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On Relation Of The Riemann Zeta Function To Its Partial Product Definitions*

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Abstract

New relations for the Riemann zeta function (RZF) by defining supplementary partial product functions are developed in this paper. Relations are based on partial products of prime numbers with recourse to product form of the RZF found by unique factorization in Z. This paper is in pursuit of generating new identities—involving RZF including summations, products, and limits—mostly in the matter of multiplicative property of Euler products and by applying Taylor series. This is done with the intention of relating some classical and newly defined functions (using multiplicative Jordan's totient function and primorial sequence) to RZF in the form of theorems and proofs.

1 Introduction

The famous Riemann zeta function $\zeta(s)$ (RZF) has a variety of applications in mathematics, in particular, in field of number theory which makes it an important special function. In addition to its direct applications, there exist several open problems in the field solvable in the event of a correct proof of the well-known Riemann hypothesis concerning real part of non-trivial zeros located in vertical line $\Re(s) = \frac{1}{2}$. The Riemann hypothesis proof as a key towards answers to many problems can be found by equivalences which transform difficulty level of the problem into a new level [7].

The paper is organized as 2 main sections: 1- A brief introductory section (2) for RZF 2- Section 3 for developing new formulas, relations, and theorems with 2 subsections concerning RZF as Taylor series coefficients in subsection 3.1, and other special sums, products, and limits for RZF in subsection 3.2. Section 2 provides fundamental, classical definitions involving RZF and other related number theory functions for the purpose of an overview.

The development of new relations for RZF is done by defining specific auxiliary functions, named and introduced as lemmas, in order to acquire new series and limit-based formulas for this function. However, other identities for RZF are developed by consideration of partial product of a special case of Euler products. The method is also useful when a relation between primorial function and RZF is needed.

2 Riemann zeta function

The RZF definition and its relation to L-functions with its product expansion are reviewed briefly in this part.

Definition 1 (Riemann zeta function [1]) *The RZF is defined by*

$$\zeta(s) \triangleq \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{1}$$

for $\Re(s) > 1$.

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Now, with respect to L-functions [1], introduced by Dirichlet (1837), which is defined by

$$L(s,\chi) \triangleq \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}, \quad \Re(s) > 1, \tag{2}$$

relation (1) is written as

$$\zeta(s) = L(s,1). \tag{3}$$

Product expansion of (1) [1] is defined as

$$\zeta(s) = \prod_{p} \frac{1}{1-p^{-s}}, \ \Re(s) > 1, \tag{4}$$

where product is taken over all primes. Identity (4) is a special case of product of the L-functions where χ is considered as a multiplicative function [16]. Equation (4) is product over all prime numbers, therefore it gives

$$-\frac{\zeta^{'}(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \Lambda(n) n^{-s}, \ \Re(s) > 1 \tag{5}$$

calculated by taking logarithmic derivative. In (5), $\Lambda(n)$ is the Mangoldt's function [1] which equals zero for $n \neq p^r$, and it equals to

$$\Lambda(n) \triangleq \log(p)$$

for $n = p^r$ where $r \in \mathbb{N}$.

3 Riemann Zeta Function in Taylor Series and Some Special Series

With an eye to other formulas involving RZF, and in contrast to previous section showing some classic relations for RZF, in this part, some identities involving RZF including sums and products are presented. Majority of these formulas are related to RZF as coefficients in power series of different functions.

3.1 Taylor Series Involving RZF and Other Special Functions

The main intention is to develop power series with ζ function separable in coefficients. Assume that left-hand side of following relation admits Taylor series expansion in neighborhood of x_0 in the form of

$$\sum_{n\geq 1} \frac{f(x/n^p)}{n^\alpha} = \sum_{n\geq 0} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n \zeta(\alpha+np), \tag{6}$$

which is found by applying Taylor series to left-side of equation and is valid for $f^{(m)}(x_0) = f^{(m)}(x_0/2^p) = ...$ where $m = 0, 1, \ldots$. In other words, these parts of coefficients should be factored for the zeta function to appear. Variable p shifts argument in right side, and condition for which $\zeta(1)$ not appearing in right side is $\alpha \nmid 1-np$ for α and p as integers. This shifting of variable is applicable on argument of zeta function in series presented in this paper, hence it is not re-mentioned in next examples.

