Applied Mathematics E-Notes, 20(2020), 367-387 © ISSN 1607-2510
Available free at mirror sites of http://www.math.nthu.edu.tw/∼amen/

Global Existence And Blow-Up Of Generalized Self-Similar Solutions To Nonlinear Degenerate Diffusion Equation Not In Divergence Form*

Bilal Basti†, Nouredine Benhamidouche‡

Received 11 August 2019

Abstract

This paper investigates the problem of existence and uniqueness of positive solutions under the general self-similar form of the degenerate parabolic partial differential equation which is known as "nonlinear diffusion equation not in divergence form". By applying the properties of Banach's fixed point theorems, we establish several results on the existence and uniqueness of the general form of self-similar solutions of this equation.

1 Introduction

Many problems and models in physics, chemistry, biology and economics are modeled by partial differential equations. In this work, we shall give an example of a class of renowned equations, which allow to describe the diffusion phenomena; that is, a parabolic PDE and known as nonlinear diffusion equation not in divergence form and is written as:

$$\frac{\partial u}{\partial t}=u^{m}\frac{\partial^{2}u}{\partial x^{2}},\ m\in(0,1)\cup(1,\infty)\,, \tag{1}$$

where $u=u(x,t)$ is a nonnegative scalar function of space variables $x\in\mathbb{R}$ and time $t>0$.

The nonlinear diffusion equation not in divergence form (1), is often studied by researchers (see for example [1, 6, 15–17]), who gave some results of existence and uniqueness of the global solutions in time and solutions which blow-up in a finite time, this is applied for a certain class of the function u which satisfied some sufficient conditions.

In general, for some PDEs which has the characterization of symmetries, (see for example [11, 12, 13]), we can determine their exact solutions with certain (finite or infinite) transformations. Here, a PDE becomes an ordinary differential equation, in this case the solutions are called self-similar solutions ([3, 7, 9, 16]), which often play a central role in the study of a PDE, since it is equivalent to these solutions to solve locally or globally.

C. Wang et al. [16], tackled in detail the existence and the uniqueness of a shrinking self-similar solution to the equation (1) for $m\geq 1$. The proposed solution was:

$$u(x,t)=\frac{1}{(t+1)^{\beta}}\omega\left((t+1)^{\alpha}|x|^{2}\right),\ t>0\text{ and }x\in\mathbb{R}^{n},$$

where ω is a positive function that satisfies some properties and

$$\alpha\geq 0,\ \beta=\frac{1+\alpha}{m},$$

*Mathematics Subject Classifications: 34A05, 35B06, 35B44, 35C06.

†Laboratory for Pure and Applied Mathematics, University of Mohamed Boudiaf M'sila, 28000, Algeria. Prof in Department of Mathematics, University of Ziane Achour Djelfa, 17000, Algeria

‡Laboratory for Pure and Applied Mathematics, University of Mohamed Boudiaf M'sila, 28000, Algeria

are constants chosen so that the solutions exist.

The equation (1) is a representation of a large class of the nonlinear parabolic equations. Obviously, for $m = 0$ we recognize the well-known heat equation, which in fact has quite different properties from the two nonlinear ranges, $m > 1$ (usually called not in divergence form) and $m \in (0, 1)$. For the case $m \in (0, 1)$, if we consider the implicit change of variables:

$$v = p^{\frac{1}{p-1}} u^{\frac{1}{p}}, \text{ with } p = \frac{1}{1-m}, \; m \in (0, 1),$$

the equation (1) can be written in divergence form of the porous medium equation (see [2, 5, 10, 14])

$$\frac{\partial v}{\partial t} = \frac{\partial^2}{\partial x^2} (v^p), \; p > 1. \tag{2}$$

The porous medium equation (2), is an equation, which admits the properties of similarity. There are several known fundamental families of self-similar solutions, maybe the most important one is formed by the Barenblatt solutions $\mathcal{B}_{(\cdot)}$, discovered independently by Barenblatt in [2] and by Zeldovich and Kompaneets in [18], which are written under the following form:

$$\mathcal{B}_C(x, t) = \begin{cases} t^{-\frac{1}{p+1}} \left(\frac{p-1}{2p(p+1)} \left(C^2 - x^2 t^{-\frac{2}{p+1}} \right) \right)^{\frac{1}{p-1}}, & \text{for } |x| < Ct^{\frac{1}{p+1}}, \; C > 0, \\ 0, & \text{otherwise.} \end{cases} \tag{3}$$

Our objective in this work is to study the existence and uniqueness of positive solutions of the degenerate parabolic PDE (1), under the generalized self-similar form which is:

$$u(x, t) = c(t) f\left(\frac{x}{a(t)}\right), \; x \in \mathbb{R}, \; t > 0.$$

The functions $a(t)$ and $c(t)$ depend on time t and the basic profile $f > 0$, are not known in advance and are to be identified.

2 Definitions and Preliminary Results

To discuss the generalized self-similar solutions:

$$u(x, t) = c(t) f(\eta), \text{ with } \eta = \frac{x}{a(t)}, \; x \in \mathbb{R}, \; t > 0. \tag{4}$$

We should first deduce the equation satisfied by the function $f(\eta)$ in (4) used for the definition of self-similar solutions.

