beta}{\alpha+\beta}\right)^{\beta}\right] \tag{18}$$

where $B(\cdot,\cdot)$ is Euler's Beta function.

Proof. If we write the inequality (14) for $f_{\alpha,\beta}$ on the interval $[0,1]$, then we have

$$\left|f_{\alpha,\beta}(x) - F(\alpha,\beta)\right| \leq \left[\frac{1}{2} + \left|x - \frac{1}{2}\right|\right] \left\|f'_{\alpha,\beta}\right\|_{[0,1],1}, \tag{19}$$

for all $x \in [0,1]$. Now, observe that

$$\begin{aligned} \left\|f'_{\alpha,\beta}\right\|_{[0,1],1} &= \int_0^1 \left|f'_{\alpha,\beta}(t)\right| dt = \int_0^1 \left|g'_{\alpha,\beta}(t)\right| \exp\left[g_{\alpha,\beta}(t)\right] dt \\ &= \int_0^1 \left|g'_{\alpha,\beta}(t)\right| f_{\alpha,\beta}(t)\, dt = \int_0^1 g_{\alpha-1,\beta-1}(t) \left|\alpha - (\alpha+\beta)t\right| f_{\alpha,\beta}(t)\, dt \\ &\leq \max_{t\in[0,1]} \left|\alpha - (\alpha+\beta)t\right| \int_0^1 g_{\alpha-1,\beta-1}(t) f_{\alpha,\beta}(t)\, dt \\ &= \max\{\alpha,\beta\} \int_0^1 g_{\alpha-1,\beta-1}(t) f_{\alpha,\beta}(t)\, dt. \end{aligned} \tag{20}$$

Since

$$\begin{aligned} \int_0^1 g_{\alpha-1,\beta-1}(t) f_{\alpha,\beta}(t)\, dt &\leq \max_{t\in[0,1]} g_{\alpha-1,\beta-1}(t) \int_0^1 f_{\alpha,\beta}(t)\, dt \\ &= \left(\frac{\alpha-1}{\alpha+\beta-2}\right)^{\alpha-1} \left(\frac{\beta-1}{\alpha+\beta-2}\right)^{\beta-1} F(\alpha,\beta) \text{ by (10)}, \end{aligned} \tag{21}$$

hence by (19)–(21) we get

$$\left|f_{\alpha,\beta}(x) - F(\alpha,\beta)\right| \leq \left[\frac{1}{2} + \left|x - \frac{1}{2}\right|\right] \left(\frac{\alpha-1}{\alpha+\beta-2}\right)^{\alpha-1} \left(\frac{\beta-1}{\alpha+\beta-2}\right)^{\beta-1} F(\alpha,\beta)$$

that is equivalent to (15).

We also have

$$\begin{aligned} \int_0^1 g_{\alpha-1,\beta-1}(t) f_{\alpha,\beta}(t)\, dt &\leq \max_{t\in[0,1]} f_{\alpha,\beta}(t) \int_0^1 g_{\alpha-1,\beta-1}(t)\, dt \\ &= \exp\left[\left(\frac{\alpha}{\alpha+\beta}\right)^{\alpha} \left(\frac{\beta}{\alpha+\beta}\right)^{\beta}\right] B(\alpha,\beta) \text{ (by 5)}, \end{aligned} \tag{22}$$

hence by (19), (20) and (22) we get (17). ∎

In 1998, Dragomir and Wang proved the following Ostrowski type inequality for p-norms of the derivative [6].