If $x = 1$ with coefficients $g(n) = f(1/n)$, left hand side of (6) is a *Dirichlet*-series shown as

$$G(s) = \sum_{n=1}^{\infty} g(n) n^{-s},$$

where $\zeta(s) = G(s)$ for $g(n) = 1$. As an example, $f(x) = \sin(x - x_0)$ is standard form of f in (6) to be valid. Equation (6) is reformulated as

$$\sum_{n\geq 1} \frac{f((x-x_0)/n^p)}{n^\alpha} = \sum_{n\geq 0} \frac{f^{(n)}(0)}{n!}(x-x_0)^n \zeta(\alpha+np)$$

to take functions like $f(x) = \sin(x)$ as well provided that zeta function is separable in coefficients as in (6). However, in this paper form of equation (6) is selected for generated Taylor series centered at $x_0 = 0$. If argument of f is inverted and x set to 1, (6) forms a generalized Euler sum which is itself a special case of *Dirichlet*-series [2], [3], represented as

$$H(s) = \sum_{n=1}^{\infty} \frac{h(n)}{n^s}, \quad \Re(s) > 1,$$

for various functions in place of $h(n)$ which is defined in [15]. However, by inverting argument of f, right-side of (6) forms a *Laurent* series. If $h(n)$ is a *Dirichlet* character $(mod\ k)$, $H(s) = L(s, \chi)$ becomes meromorphic continuation of it [8], [14].

Example 1 *For $f(x) = \sin(x)$ near $x_0 = 0$, (6) yields*

$$\sum_{n\geq 1} \frac{\sin(x/n^p)}{n^\alpha} = \sum_{n\geq 0} \frac{(-1)^n x^{2n+1}}{(2n+1)!}\zeta(\alpha + (2n+1)p).$$

Next, Lemma 1 is introduced:

Lemma 1 *The defined Upsilon-function*

$$\Upsilon_p(\alpha) \triangleq \sum_{n\geq 0} \frac{n^p}{\alpha^n n!}, \ \alpha \neq 0, \tag{7}$$

where for $p = 0$ it gives

$$\Upsilon_0(\alpha) = e^{\frac{1}{\alpha}},$$

can be calculated for $p = 1, 2, ...$ by recursive relation

$$\Upsilon_p(\alpha) = -\alpha \Upsilon'_{p-1}(\alpha).$$

Proof. This result is provable by induction. ∎

In view of $\Upsilon_p(\alpha)$, assume following series expansion

$$\sum_{n\geq 0} \frac{f(nx)}{\alpha^n n!} = \sum_{n\geq 0} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n \Upsilon_n(\alpha), \ \alpha \neq 0, \tag{8}$$

which is valid if $f^{(m)}(x_0) = f^{(m)}(2x_0) = ...$ for $m = 0, 1, \ldots$. By applying (6)-(8), following double sum involving ζ function is found as

$$\sum_{m\geq 1}\sum_{n\geq 0} \frac{f(nx/m^p)}{m^\beta \alpha^n n!} = \sum_{n\geq 0} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n \Upsilon_n(\alpha)\zeta(\beta + np), \ \alpha \neq 0. \tag{9}$$

Next, some examples with Lemma 2 are presented.

Example 2 *Sums*

$$\sum_{n\geq 0} \frac{\sin(nx)}{\alpha^n} = \frac{\alpha \sin(x)}{\alpha^2 + 1 - 2\alpha \cos(x)}, \ |\alpha| > 1 \tag{10}$$

and

$$\sum_{n\geq 0} \frac{\cos(nx)}{\alpha^n} = \frac{\alpha^2 - \alpha \cos(x)}{\alpha^2 + 1 - 2\alpha \cos(x)}, \ |\alpha| > 1 \tag{11}$$

have closed form expressions evaluated using Euler identities for $\sin(x)$ *and* $\cos(x)$ *functions. Integration of* (10) *is calculable explicitly after removing first zero term from left-side and dividing*[1] *right-side nominator and denominator integrand by* α^2 *as*

$$-\sum_{n\geq 1}\frac{\cos(nx)}{n\alpha^n}=\int\frac{\alpha^{-1}\sin(x)}{1+\alpha^{-2}-2\alpha^{-1}\cos(x)}\mathrm{d}x=\frac{1}{2}\ln(1+\frac{1}{\alpha^2}-\frac{2}{\alpha}\cos(x)). \tag{12}$$