2.1 Basic Idea to Compute the Self-Similar Solutions

The functional-differential equation resulting from the substitution of expression (4) in the original PDE (1), should be reduced to the standard bilinear functional equation (see [13]); in this case we obtain the following equation:

$$\dot{c}(t) f(\eta) - c(t) \frac{\dot{a}(t)}{a^2(t)} x f'_\eta(\eta) = \frac{c^{m+1}(t)}{a^2(t)} f^m(\eta) f''_{\eta\eta}(\eta), \text{ with } a, c \in \mathbb{R}_+. \tag{5}$$

By expressing x from (4) in terms of η, substituting into (5) and divide by $c(t)$, we get the functional equation in two variables t and η, as follows:

$$\frac{\dot{c}(t)}{c(t)} f(\eta) - \frac{\dot{a}(t)}{a(t)} \eta f'_\eta(\eta) - \frac{c^m(t)}{a^2(t)} f^m(\eta) f''_{\eta\eta}(\eta) = 0.$$

This can be rewritten as the standard bilinear functional equation:

$$\varphi_1\psi_1+\varphi_2\psi_2+\varphi_3\psi_3=0,$$

with a direct calculation shows that $a=a(t)$, $c=c(t)$ and $f=f(\eta)$, we have:

$$\varphi_1=\frac{\dot{c}}{c},\ \varphi_2=-\frac{\dot{a}}{a},\ \varphi_3=-\frac{c^m}{a^2}$$

and

$$\psi_1=f,\ \psi_2=\eta f'_\eta,\ \psi_3=f^m f''_{\eta\eta}.$$

Substituting these expressions into the solution of the three-term functional equation [13], yields the determining system of ordinary differential equations (see also [3, 4])

$$\begin{cases} \frac{\dot{c}}{c}=k_1\frac{c^m}{a^2}, \\ \frac{\dot{a}}{a}=k_2\frac{c^m}{a^2}, \\ f^m f''_{\eta\eta}=\alpha f+\beta\eta f'_\eta. \end{cases} \tag{6}$$

Where α,β,k_1 and k_2 are arbitrary constants.

The system of ordinary differential equations (6) depends on many unknown parameters. The functions $a(t)$, $c(t)$ and $f(\eta)$ are explicitly determined.

2.2 Statement of the Problem

In this part, we first attempt to find the equivalent approximate to the following problem of the nonlinear diffusion equation not in divergence form:

$$\begin{cases} \frac{\partial u}{\partial t}=u^m\frac{\partial^2 u}{\partial x^2}, & x\geq 0,\ m\in(0,1)\cup(1,\infty), \\ u(0,t)=c(t)U, & U\geq 0,\ t>0, \\ \frac{\partial u(\lambda a(t),t)}{\partial x}=\frac{\lambda\beta c^m(t)}{(1-m)a(t)}\lim\limits_{x\to\lambda a(t)}[u(x,t)]^{1-m}, & \lambda>0, \\ \int\limits_x^{\lambda a(t)}\left(\frac{u(x,t)}{u(s,t)}\right)^{m-1}ds\leq\frac{\lambda}{2}a(t), & \forall m>1,\ \forall x\in[0,\lambda a(t)). \end{cases} \tag{7}$$

Under the generalized self-similar form which is:

$$u(x,t)=c(t)f(\eta),\ \text{with } \eta=\frac{x}{a(t)} \text{ and } a,c\in\mathbb{R}_+. \tag{8}$$

According to the preceding part (the system (6)), we consider this problem:

$$\begin{cases} f^m(\eta)f''(\eta)=\alpha f(\eta)+\beta\eta f'(\eta), & \eta\geq 0,\ m\in(0,1)\cup(1,\infty), \\ f(0)=U, & U\geq 0, \\ f'(\lambda)=\frac{\lambda\beta}{1-m}\lim\limits_{\eta\to\lambda^-}f^{1-m}(\eta), & \lambda>0, \\ \int\limits_\eta^\lambda f^{1-m}(\xi)d\xi\leq\frac{\lambda}{2}f^{1-m}(\eta), & \forall m>1,\ \forall\eta\in[0,\lambda), \end{cases} \tag{9}$$

in which α,β are arbitrary real constants.

3 Existence and Uniqueness of Basic Profile

In this section we study the existence and uniqueness of a class of positive solutions under the generalized self-similar form (8) for the problem of the nonlinear diffusion equation not in divergence form (7).

We find also exact solutions under certain conditions, which depend on the similarity coefficients α and β.

As in [7] it is necessary to consider the weak solutions of the problem (9).

Definition 1 (Weak solution) *A function f will be called a weak solution with compact support of the problem (9) if and only if*

1) *f is a continuous function, bounded and nonnegative on $[0,\infty)$.*

2) *f has a continuous derivative in a left neighborhood of $\eta=\lambda$.*

3) *f satisfies the identity:*

$$\int_0^\infty \left(f'(\xi)+\frac{\beta}{m-1}\xi f^{1-m}(\xi)\right)\varphi'(\xi)\,d\xi+\left(\alpha+\frac{\beta}{m-1}\right)\int_0^\infty f^{1-m}(\xi)\,\varphi(\xi)\,d\xi=0,$$

for all $\varphi\in C_0^1(0,\infty)$.