Lemma 10 *Let $f : [a,b] \to \mathbb{R}$ be an absolutely continuous function on $[a,b]$. If $f' \in L_p[a,b]$, then we have the inequality*

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)\, dt\right| \leq \frac{1}{(q+1)^{1/q}} \left[\left(\frac{x-a}{b-a}\right)^{q+1} + \left(\frac{b-x}{b-a}\right)^{q+1}\right]^{1/q} (b-a)^{1/q} \left\|f'\right\|_{[a,b],p}, \tag{23}$$

for all $x \in [a,b]$, where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and $\|\cdot\|_{[a,b],p}$ is the p-Lebesgue norm on $L_p[a,b]$, i.e., we recall it

$$\|g\|_{[a,b],p} := \left(\int_a^b |g(t)|^p\, dt\right)^{1/p}.$$

Using this tool we can prove the following result as well:

Theorem 11 *For α, $\beta > 1$ we have*

$$|F(\alpha,\beta) - f_{\alpha,\beta}(x)| \le \frac{1}{(q+1)^{1/q}}\left[x^{q+1} + (1-x)^{q+1}\right]^{1/q} \times \max\{\alpha,\beta\}\exp\left[\left(\frac{\alpha}{\alpha+\beta}\right)^{\alpha}\left(\frac{\beta}{\alpha+\beta}\right)^{\beta}\right] \times \left[B(p(\alpha-1)+1, p(\beta-1)+1)\right]^{1/p} \tag{24}$$

for all $x \in [0,1]$, where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$. In particular,

$$\left|F(\alpha,\beta) - \exp\left(\frac{1}{2^{\alpha+\beta}}\right)\right| \le \frac{1}{2(q+1)^{1/q}}\max\{\alpha,\beta\}\exp\left[\left(\frac{\alpha}{\alpha+\beta}\right)^{\alpha}\left(\frac{\beta}{\alpha+\beta}\right)^{\beta}\right] \times \left[B(p(\alpha-1)+1, p(\beta-1)+1)\right]^{1/p}, \tag{25}$$

where $B(\cdot,\cdot)$ is Euler's Beta function.

Proof. If we write the inequality (23) for the function for $f_{\alpha,\beta}$ on the interval $[0,1]$, then we have

$$|f_{\alpha,\beta}(x) - F(\alpha,\beta)| \le \frac{1}{(q+1)^{1/q}}\left[x^{q+1} + (1-x)^{q+1}\right]^{1/q}\left\|f'_{\alpha,\beta}\right\|_{[0,1],p}, \tag{26}$$

for all $x \in [a,b]$, where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$.

Observe that

$$\begin{aligned}\left\|f'_{\alpha,\beta}\right\|^p_{[0,1],p} &= \int_0^1 \left|f'_{\alpha,\beta}(t)\right|^p dt = \int_0^1 \left|g'_{\alpha,\beta}(t)\right|^p \left(\exp\left[g_{\alpha,\beta}(t)\right]\right)^p dt \\ &= \int_0^1 \left|g'_{\alpha,\beta}(t)\right|^p f^p_{\alpha,\beta}(t)\,dt = \int_0^1 g^p_{\alpha-1,\beta-1}(t)\left|\alpha - (\alpha+\beta)t\right|^p f^p_{\alpha,\beta}(t)\,dt \\ &\le \max\{\alpha^p,\beta^p\}\int_0^1 x^{p(\alpha-1)}(1-x)^{p(\beta-1)} f^p_{\alpha,\beta}(t)\,dt.\end{aligned} \tag{27}$$

Since

$$\max_{t\in[0,1]} f^p_{\alpha,\beta}(t) = f^p_{\alpha,\beta}\left(\frac{\alpha}{\alpha+\beta}\right) = \exp\left[p\left(\frac{\alpha}{\alpha+\beta}\right)^{\alpha}\left(\frac{\beta}{\alpha+\beta}\right)^{\beta}\right],$$

and by (27), we get

$$\begin{aligned}\left\|f'_{\alpha,\beta}\right\|^p_{[0,1],p} &\le \max\{\alpha^p,\beta^p\}\exp\left[p\left(\frac{\alpha}{\alpha+\beta}\right)^{\alpha}\left(\frac{\beta}{\alpha+\beta}\right)^{\beta}\right]\int_0^1 x^{p(\alpha-1)}(1-x)^{p(\beta-1)}\,dt \\ &= \max\{\alpha^p,\beta^p\}\exp\left[p\left(\frac{\alpha}{\alpha+\beta}\right)^{\alpha}\left(\frac{\beta}{\alpha+\beta}\right)^{\beta}\right] B(p(\alpha-1)+1, p(\beta-1)+1),\end{aligned}$$