Integration of (12) *gives*

$$\sum_{n\geq 1}\frac{\sin(nx)}{n^2\alpha^n}=-\int_0^x\frac{1}{2}\ln(1+\frac{1}{\alpha^2}-\frac{2}{\alpha}\cos(\theta))\mathrm{d}\theta\Big|_{\alpha=1}=-\int_0^x\ln(2\sin(\frac{x}{2}))\mathrm{d}x \tag{13}$$

which is a generalized form for the Clausens's integral [10] originally defined for $\alpha=1$*. Another relation is cosine series [10] of the form*

$$\sum_{n\geq 1}\frac{\cos(nx)}{n^2}=\frac{x^2}{4}-\frac{\pi x}{2}+\frac{\pi^2}{6},$$

which is equal to left-hand side of (11) *for* $\alpha=1$ *after integrating twice [10].*

For integration of (11), Lemma 2 is introduced:

Lemma 2 *The defined sigma-function*

$$\varsigma_p(\alpha)\triangleq\sum_{n\geq 0}\frac{n^p}{\alpha^n},\qquad |\alpha|>1, \tag{14}$$

where for $p=0$ *it equals*

$$\varsigma_0(\alpha)=\frac{\alpha}{\alpha-1},$$

is calculated for $p=1,2,...$ *by recursive relation*

$$\varsigma_p(\alpha)=-\alpha\varsigma_{p-1}^{'}(\alpha).$$

Proof. This result is provable by induction. ∎

Sigma-function is related to the *Polylogarithm*-function by following relation:

$$\varsigma_m(\alpha)=(-1)^{m+1}Li_{-m}(\alpha),\ m=0,1,2,.... \tag{15}$$

A formula similar to (8) using Lemma 2 is expansion

$$\sum_{n\geq 0}\frac{f(nx)}{\alpha^n}=\sum_{n\geq 0}\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n\varsigma_n(\alpha). \tag{16}$$

The generating function for $\varsigma_n(\alpha)$ using (16) where $f(x)=e^x$ is found as

$$\frac{1}{1-\alpha^{-1}e^x}=\sum_{n\geq 0}\frac{x^n}{n!}\varsigma_n(\alpha),\ x<\ln|\alpha|.$$

For $\alpha=-1$, (16) diverges according to (14), therefore, using relation $Li_m(-1)=\zeta(m)(2^{1-m}-1)$ extracted based on Hardy's series for RZF [11],

$$\zeta(s)=\frac{1}{1-2^{1-s}}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^s},\ \ \Re(s)>0,$$

[1] The integrand is reformed according to desired domain in order to result in correct integration as for right-side of (10), $|\alpha|<1$ is part of domain where left-side diverges, thus, the integrand was changed to fit left-side domain as well.

where $m \neq 1$ with (15), an alternative series results in

$$\sum_{n\geq 0}(-1)^n f(nx) = \sum_{n\geq 0}\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n(-1)^{n+1}Li_{-n}(-1). \tag{17}$$

Moreover, (17) is reducible to

$$\sum_{n\geq 0}(-1)^n f(nx) = \frac{1}{2}f(x_0) + \sum_{n\geq 0}\frac{f^{(2n+1)}(x_0)}{(2n+1)!}(x-x_0)^{2n+1}\zeta(-2n-1)(4^{n+1}-1), \tag{18}$$

where even powers—in general form of series—are removed by $\zeta(-2n) = 0$ and first term equals $-\zeta(0)f(x_0)$. To rewrite series in terms of the *Bernoulli*-numbers [1], $\zeta(-n)$ should be replaced by $-\frac{B_{n+1}}{n+1}$. Many divergent series are evaluated using various methods including *Ramanujan's* summation. In fact, *Ramanujan's* summation equals (18) with little variations for $x_0 = 0$ and $x = 1$ in terms of *Bernoulli*-numbers [6].