We discuss the existence of basic profile f of the weak solutions with compact support, then as we shall see later, f is positive in a right neighborhood of $\eta=0$; more specifically for some $\varepsilon>0$, there exists a number $\varepsilon<\lambda<\infty$ such that:

$$f>0, \text{ on } (0,\lambda)$$

and

$$f\equiv 0, \text{ on } [\lambda,\infty).$$

Assume that f is a weak solution of the problem (9) with compact support and let λ be an arbitrary positive number, we shall be mainly concerned with proving the existence and uniqueness of a positive solution of problem (9) on an interval $(0,\lambda)$ which satisfies the boundary conditions:

$$f(0)=U,\ f(\lambda)=0. \tag{10}$$

Then we shall show that for suitable α and β there exists a unique positive solution of the equation:

$$f^m(\eta)\,f''(\eta)=\alpha f(\eta)+\beta\eta f'(\eta), \tag{11}$$

in the left neighborhood of $\eta=\lambda$ and that this solution can be continued back to $\eta=0$. We then ask whether λ can be chosen so that condition (10) is satisfied.

Before dealing with the question of existence, we give the necessary conditions of the parameters α and β for the existence of a nontrivial weak solution with compact support of the problem (9).

Lemma 1 *The function f is a nontrivial weak solution with compact support of the boundary value problem (9)–(10), if and only if*

(i) *β is a negative coefficient for $m\in(0,1)$.*

(ii) *β is a positive coefficient for $m>1$.*

(iii) *$\beta=0$ and α takes strictly positive value.*

Proof. Suppose f is a nontrivial weak solution of problem (9) with compact support. Then for some $\varepsilon > 0$, we have:

$$f\begin{cases} > 0 & \text{in } (\lambda-\varepsilon,\lambda) \\ = 0 & \text{in } [\lambda,\infty) \end{cases}, \quad \lambda > 0.$$

It results that f is a weak solution of (9) which satisfies (10) in the left neighborhood of $\eta = \lambda$. The integration of (11) starting from η to λ, where $\lambda - \varepsilon < \eta < \lambda$ gives:

$$(1-m) f'(\eta) = \beta\eta f^{1-m}(\eta) + [\alpha(m-1)+\beta] \int_{\eta}^{\lambda} f^{1-m}(\xi)\, d\xi. \tag{12}$$

The continuity of f and f' left of λ, ensures the existence of $\eta^* \in (\lambda-\varepsilon,\lambda)$ such that $f'(\eta^*) < 0$.

In fact, if we apply the mean value theorem on the interval $(\lambda-\varepsilon,\lambda)$, we have:

$$\exists \eta^* \in (\lambda-\varepsilon,\lambda) \text{ such that } f'(\eta^*) = \frac{f(\lambda) - f(\lambda-\varepsilon)}{\lambda - (\lambda-\varepsilon)} = -\frac{f(\lambda-\varepsilon)}{\varepsilon} < 0.$$

This implies that the member in the left of (12) at the point $\eta = \eta^*$ is negative for $m \in (0,1)$ and positive for $m > 1$. Hence, β and $\alpha(m-1)+\beta$ cannot both be greater than zero for $m \in (0,1)$, (resp. less than zero for $m > 1$) and $\beta = 0$ implies that $\alpha > 0$. Hence the estimate (iii).

In what follows, because the function f is positive and tends to zero if $\eta \to \lambda$, the mean value theorem enables us to prove that there exists $\lambda - \varepsilon \le \lambda_0 < \lambda$, such that $f'(\eta) < 0$, $\forall \eta \in (\lambda_0,\lambda)$. We divide the proof into two cases:

(i) For $m \in (0,1)$, let us consider $\beta > 0$, for all $\eta \in (\lambda_0,\lambda)$, this implies that $\alpha(m-1)+\beta < 0$. It results from (12) that:

$$(1-m) f'(\eta) f^{m-1}(\eta) - \beta\eta > [\alpha(m-1)+\beta](\lambda-\eta). \tag{13}$$

In fact, we will use:

$$\forall \eta \in [\lambda_0,\lambda), \ \forall \xi \in (\eta,\lambda), \text{ we have } f(\xi) < f(\eta),$$

which implies for any $m \in (0,1)$, that:

$$f^{1-m}(\xi) < f^{1-m}(\eta) \text{ and } f^{1-m}(\xi) f^{m-1}(\eta) < 1,$$

also

$$\int_{\eta}^{\lambda} f^{1-m}(\xi) f^{m-1}(\eta)\, d\xi < \lambda - \eta.$$

If $\eta \to \lambda$ in (13), the left part is negative and the right part tends towards zero, contradiction, which implies that $\beta < 0$ for $m \in (0,1)$.

(ii) For $m > 1$, we consider $\beta < 0$, for all $\eta \in (\lambda_0,\lambda)$, this implies that $\alpha(m-1)+\beta > 0$, consequently $\alpha > 0$. It results from (12) that:

$$(1-m) f'(\eta) - \beta\eta f^{1-m}(\eta) < \alpha(m-1) \int_{\eta}^{\lambda} f^{1-m}(\xi)\, d\xi. \tag{14}$$

If $\eta \to \lambda$ in (14), the left part is positive and the right part tends towards zero, contradiction.