namely

$$\left\|f'_{\alpha,\beta}\right\|_{[0,1],p} \le \max\{\alpha,\beta\}\exp\left[\left(\frac{\alpha}{\alpha+\beta}\right)^{\alpha}\left(\frac{\beta}{\alpha+\beta}\right)^{\beta}\right]\left[B(p(\alpha-1)+1, p(\beta-1)+1)\right]^{1/p},$$

Therefore, by (26), we get the desired result (24). ■

3 Quadrature Rules of Ostrowski and Trapezoid Type

Let

$$I_k : a = x_0 < x_1 < ... < x_{k-1} < x_k = b$$

be a *division* of the interval $[a,b]$, α_i $(i = 0,...,k+1)$ be "$k+2$" points so that $\alpha_0 = a$, $\alpha_i \in [x_{i-1}, x_i]$ $(i = 1,...,k)$ and $\alpha_{k+1} = b$. Define

$$h_i := x_{i+1} - x_i \ \ (i = 0,...,k-1) \text{ and } \nu(h) := \max\{h_i \mid i = 0,...,k-1\}.$$

Consider the equality

$$\int_a^b f(t)\,dt = \Omega_k(f, I_k, \boldsymbol{\alpha}_{k+1}) + R_k(f, I_k, \boldsymbol{\alpha}_{k+1}) \tag{28}$$

where

$$\Omega_k(f, I_k, \boldsymbol{\alpha}) := \sum_{i=0}^{k} (\alpha_{i+1} - \alpha_i) f(x_i) \tag{29}$$

is the *Ostrowski quadrature rule* associated to the division I_k and the "k+2" points $\boldsymbol{\alpha}_{k+1} := (\alpha_0, \alpha_1, ..., \alpha_k, \alpha_{k+1})$ while $R_k(f, I_k, \boldsymbol{\alpha}_{k+1})$ is the error in approximating the integral $\int_a^b f(t)\,dt$ by the quadrature $\Omega_k(f, I_k, \boldsymbol{\alpha})$.

If we chose in (29)

$$\alpha_0 = a,\ \alpha_1 = \frac{a + x_1}{2},\ \alpha_2 = \frac{x_1 + x_2}{2}, ..., \alpha_{k-1} = \frac{x_{k-2} + x_{k-1}}{2},\ \alpha_k = \frac{x_{k-1} + x_k}{2},\ \alpha_{k+1} = b,$$

then we get after some arrangements that

$$\begin{aligned}\Omega_k(f, I_k, \boldsymbol{\alpha}) &= \frac{1}{2}\left[(x_1 - a) f(a) + \sum_{i=1}^{k-1} (x_{i+1} - x_{i-1}) f(x_i) + (b - x_{k-1}) f(b)\right] \\ &=: T_k(f, I_k),\end{aligned}$$

where $T_k(f, I_k)$ is called the *Trapezoid quadrature rule* associated to the function f and the division I_k.

In this situation we have

$$\int_a^b f(t)\,dt = T_k(f, I_k) + R_k(f, I_k), \tag{30}$$

where $R_k(f, I_k)$ is the error in approximation the integral by the trapezoid rule $T_k(f, I_k)$.

Let

$$I_k :\ x_i := a + (b-a)\frac{i}{k},\ i = 0,...,k$$

be the equidistant partitioning of $[a,b]$. We can consider then the *equidistant Trapezoid rule* given by

$$T_k(f) := \frac{1}{k}\frac{f(a) + f(b)}{2}(b-a) + \frac{b-a}{k}\sum_{i=1}^{k-1} f\left(a + (b-a)\frac{i}{k}\right)$$

for $k \geq 2$.

Further, we can approximate the integral as

$$\int_a^b f(t)\,dt = T_k(f) + R_k(f), \tag{31}$$

where $R_k(f)$ is the error in this equidistant approximation.