Example 3 *By* (16) *where* $f(x) = \cos(x)$, *series expansion of (real part) integration of right-hand side of* (11) *is evaluated by separating first term as*

$$x + \sum_{n\geq 1}\frac{\sin(nx)}{n\alpha^n} = \int \frac{1-\alpha^{-1}\cos(x)}{1+\alpha^{-2}-2\alpha^{-1}\cos(x)}\mathrm{d}x = \sum_{n\geq 0}\frac{(-1)^n}{(2n+1)!}x^{2n+1}\varsigma_{2n}(\alpha), \tag{19}$$

for $|\alpha| \geq 1$. *The power series for* (10) *and* (11) *are equivalent to*

$$\frac{\alpha\sin(x)}{\alpha^2+1-2\alpha\cos(x)} = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}\varsigma_{2n+1}(\alpha)$$

and

$$\frac{\alpha^2-\alpha\cos(x)}{\alpha^2+1-2\alpha\cos(x)} = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}\varsigma_{2n}(\alpha)$$

for $|\alpha| > 1$.

A new double sum formula similar to (9) is found by Lemma 2 as

$$\sum_{m\geq 1}\sum_{n\geq 0}\frac{f(nx/m^p)}{m^\beta\alpha^n} = \sum_{n\geq 0}\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n\varsigma_n(\alpha)\zeta(\beta+np),\ \alpha > 1. \tag{20}$$

Next examples illustrate use of formula (20).

Example 4 *Letting* $f(x) = \sin(x)$ *and* $f(x) = \cos(x)$ *with* $p = 1$ *in* (20) *using previous formulas gives*

$$\sum_{m\geq 1}\frac{\alpha\sin(x/m)}{m^\beta(\alpha^2+1-2\alpha\cos(x/m))} = \sum_{n\geq 0}\frac{(-1)^n}{(2n+1)!}x^{2n+1}\varsigma_{2n+1}(\alpha)\zeta(2n+\beta+1), \quad \alpha > 1$$

and

$$\sum_{m\geq 1}\frac{\alpha^2-\alpha\cos(x/m)}{m^\beta(\alpha^2+1-2\alpha\cos(x/m))} = \sum_{n\geq 0}\frac{(-1)^n}{(2n)!}x^{2n}\varsigma_{2n}(\alpha)\zeta(2n+\beta), \quad \alpha > 1.$$

To develop more formulas in following, Lemma 3 is introduced:

Lemma 3 *The defined tau-function*

$$\tau_p(\alpha) \triangleq \sum_{n\geq 1} \frac{1}{n^p \alpha^n n!}, \ \alpha \neq 0 \tag{21}$$

where for $p = 0$ *it equals*

$$\tau_0(\alpha) = e^{\frac{1}{\alpha}} - 1,$$

is calculated for $p = -1, 0, \ldots$ *by recursive relation*

$$\tau_p(\alpha) = -\int \frac{\tau_{p-1}(\alpha)}{\alpha} \mathrm{d}\alpha.$$

Proof. This result can be verified by induction. ∎

By using Lemma 3, series for left-side of equation (22) is assumed as its right-side:

$$\sum_{n\geq 1} \frac{f(nx)}{n^p \alpha^n n!} = \sum_{n\geq 0} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n \tau_{p-n}(\alpha). \tag{22}$$

For ζ to appear, double summation method applied on (22) as before results in

$$\sum_{m\geq 1} \sum_{n\geq 1} \frac{f(nx/m^{p_1})}{m^\beta n^p \alpha^n n!} = \sum_{n\geq 0} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n \tau_{p-n}(\alpha) \zeta(\beta + np_1), \ \beta > 1. \tag{23}$$

Tau function satisfies $\tau_{-n}(1) = eB(n)$ for $n > 1$, where $B(n)$ is the *Bell*-number [4], [5] appearing in series [9] of the form

$$\sum_{n\geq 0} \frac{f(nx)}{n!} = e \sum_{n\geq 0} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n B(n). \tag{24}$$

Next example illustrates a property of these numbers.

Example 5 *Assume that a set of* $m + 1$ *recurrence functions defined as*

$$f_0(x) = \sum_{n\geq 0} \frac{a_n}{n!} (x - x_0)^n,$$

$$f_m(x) = \sum_{n\geq 0} \frac{f_{m-1}(nx)}{n!},$$

exists. Then, using Bell-numbers, the Taylor series expansion of $f_m(x)$ *near* x_0 *for* $m \in \mathbb{R}$ *is equal to*

$$f_m(x) = e^m \sum_{n\geq 0} \frac{a_n}{n!} (x - x_0)^n B^m(n) \tag{25}$$

for

$$x < x_0 + \lim_{n\to\infty} \left(\frac{n!}{a_n B^m(n)}\right)^{1/n}.$$

3.2 Special Products and Sums Representing RZF

In this section, Theorem 1 with the intention of relating partial zeta function and partial primes' product function (Definition 2) to RZF is presented. However, a simple average sum based on limit operator is also introduced in Theorem 6.