Thus, we have already proved that the coefficients

$$\beta > 0 \text{ for } m > 1, \text{ or } \beta < 0 \text{ for } m \in (0,1), \text{ or } \beta = 0 \text{ and } \alpha > 0,$$

are the only cases for which the nontrivial weak solution with compact support of the boundary value problem (9)–(10) exists. ■

New explicit solutions for $\beta = 0$ **and** $m \in (0,1) \cup (1,2)$

Let λ be an arbitrary positive number. It is clear from the proof of Lemma 1 that a necessary condition for the existence of a positive solution of problem (9)–(10) in a left neighborhood of $\eta = \lambda$ is either $\beta > 0$ for $m > 1$, $\beta < 0$ for $m \in (0,1)$ or $\beta = 0$ and $\alpha > 0$.

The objective of this part is to show that this condition is also sufficient.

We begin by assuming that $\beta = 0$ and $\alpha > 0$, we can then solve problem (9)–(10) uniquely. After an elementary computation that the function:

$$f(\eta;\lambda) = \left(\frac{\alpha m^2}{2(2-m)}(\lambda-\eta)^2\right)^{\frac{1}{m}}, \ m \in (0,1) \cup (1,2), \ 0 \le \eta < \lambda, \tag{15}$$

is the unique solution of the problem (9)–(10).

In fact, starting from (11), we put $\beta = 0$ and $\alpha > 0$, we get

$$f^m(\eta) f''(\eta) = \alpha f(\eta),$$

also

$$f''(\eta) f'(\eta) = \alpha f^{1-m}(\eta) f'(\eta). \tag{16}$$

For $m \in (0,1) \cup (1,2)$, we have from (9) that $f'(\lambda) = 0$. Then, the integration of (16) starting from η to λ gives:

$$\frac{1}{2}[f'(\eta)]^2 = \frac{\alpha}{2-m} f^{2-m}(\eta).$$

Also, after a simple integration from η to λ, we have the function $f(\eta;\lambda)$ which is presented in (15).

Because $f(0;\lambda)$ is a continuous, monotonically increasing function of λ, such that

$$f(0;0) = 0 \text{ and } f(0;\infty) = \infty,$$

the equation $f(0;\lambda) = U$ is uniquely solvable for every $U > 0$.

Let $\lambda(U)$ be its solution, then $f = f(\eta;\lambda(U))$ is the unique solution of the problem (9)–(10), with compact support define for all $0 \le \eta < \lambda$ and satisfies:

$$f(0;\lambda) = U \text{ and } f(\lambda;\lambda) = 0,$$

where

$$\lambda(U) = \sqrt{\frac{2(2-m)}{\alpha m^2} U^m}.$$

In addition, f is a continuous and monotonous decreasing function.

Next, we discuss the case $\beta \ne 0$ and we give some elementary lemmas for the two following cases $\beta < 0$ for $m \in (0,1)$ and $\beta > 0$ for $m > 1$.

Lemma 2 *Suppose that* $0 \le \lambda_1 < \lambda$ *and* f *is a positive solution of the boundary value problem (9)–(10) on* $[\lambda_1, \lambda)$. *Then*

1. $f'(\eta) < 0$ *on* $[\lambda_1, \lambda)$, *if and only if* $\alpha > \frac{\beta}{1-m}$.

2. *If* $\alpha < \frac{\beta}{1-m}$ *and* $f'(\eta_0) = 0$, *for some* $\eta_0 \in [\lambda_1, \lambda)$. *Then* f *has a maximum at* η_0 *such that:*

$$\eta_0 \le \frac{\lambda[\alpha(m-1)+\beta]}{\alpha(m-1)}, \text{ for the case } m \in (0,1)$$

and

$$\eta_0 \ge \frac{\lambda[\alpha(m-1)+\beta]}{\alpha(m-1)}, \text{ for the case } m \in (1,\infty).$$

3. *Suppose that f is a positive solution of (9) on $[0,\lambda)$. Then:*

$$f'(0)<0,\ \textit{for}\ \alpha>\frac{\beta}{1-m}.$$

4. *If*

$$\alpha=\frac{\beta}{1-m},$$

we have:

$$f'(0)=0\ \textit{and}\ f'(\eta)<0,\ \forall\eta\in(0,\lambda),\ \textit{i.e.}\ f(\eta)\le f(0)=U,\ \forall\eta\in[0,\lambda]. \tag{17}$$

In this case, we find a new explicit solution of the problem (9)–(10) as follows:

$$f(\eta;\lambda)=\left(\frac{\beta m}{2(m-1)}\left(\lambda^2-\eta^2\right)\right)^{\frac{1}{m}},\ m\in(0,1)\cup(1,\infty),\ |\eta|<\lambda. \tag{18}$$

Proof. For $\eta\in[\lambda_1,\lambda)$, the integral equation (12) gives:

$$(1-m)f'(\eta)=\beta\eta f^{1-m}(\eta)+[\alpha(m-1)+\beta]\int_{\eta}^{\lambda}f^{1-m}(\xi)\,d\xi. \tag{19}$$