Assume that f is absolutely continuous on $[a,b]$.

If f' is essentially bounded on $[a,b]$, namely, $f' \in L_\infty[a,b]$, then we have the error bounds [7, p. 19]

$$
\begin{aligned}
|R_k(f, I_k, \boldsymbol{\alpha}_{k+1})| &\le \left[\frac{1}{4}\sum_{i=0}^{k-1} h_i^2 + \sum_{i=0}^{k-1}\left(\alpha_{i+1} - \frac{x_i + x_{i+1}}{2}\right)^2\right] \|f'\|_{\infty,[a,b]} \\
&\le \frac{1}{2}\sum_{i=0}^{k-1} h_i^2 \|f'\|_{\infty,[a,b]} \le \frac{1}{2}(b-a)\|f'\|_{\infty,[a,b]}\,\nu(h). \qquad (32)
\end{aligned}
$$

The trapezoid rule error $R_k(f, I_k)$ satisfies the better bounds

$$
|R_k(f, I_k)| \le \frac{1}{4}\left(\sum_{i=0}^{k-1} h_i^2\right)\|f'\|_{\infty,[a,b]} \le \frac{1}{4}(b-a)\|f'\|_{\infty,[a,b]}\,\nu(h)
$$

and the equidistant error $R_k(f)$ satisfies the inequality

$$
|R_k(f)| \le \frac{1}{4k}(b-a)^2\|f'\|_{\infty,[a,b]}.
$$

In terms of 1-norm we have the error bounds [2], see also [7, p. 51],

$$
|R_k(f, I_k, \boldsymbol{\alpha}_{k+1})| \le \left[\frac{1}{2}\nu(h) + \max_{i=1,\ldots,n}\left|\alpha_{i+1} - \frac{x_i + x_{i+1}}{2}\right|\right]\|f'\|_{1,[a,b]} \le \|f'\|_{1,[a,b]}\,\nu(h). \qquad (33)
$$

In particular, we have

$$
|R_k(f, I_k)| \le \frac{1}{2}\nu(h)\|f'\|_{1,[a,b]}
$$

and

$$
|R_k(f)| \le \frac{1}{2k}(b-a)\|f'\|_{1,[a,b]}.
$$

If $f' \in L_p[a,b]$, $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then [3], see also [7, p. 35],

$$
\begin{aligned}
|R_k(f, I_k, \boldsymbol{\alpha}_{k+1})| &\le \frac{1}{(q+1)^{1/q}}\left[\sum_{i=0}^{k-1}(\alpha_{i+1} - x_i)^{q+1} + (x_{i+1} - \alpha_{i+1})^{q+1}\right]^{1/q}\|f'\|_{p,[a,b]} \\
&\le \frac{1}{(q+1)^{1/q}}\|f'\|_{p,[a,b]}\left(\sum_{i=0}^{k-1} h_i^{q+1}\right)^{1/q} \\
&\le \frac{1}{(q+1)^{1/q}}(b-a)^{1/q}\|f'\|_{p,[a,b]}\,\nu(h). \qquad (34)
\end{aligned}
$$

Moreover, we have

$$
|R_k(f, I_k)| \le \frac{1}{2(q+1)^{1/q}}\|f'\|_{p,[a,b]}\left(\sum_{i=0}^{k-1} h_i^{q+1}\right)^{1/q} \le \frac{1}{2(q+1)^{1/q}}(b-a)^{1/q}\|f'\|_{p,[a,b]}\,\nu(h)
$$

and

$$
|R_k(f)| \le \frac{1}{2k(q+1)^{1/q}}(b-a)^{1+1/q}\|f'\|_{p,[a,b]}.
$$

Let

$$
I_k : 0 = x_0 < x_1 < \ldots < x_{k-1} < x_k = 1
$$

be a division of the interval $[0,1]$ and $\alpha_0 = 0$, $\alpha_i \in [x_{i-1}, x_i]$ $(i = 1, ..., k)$ and $\alpha_{k+1} = 1$. We define the following Ostrowski type quadrature rule for the Exponential Beta function by