A limit-based relation for ζ_n is

$$\zeta(s) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \zeta_n(s), \tag{26}$$

which is proved by a relation presented in Lemma 4.

Lemma 4 *Let S be a Cesàro mean [12] (sum or product) and S_n its partial value. Then*

$$S = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} S_n,$$

where

$$S_n = \sum_{m=1}^{n} a_n = \prod_{m=1}^{n} b_n,$$

under condition

$$S = \lim_{n\to\infty} S_n.$$

Proof. The sum in statement of Lemma 4 is written as

$$S = \lim_{N\to\infty} \Big\{ \underbrace{\frac{S_1}{N}}_{0} + \underbrace{\frac{S_2}{N}}_{0} + \cdots + \underbrace{\frac{S_n}{N} + \frac{S_{n+1}}{N} + \cdots + \frac{S_\infty}{N}}_{average\ of\ infinite\ terms} \Big\} = S_\infty,$$

which ends proof under condition of lemma with assumption of

$$\lim_{n\to\infty} \frac{s_n}{n} = 0.$$

■

Definition 2 *Define partial primes' product function and partial zeta function as*

$$P_n(s) = \left\{ \prod_{i=1}^{n} p_i \right\}^{-s} = \{p_n\#\}^{-s} \tag{27}$$

and

$$\zeta_n(s) = \prod_{i=1}^{n} \frac{1}{1 - p_i^{-s}}, \tag{28}$$

where $p_n\#$ is the primorial function.

Next, using these functions, Theorem 5 is introduced.

Theorem 5 *RZF can be determined in terms of partial product functions P_n and ζ_n as*

$$\zeta(s) = \lim_{n\to\infty} \frac{1}{P_{2\lfloor n\rfloor+1}(s)} \left\{ \frac{2^{-s}}{1-2^{-s}} - \sum_{k=1}^{2\lfloor n\rfloor} (-1)^{k+1} P_k(s)\zeta_{k+1}(s) \right\},\ \Re(s) > 1 \tag{29}$$

and

$$\zeta(s) = \frac{2^s}{2^s - 1} + \sum_{n=2}^{\infty} \frac{P_n(s)\zeta_n(s)}{P_{n-1}(s)},\ \Re(s) > 1. \tag{30}$$

Proof. By assuming p_i as sequence of prime numbers, left-side of equality

$$\frac{p_1^{-s}}{1-p_1^{-s}} = \frac{p_1^{-s}}{(1-p_1^{-s})(1-p_2^{-s})} - \frac{(p_1p_2)^{-s}}{(1-p_1^{-s})(1-p_2^{-s})}$$

is expanded as its right-side by multiplying numerator and denominator by $1-{p_2}^{-s}$. If this is done for n prime numbers, it yields

$$\frac{p_1^{-s}}{1-p_1^{-s}} = \frac{p_1^{-s}}{(1-p_1^{-s})(1-p_2^{-s})} - \frac{(p_1p_2)^{-s}}{(1-p_1^{-s})\dots(1-p_3^{-s})} + \dots + \frac{(-1)^n(p_1\dots p_{n-1})^{-s}}{(1-p_1^{-s})\dots(1-p_n^{-s})} + \frac{(-1)^{n+1}(p_1\dots p_n)^{-s}}{(1-p_1^{-s})\dots(1-p_n^{-s})}.$$

This can be rewritten on the basis of partial product functions P_n and ζ_n as

$$\frac{p_1^{-s}}{1-p_1^{-s}} = P_1(s)\zeta_2(s) - P_2(s)\zeta_3(s) + \dots + (-1)^n P_{n-1}(s)\zeta_n(s) + (-1)^{n+1}P_n(s)\zeta_n(s).$$

In this step, n is replaced by $2n+1$ in order to eliminate last term sign, then equation is solved for $\zeta_{2n+1}(s)$, therefore it gives

$$\zeta_{2n+1}(s) = \frac{1}{P_{2n+1}(s)}\Big\{\frac{p_1^{-s}}{1-p_1^{-s}} - \sum_{k=1}^{2n}(-1)^{k+1}P_k(s)\zeta_{k+1}(s)\Big\}$$

which equals Theorem 5 statement by setting $p_1 = 2$ and limiting equality as n approaches infinity.