1. Because $\beta<0$ for $m\in(0,1)$, the right member in (19) is always negative i.e. $f'(\eta)<0$ if and only if $\alpha(m-1)+\beta$ takes a negative value. In the same way, we have $\beta>0$ for $m\in(1,\infty)$, then the left member in (19) is always positive i.e. $f'(\eta)<0$ if and only if $\alpha(m-1)+\beta$ takes a positive value, therefore

$$f'(\eta)<0\ \text{on}\ [\lambda_1,\lambda)\ \text{when}\ \alpha>\frac{\beta}{1-m}.$$

2. If $\alpha<\frac{\beta}{1-m}$ then $\alpha<0$ for any $m\in(0,1)\cup(1,\infty)$. According to (11);

$$f''(\eta_0)<0\ \text{when}\ f'(\eta_0)=0.$$

So f has a maximum at $\eta=\eta_0$ and strictly decreasing on (η_0,λ), i.e. $f'(\eta)<0$ on (η_0,λ). We put $\eta=\eta_0$ in (19), we get:

$$0=\beta\eta_0 f^{1-m}(\eta_0)+[\alpha(m-1)+\beta]\int_{\eta_0}^{\lambda}f^{1-m}(\xi)\,d\xi.$$

For $m\in(0,1)$, we have $\beta<0$ and $\alpha(m-1)+\beta>0$, then:

$$\beta\eta_0 f^{1-m}(\eta_0)+[\alpha(m-1)+\beta](\lambda-\eta_0)f^{1-m}(\eta_0)\ge 0.$$

So

$$\eta_0\le\frac{\lambda[\alpha(m-1)+\beta]}{\alpha(m-1)}.$$

In the same way, for $m\in(1,\infty)$, we have $\beta>0$ and $\alpha(m-1)+\beta<0$, then we find:

$$\eta_0\ge\frac{\lambda[\alpha(m-1)+\beta]}{\alpha(m-1)}.$$

3. With $\eta=0$, (19) becomes:

$$(1-m)f'(0)=[\alpha(m-1)+\beta]\int_{0}^{\lambda}f^{1-m}(\xi)\,d\xi. \tag{20}$$

The sign of $f'(0)$ results immediately from (20).

4. If

$$\alpha(m-1)+\beta=0 \text{ for any } m\in(0,1)\cup(1,\infty),$$

we have from (19):

$$(1-m)\,f'(\eta)=\beta\eta f^{1-m}(\eta). \tag{21}$$

Which implies easily (17).

The integration of (21) starting from η to λ gives:

$$-\frac{1}{m}f^{m}(\eta)=\frac{\beta}{2(1-m)}\left(\lambda^{2}-\eta^{2}\right),$$

which implies the function $f(\eta;\lambda)$ presents in (18) by:

$$f(\eta;\lambda)=\left(\frac{\beta m}{2(m-1)}\left(\lambda^{2}-\eta^{2}\right)\right)^{\frac{1}{m}},\ m\in(0,1)\cup(1,\infty),\ \ |\eta|<\lambda.$$

Let $\lambda(U)$ be its solution.

Because $f(0;\lambda(U))$ is a continuous, monotonically increasing function of $\lambda(U)$, such that:

$$f(0;0)=0 \text{ and } f(0;\infty)=\infty.$$

Then the equation $f(0;\lambda(U))=U$ is uniquely solvable for every $U\geq 0$ and $f=f(\eta;\lambda(U))$ is the unique solution of problem (9)–(10), with compact support definite for all $|\eta|<\lambda$ and it satisfies:

$$f(0;\lambda)=U \text{ and } f(\lambda;\lambda)=0,$$

where

$$\lambda(U)=\sqrt{\frac{2(m-1)}{\beta m}U^{m}}.$$

In addition, f is a continuous and monotonous decreasing function. ■

We now turn to the question of existence.

Lemma 3 *For any $m\in(0,1)\cup(1,\infty)$ and any $\lambda>0$, the problem (9) with the boundary conditions (10), has a unique positive solution in the left neighborhood of $\eta=\lambda$.*

The proof of this lemma concerns the analysis of the of Banach's fixed point theorem, which is:

Theorem 4 (Banach's fixed point [8]) *Let X be a non-empty closed subset of a Banach space E, then any contraction mapping M of X into itself has a unique fixed point.*

Now, we prove the Lemma 3.

Proof. Assume that f is a positive solution in the left neighborhood of $\eta=\lambda$. By Lemma 2, $f'(\eta)<0$ for $\eta\in(\lambda-\varepsilon,\lambda)$ for some $\varepsilon>0$.