$$\Omega_k\left(f_{\alpha,\beta}, I_k, \boldsymbol{\alpha}\right) := \sum_{i=0}^{k} \left(\alpha_{i+1} - \alpha_i\right) \exp\left[x_i^{\alpha}\left(1 - x_i\right)^{\beta}\right]$$

and the Trapezoid rule by

$$T_k\left(f_{\alpha,\beta}, I_k\right) := \frac{1}{2}\left[x_1 + \sum_{i=1}^{k-1}\left(x_{i+1} - x_{i-1}\right)\exp\left[x_i^{\alpha}\left(1 - x_i\right)^{\beta}\right] + 1 - x_{k-1}\right].$$

Consider also the *equidistant Trapezoid rule* given by

$$T_k\left(f_{\alpha,\beta}\right) := \frac{1}{k} + \frac{1}{k}\sum_{i=1}^{k-1}\exp\left[\left(\frac{i}{k}\right)^{\alpha}\left(1 - \frac{i}{k}\right)^{\beta}\right]$$

for $k \geq 2$.

Theorem 12 *Let I_k, $\boldsymbol{\alpha}$ be as defined above. Then*

$$F(\alpha, \beta) = \Omega_k\left(f_{\alpha,\beta}, I_k, \boldsymbol{\alpha}\right) + R_k\left(f_{\alpha,\beta}, I_k, \boldsymbol{\alpha}_{k+1}\right),$$

where the remainder $R_k\left(f_{\alpha,\beta}, I_k, \boldsymbol{\alpha}_{k+1}\right)$ satisfies the bounds

$$\begin{aligned}
\left|R_k\left(f_{\alpha,\beta}, I_k, \boldsymbol{\alpha}_{k+1}\right)\right| &\leq \left[\frac{1}{4}\sum_{i=0}^{k-1} h_i^2 + \sum_{i=0}^{k-1}\left(\alpha_{i+1} - \frac{x_i + x_{i+1}}{2}\right)^2\right]\left\|f'_{\alpha,\beta}\right\|_{\infty,[0,1]} \\
&\leq \left[\frac{1}{4}\sum_{i=0}^{k-1} h_i^2 + \sum_{i=0}^{k-1}\left(\alpha_{i+1} - \frac{x_i + x_{i+1}}{2}\right)^2\right] \\
&\times \max\{\alpha, \beta\}\left(\frac{\alpha - 1}{\alpha + \beta - 2}\right)^{\alpha-1}\left(\frac{\beta - 1}{\alpha + \beta - 2}\right)^{\beta-1} \\
&\times \exp\left[\left(\frac{\alpha}{\alpha + \beta}\right)^{\alpha}\left(\frac{\beta}{\alpha + \beta}\right)^{\beta}\right], \ \alpha, \ \beta > 1, \qquad (35)
\end{aligned}$$

$$\begin{aligned}
\left|R_k\left(f_{\alpha,\beta}, I_k, \boldsymbol{\alpha}_{k+1}\right)\right| &\leq \left[\frac{1}{2}\nu(h) + \max_{i=1,\dots,n}\left|\alpha_{i+1} - \frac{x_i + x_{i+1}}{2}\right|\right]\left\|f'_{\alpha,\beta}\right\|_{1,[0,1]} \\
&\leq \left[\frac{1}{2}\nu(h) + \max_{i=1,\dots,n}\left|\alpha_{i+1} - \frac{x_i + x_{i+1}}{2}\right|\right] \\
&\times \max\{\alpha, \beta\} B(\alpha, \beta)\exp\left[\left(\frac{\alpha}{\alpha + \beta}\right)^{\alpha}\left(\frac{\beta}{\alpha + \beta}\right)^{\beta}\right], \ \alpha, \ \beta > 0 \qquad (36)
\end{aligned}$$