To prove second relation, same technique by interchanging first and second terms is applied which results in

$$\frac{p_1^{-s}}{1-p_1^{-s}} = \frac{-p_1^{-s}p_2^{-s}}{(1-p_1^{-s})(1-p_2^{-s})} + \frac{{p_1}^{-s}}{(1-p_1^{-s})(1-p_2^{-s})}.$$

This is done for n prime numbers, therefore it gives

$$\frac{1}{1-p_1^{-s}} + \dots + \frac{{p_n}^{-s}}{(1-p_1^{-s})\dots(1-p_n^{-s})} = \frac{1}{(1-p_1^{-s})\dots(1-p_n^{-s})},$$

which can be reformed as (30) by $p_k = P_k(s)/P_{k-1}(s)$.

A familiar relation [13] for $s = 2, 3, \dots$ is

$$\zeta(s) = \frac{2^s}{2^s-1} + \sum_{n=2}^{\infty}\frac{(p_{n-1}\#)^s}{J_s(p_n\#)}, s = 2, 3, \dots$$

in terms of primorial sequence $p_n\#$ with multiplicative Jordan's totient function defined as

$$p_n\# \triangleq \prod_{k=1}^{n} p_k$$

and

$$J_k(n) \triangleq n^k \prod_{p|n}(1-\frac{1}{p^k}).$$

However, equation (29) is also valid for non-integer values of s in the domain of convergence. ∎

Theorem 6 *The average identity*

$$\zeta(s) = \frac{1}{q}\Big\{\sum_{i=1}^{q}\Big\{\frac{1}{1-p_i^{-s}} + \sum_{n=0}^{\infty}\sum_{m=p_i^n+1}^{p_i^{n+1}-1}\frac{1}{m^s}\Big\}\Big\} \tag{31}$$

represents RZF, where sequence p_i denotes prime numbers.

Proof. The set

$$\bigcup_{i=0}^{n}\Big\{m^i+1\ ,m^i+2,\dots,m^{i+1}-1\Big\},$$

where $1<m\in\mathbb{N}$, is equal to $\{1,\ 2,\dots,m^{n+1}-1\}$ excluded by $\{1,m^1,m^2\dots,m^{n+1}\ \}$, so the sum

$$\sum_{n=0}^{\infty}\sum_{m=p_i^n+1}^{p_i^{n+1}-1}\frac{1}{m^s}=\sum_{m\in\mathbb{N}-\{1,\ p_i,\ p_i^2,\ \dots\}}\frac{1}{m^s}$$

in (31) equals

$$\zeta(s)-\big(1+\frac{1}{p_i^s}+\frac{1}{p_i^{2s}}+\dots\big)=\zeta(s)-\frac{1}{1-p_i^{-s}},$$

and it finally follows that

$$\zeta(s)=\frac{1}{q}\Big\{\sum_{i=1}^{q}\Big\{\frac{1}{1-p_i^{-s}}+\zeta(s)-\frac{1}{1-p_i^{-s}}\Big\}\Big\}.$$

The last relation is theorem's statement.

Equation (31) is valid for other unique sequences satisfying $1<\alpha_i\in\mathbb{N}$ in place of p_i, but using prime numbers p_i with theorem 2 and relation (31), a functional equation for Euler product is found as

$$\prod_{i=1}^{\infty}\frac{1}{1-p_i^{-s}}=\prod_{i=1}^{\infty}\Big\{\zeta(s)-\sum_{n=0}^{\infty}\sum_{m=p_i^n+1}^{p_i^{n+1}-1}\frac{1}{m^s}\Big\},\tag{32}$$

where terms in left-hand side correspond to terms in right-hand side respectively. ■

4 Summary

Basic definitions related to the RZF were reviewed briefly at beginning part of this paper. Then, several lemmas with their proofs and relations to the RZF were presented in order to develop some power series (with RZF in their coefficients) and some limit-based identities. Finally, a theorem providing a familiar identity for RZF, and another one for Euler product were introduced.

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