1. For $m\in(0,1)$, we have $\beta<0$. In this case, the writing of (19) is given as follows:

$$f'(\eta)=\alpha\eta f^{1-m}(\eta)+\left[\alpha(1-m)-\beta\right]\int_{\eta}^{\lambda}\xi\frac{f'(\xi)}{f^{m}(\xi)}d\xi. \tag{22}$$

With $G(f)=\eta$ we have

$$\frac{dG}{df}=\frac{1}{\alpha G f^{1-m}-\left[\alpha(1-m)-\beta\right]\int_{0}^{f}\frac{G(\varphi)}{\varphi^{m}}d\varphi}. \tag{23}$$

By integrating the differ-integral equation (23) from 0 to f, we obtain

$$G(f) - \lambda = \int_0^f \frac{d\varphi}{\alpha G \varphi^{1-m} - [\alpha(1-m) - \beta] \int_0^{\varphi} \frac{G(g)}{g^m} dg}. \tag{24}$$

We put $H(f) = 1 - \lambda^{-1} G(f)$, then the equation (24) becomes

$$H(f) = \frac{1}{\lambda^2} \int_0^f \frac{d\varphi}{\alpha H \varphi^{1-m} - \frac{\beta}{1-m}\varphi^{1-m} - [\alpha(1-m) - \beta] \int_0^{\varphi} \frac{H(g)}{g^m} dg}. \tag{25}$$

Using the principle of contraction of Banach Theorem, we prove now that the equation (25) admits a unique positive solution to the right of $f = 0$.

Let X be the set of bounded functions $H(f)$ on $[0, h]$, $h > 0$, satisfying:

$$0 \le H(f) \le \rho = -\frac{\beta m}{|\alpha|(1-m) + |\alpha(1-m) - \beta|}.$$

Let $\|\cdot\|$ be the norm sup defined on X, then X is a complete metric space. On X we define the operator:

$$M(H)(f) = \frac{1}{\lambda^2} \int_0^f \frac{d\varphi}{\alpha H \varphi^{1-m} - \frac{\beta}{1-m}\varphi^{1-m} - [\alpha(1-m) - \beta] \int_0^{\varphi} \frac{H(g)}{g^m} dg}. \tag{26}$$

First, we prove that M is a mapping from X to X.

Let $H \in X$. It is clear that

$$\begin{aligned} & \alpha H(\varphi)\varphi^{1-m} - \frac{\beta}{1-m}\varphi^{1-m} - [\alpha(1-m) - \beta] \int_0^{\varphi} \frac{H(g)}{g^m} dg \\ \ge\ & -\frac{\beta}{1-m}\varphi^{1-m} - |\alpha| H(\varphi)\varphi^{1-m} - |\alpha(1-m) - \beta| \, \|H\| \int_0^{\varphi} g^{-m} dg \\ \ge\ & -\frac{\beta}{1-m}\varphi^{1-m}. \end{aligned} \tag{27}$$

Therefore, starting from (26), we have:

$$M(H)(f) \le \frac{1}{\lambda^2} \int_0^f \frac{d\varphi}{-\frac{\beta}{1-m}\varphi^{1-m}} \le \frac{(m-1)h^m}{\beta \lambda^2 m}. \tag{28}$$

Thus, $M(H)$ is well defined on X and $M(H) : [0, h] \to \mathbb{R}$ is nonnegative and continuous. The right side of (28) shows that we can find $h_0 > 0$, such that $h \le h_0$ and

$$\|M(H)\| \le \rho, \text{ for all } H \in X.$$

So M is a mapping from X to X for some $h \le h_0$.

In the next step, we show that M is a contracting mapping on X. Let $H_1, H_2 \in X$ and $h \le h_0$, be a positive real number, we put:

$$K(H) = \alpha H(\varphi)\varphi^{1-m} - \frac{\beta}{1-m}\varphi^{1-m} - [\alpha(1-m) - \beta] \int_0^{\varphi} \frac{H(g)}{g^m} dg.$$

Then

$$\begin{aligned} \|K(H_2) - K(H_1)\| &\leq |\alpha| \|H_2(\varphi) - H_1(\varphi)\| \varphi^{1-m} \\ &\quad + \frac{|\alpha(1-m) - \beta|}{1-m} \|H_2(\varphi) - H_1(\varphi)\| \varphi^{1-m} \\ &\leq \left(|\alpha| + \frac{|\alpha(1-m) - \beta|}{1-m}\right) \|H_2(\varphi) - H_1(\varphi)\| \varphi^{1-m}. \end{aligned}$$

So

$$\begin{aligned} \|M(H_1) - M(H_2)\| &= \left\| \frac{1}{\lambda^2} \int_0^f \frac{d\varphi}{K(H_1)} - \frac{1}{\lambda^2} \int_0^f \frac{d\varphi}{K(H_2)} \right\| \\ &= \frac{1}{\lambda^2} \left\| \int_0^f \frac{K(H_2) - K(H_1)}{K(H_1) K(H_2)} d\varphi \right\| \\ &\leq \frac{(1-m)^2}{\lambda^2 \beta^2} \int_0^f \frac{\|K(H_2) - K(H_1)\|}{\varphi^{2(1-m)}} d\varphi \\ &\leq \frac{(1-m)[|\alpha|(1-m) + |\alpha(1-m) - \beta|]}{m h^{-m} \lambda^2 \beta^2} \|H_2(\varphi) - H_1(\varphi)\|. \end{aligned}$$

Hence, there exists a $h_1 \in (0, h_0]$ such that if $h \leq h_1$, M is a contraction on X. After the principle of contraction of Banach [8], M has a unique fixed point in X, consequently, there exists a unique positive solution for the problem (9) in the interval $(\lambda - \varepsilon, \lambda)$ for some $\varepsilon > 0$.