and

$$\begin{aligned}
\left|R_k\left(f_{\alpha,\beta}, I_k, \boldsymbol{\alpha}_{k+1}\right)\right| &\leq \frac{1}{(q+1)^{1/q}}\left[\sum_{i=0}^{k-1}\left(\alpha_{i+1} - x_i\right)^{q+1} + \left(x_{i+1} - \alpha_{i+1}\right)^{q+1}\right]^{1/q}\left\|f'_{\alpha,\beta}\right\|_{p,[0,1]} \\
&\leq \frac{1}{(q+1)^{1/q}}\left[\sum_{i=0}^{k-1}\left(\alpha_{i+1} - x_i\right)^{q+1} + \left(x_{i+1} - \alpha_{i+1}\right)^{q+1}\right]^{1/q} \\
&\times \max\{\alpha, \beta\}\exp\left[\left(\frac{\alpha}{\alpha + \beta}\right)^{\alpha}\left(\frac{\beta}{\alpha + \beta}\right)^{\beta}\right] \\
&\times \left[B\left(p(\alpha - 1) + 1, p(\beta - 1) + 1\right)\right]^{1/p}, \ \alpha, \ \beta > 1, \qquad (37)
\end{aligned}$$

where $p>1$ *and* $\frac{1}{p}+\frac{1}{q}=1$.

The proof follows from the inequalities (32), (33) and (34), and the fact that from the previous section we have the following upper bounds for the norms of $f'_{\alpha,\beta}$

$$\left\|f'_{\alpha,\beta}\right\|_{\infty,[0,1]} \le \max\{\alpha,\beta\}\left(\frac{\alpha-1}{\alpha+\beta-2}\right)^{\alpha-1}\left(\frac{\beta-1}{\alpha+\beta-2}\right)^{\beta-1}\exp\left[\left(\frac{\alpha}{\alpha+\beta}\right)^{\alpha}\left(\frac{\beta}{\alpha+\beta}\right)^{\beta}\right],\ \alpha,\beta>1,$$

$$\left\|f'_{\alpha,\beta}\right\|_{1,[0,1]} \le \max\{\alpha,\beta\}\exp\left[\left(\frac{\alpha}{\alpha+\beta}\right)^{\alpha}\left(\frac{\beta}{\alpha+\beta}\right)^{\beta}\right]B(\alpha,\beta),\ \alpha,\beta>0$$

and

$$\left\|f'_{\alpha,\beta}\right\|_{[0,1],p} \le \max\{\alpha,\beta\}\exp\left[\left(\frac{\alpha}{\alpha+\beta}\right)^{\alpha}\left(\frac{\beta}{\alpha+\beta}\right)^{\beta}\right]\left[B\left(p(\alpha-1)+1,p(\beta-1)+1\right)\right]^{1/p},\ \alpha,\beta>1.$$

Corollary 13 *Let* I_k *be as defined above. Then*

$$F(\alpha,\beta)=T_k(f_{\alpha,\beta},I_k)+R_k(f_{\alpha,\beta},I_k),$$

where the remainder $R_k(f_{\alpha,\beta},I_k)$ *satisfies the bounds*

$$\begin{aligned}\left|R_k(f_{\alpha,\beta},I_k)\right| &\le \frac{1}{4}\sum_{i=0}^{k-1}h_i^2\left\|f'_{\alpha,\beta}\right\|_{\infty,[0,1]}\\ &\le \left[\frac{1}{4}\sum_{i=0}^{k-1}h_i^2\right]\times\max\{\alpha,\beta\}\left(\frac{\alpha-1}{\alpha+\beta-2}\right)^{\alpha-1}\left(\frac{\beta-1}{\alpha+\beta-2}\right)^{\beta-1}\\ &\quad\times\exp\left[\left(\frac{\alpha}{\alpha+\beta}\right)^{\alpha}\left(\frac{\beta}{\alpha+\beta}\right)^{\beta}\right],\ \alpha,\ \beta>1,\end{aligned}\tag{38}$$