2. After using some techniques, we use the same steps for the case $m \in (1, \infty)$, to prove the existence of a single positive solution for the problem (9). ■

We now continue f backwards as a function of η. By the standard theory, this can be done uniquely so long as f remains positive and bounded. There are now three possibilities:

(A) $f(\eta) \to \infty$ as $\eta \to \eta_1$ for some $\eta_1 \in [0, \lambda)$,

(B) $f(\eta)$ can be continued back to $\eta = 0$,

(C) $f(\eta) \to 0$ as $\eta \to \eta_2$ for some $\eta_2 \in [0, \lambda)$.

We begin by ruling out possibility (A).

Lemma 5 *Let $\mu \in [0, \lambda)$ be a positive real number. If f is a positive solution of the problem (9) on (μ, λ), then f is bounded on (μ, λ) and*

$$\sup_{\mu < \eta < \lambda} f(\eta) \leq \left[\frac{\lambda^2 m}{2|m-1|} \max\{|\alpha(m-1) + 2\beta|, |\beta|\}\right]^{\frac{1}{m}}, \text{ for any } m \in (0,1) \cup (1, \infty).$$

Proof. Assume that f is a positive solution of (9) on (μ, λ). We prove this lemma for the following two cases: (i) $\alpha(m-1) + \beta < 0$, (ii) $\alpha(m-1) + \beta \geq 0$.

1. For the case $m \in (0,1)$; $\beta < 0$:

(i) If $\alpha(m-1) + \beta < 0$, in this case $f'(\eta) < 0$, $\forall \eta \in (\mu, \lambda)$ by Lemma 2, i.e $f(\xi) < f(\eta)$, $\forall \xi \in (\eta, \lambda)$ and from (19),

$$(1-m) f'(\eta) \geq \beta \eta f^{1-m}(\eta) + [\alpha(m-1) + \beta] f^{1-m}(\eta)(\lambda - \eta), \ \forall \eta \in (\mu, \lambda).$$

So, for any $\mu < \eta < \lambda$, we have:

$$\begin{aligned}(1-m) f^{m-1}(\eta) f'(\eta) &\geq \beta\eta + \alpha(m-1)(\lambda-\eta) + \beta(\lambda-\eta) \\ &\geq \lambda\beta + \alpha(m-1)(\lambda-\eta). \end{aligned} \tag{29}$$

The integration of (29) from η to λ gives:

$$\begin{aligned}-\frac{1-m}{m} f^m(\eta) &\geq \left[\lambda\beta + \frac{1}{2}\alpha(m-1)(\lambda-\eta)\right](\lambda-\eta) \\ f^m(\eta) &\leq \frac{m}{2(1-m)}\left[\alpha(1-m)(\lambda-\eta) - 2\lambda\beta\right](\lambda-\eta).\end{aligned}$$

So

$$\sup_{\mu<\eta<\lambda} f^m(\eta) \leq -\frac{\lambda^2 m}{2(1-m)}\left[\alpha(m-1) + 2\beta\right]. \tag{30}$$

(ii) If $\alpha(m-1) + \beta \geq 0$, by equation (19), we have for any $\mu < \eta < \lambda$,

$$(1-m) f^{m-1}(\eta) f'(\eta) \geq \beta\eta. \tag{31}$$

The integration of (31) from η to λ gives:

$$-\frac{1-m}{m} f^m(\eta) \geq \frac{\beta}{2}\left(\lambda^2 - \eta^2\right).$$

This implies that:

$$\sup_{\mu<\eta<\lambda} f^m(\eta) \leq -\frac{\beta\lambda^2 m}{2(1-m)}. \tag{32}$$

2. The case $m \in (1, \infty)$; $\beta > 0$.

(i) If $\alpha(m-1) + \beta < 0$, by equation (19), we have for any $\mu < \eta < \lambda$,

$$(1-m) f^{m-1}(\eta) f'(\eta) \leq \beta\eta. \tag{33}$$

After the integration of (33) from η to λ, we have:

$$\sup_{\mu<\eta<\lambda} f^m(\eta) \leq \frac{\beta\lambda^2 m}{2(m-1)}. \tag{34}$$

(ii) If $\alpha(m-1) + \beta \geq 0$, $\forall\eta \in (\mu, \lambda)$, we have from (19):

$$\begin{aligned}(1-m) f'(\eta) &= \beta\eta f^{1-m}(\eta) + \left[\alpha(m-1) + \beta\right]\int_\eta^\lambda f^{1-m}(\xi)\, d\xi, \\ &\leq \beta\eta f^{1-m}(\eta) + \frac{\lambda}{2}\left[\alpha(m-1) + \beta\right] f^{1-m}(\eta).\end{aligned}$$

Then

$$(1-m) f^{m-1}(\eta) f'(\eta) \leq \beta\eta + \frac{\lambda}{2}\left[\alpha(m-1) + \beta\right]. \tag{35}$$

After the integration (35) from η to λ, we have:

$$\sup_{\mu<\eta<\lambda} f^m(\eta) \leq \frac{\lambda^2 m}{2(m-1)}\left[\alpha(m-1) + 2\beta\right]. \tag{36}$$

Note that the bounds of (30), (32), (34) and (36) are independent of μ and consequently, $f(\eta)$ can not be unlimited at $\eta \to \lambda$. ■

The following lemmas distinguish between the possibilities (B) and (C).