$$\begin{aligned}\left|R_k(f_{\alpha,\beta},I_k)\right| &\le \left[\frac{1}{2}\nu(h)\right]\left\|f'_{\alpha,\beta}\right\|_{1,[0,1]} \le \left[\frac{1}{2}\nu(h)\right]\\ &\quad\times\max\{\alpha,\beta\}B(\alpha,\beta)\exp\left[\left(\frac{\alpha}{\alpha+\beta}\right)^{\alpha}\left(\frac{\beta}{\alpha+\beta}\right)^{\beta}\right],\ \alpha,\ \beta>0\end{aligned}\tag{39}$$

and

$$\begin{aligned}\left|R_k(f_{\alpha,\beta},I_k)\right| &\le \frac{1}{2(q+1)^{1/q}}\left(\sum_{i=0}^{k-1}h_i^{q+1}\right)^{1/q}\left\|f'_{\alpha,\beta}\right\|_{p,[0,1]}\\ &\le \frac{1}{2(q+1)^{1/q}}\left(\sum_{i=0}^{k-1}h_i^{q+1}\right)^{1/q}\\ &\quad\times\max\{\alpha,\beta\}\exp\left[\left(\frac{\alpha}{\alpha+\beta}\right)^{\alpha}\left(\frac{\beta}{\alpha+\beta}\right)^{\beta}\right]\\ &\quad\times\left[B\left(p(\alpha-1)+1,p(\beta-1)+1\right)\right]^{1/p},\ \alpha,\ \beta>1.\end{aligned}\tag{40}$$

Remark 1 *Finally, we mention the following simple trapezoid quadrature rule*

$$F(\alpha,\beta)=T_k(f_{\alpha,\beta})+R_k(f_{\alpha,\beta}),$$

where the remainder $R_k(f_{\alpha,\beta})$ satisfies the bounds

$$\begin{aligned}|R_k(f_{\alpha,\beta})| &\le \frac{1}{4k}\left\|f'_{\alpha,\beta}\right\|_{\infty,[0,1]} \\ &\le \frac{1}{4k}\max\{\alpha,\beta\}\left(\frac{\alpha-1}{\alpha+\beta-2}\right)^{\alpha-1}\left(\frac{\beta-1}{\alpha+\beta-2}\right)^{\beta-1} \\ &\times \exp\left[\left(\frac{\alpha}{\alpha+\beta}\right)^{\alpha}\left(\frac{\beta}{\alpha+\beta}\right)^{\beta}\right],\ \alpha,\ \beta>1, \end{aligned} \tag{41}$$

$$\begin{aligned}|R_k(f_{\alpha,\beta})| &\le \frac{1}{2k}\left\|f'_{\alpha,\beta}\right\|_{1,[0,1]} \\ &\le \frac{1}{2k}\max\{\alpha,\beta\}\exp\left[\left(\frac{\alpha}{\alpha+\beta}\right)^{\alpha}\left(\frac{\beta}{\alpha+\beta}\right)^{\beta}\right]B(\alpha,\beta),\ \alpha,\ \beta>0\end{aligned} \tag{42}$$

and

$$\begin{aligned}|R_k(f_{\alpha,\beta})| &\le \frac{1}{2k(q+1)^{1/q}}\left\|f'_{\alpha,\beta}\right\|_{p,[0,1]} \\ &\le \frac{1}{2k(q+1)^{1/q}}\max\{\alpha,\beta\}\exp\left[\left(\frac{\alpha}{\alpha+\beta}\right)^{\alpha}\left(\frac{\beta}{\alpha+\beta}\right)^{\beta}\right] \\ &\times\left[B(p(\alpha-1)+1,p(\beta-1)+1)\right]^{1/p},\ \alpha,\ \beta>1.\end{aligned} \tag{43}$$

The bounds above show that $R_k(f_{\alpha,\beta})\to 0$ when $k\to\infty$ and therefore $F(\alpha,\beta)=\lim_{k\to\infty}T_k(f_{\alpha,\beta})$ for α, $\beta>1$.

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