First, we give an elementary lemma for the case $m \in (0, 1)$; $\beta < 0$:

Lemma 6 *Assume that f is a positive solution of problem (9)–(10) in the left neighborhood of $\eta = \lambda$. Then*

(i) If $\alpha(m-1) + 2\beta < 0$, then $f(\eta) > 0$ on $[0, \lambda)$.

(ii) If $\alpha(m-1) + 2\beta = 0$, then $f(\eta) > 0$ on $(0, \lambda)$ and $f(0) = 0$.

(iii) If $\alpha(m-1) + 2\beta > 0$, there exists an $\bar{\eta} \in (0, \lambda)$ such that $f(\eta) > 0$ on $(\bar{\eta}, \lambda)$ and $f(\bar{\eta}) = 0$.

Proof. For the case $m \in (0,1); \beta < 0$, assume that f is a positive solution of the problem (9)–(10) on $[0, \lambda)$. The integration of (19) from η to λ gives:

$$f(\eta) = \frac{\beta\eta}{m-1} \int_{\eta}^{\lambda} \frac{d\xi}{f^{m-1}(\xi)} + \frac{\alpha(m-1) + 2\beta}{m-1} \int_{\eta}^{\lambda} \frac{\xi - \eta}{f^{m-1}(\xi)} d\xi. \tag{37}$$

Hence the lemma. ∎

Now, we give an elementary lemma for the case $m \in (1, \infty); \beta > 0$:

Lemma 7 *Assume that f is a positive solution of the problem (9)–(10) in the left neighborhood of $\eta = \lambda$. Then*

(i) If $\alpha(m-1) + \beta \geq 0$, then $f(\eta) > 0$ on $[0, \lambda)$.

(ii) If $\alpha(m-1) + \beta \leq 0$ and $\alpha(m-1) + 2\beta > 0$, then $f(\eta) > 0$ on $[0, \lambda)$.

(iii) If $\alpha(m-1) + 2\beta \leq 0$, then there exists an $\bar{\eta} \in (0, \lambda)$ such that $f(\eta) > 0$ on $(\bar{\eta}, \lambda)$ and $f(\bar{\eta}) = 0$.

Proof. For the case $m \in (1, \infty); \beta > 0$, assume that f is a positive solution of the problem (9)–(10) on $[0, \lambda)$.

(i) If $\alpha(m-1) + \beta \geq 0$, then $\alpha(m-1) + 2\beta > 0$, also we have from (19) that:

$$(1-m) f'(\eta) \geq \beta\eta f^{1-m}(\eta) \tag{38}$$

and the integration of (38) from η to λ gives:

$$f^m(\eta) \geq \frac{\beta m}{2(m-1)} \left(\lambda^2 - \eta^2\right). \tag{39}$$

It is easy to see from (39) that $f(\eta)$ is always positive on $[0, \lambda)$.

Now we prove the estimate (ii) and (iii). If $\alpha(m-1) + \beta \leq 0$, we have from (19) that:

$$(1-m) f'(\eta) \geq \beta\eta f^{1-m}(\eta) + \frac{\lambda}{2} [\alpha(m-1) + \beta] f^{1-m}(\eta).$$

Which implies after an integration from η to λ, that

$$f^m(\eta) \geq \frac{m}{2(m-1)} (\beta\eta + \lambda[\alpha(m-1) + 2\beta]) (\lambda - \eta).$$

We observe in this case the validity of estimate (ii) and (iii). We can fix $\bar{\eta}$ as follows:

$$\bar{\eta} \geq -\frac{\lambda[\alpha(m-1) + 2\beta]}{\beta}, \text{ for the case } -\beta < \alpha(m-1) + 2\beta \leq 0.$$

Hence the lemma. ∎

New explicit solutions for $\alpha(m-1) + 2\beta = 0$ and $m \in (0,1) \cup (1,2)$

Let λ be an arbitrary positive number.

Starting from (37) for $m \in (0,1)$, if we put $\alpha(m-1)+2\beta=0$, we have:

$$f(\eta) = \frac{\beta\eta}{m-1}\int_{\eta}^{\lambda}\frac{d\xi}{f^{m-1}(\xi)}. \tag{40}$$

The change of variable $g(\eta) = \int_{\eta}^{\lambda} f^{1-m}(\xi)\,d\xi$, gives $g(\lambda)=0$ and $g'(\eta) = -\, f^{1-m}(\eta) < 0$, $\forall \eta \in (0,\lambda)$. Then the equation (40) becomes:

$$f(\eta) = \left[-g'(\eta)\right]^{\frac{1}{1-m}} = \frac{\beta}{m-1}\eta g(\eta). \tag{41}$$

The last equation can be resolved for any $m \in (0,1)\cup(1,2)$.

In fact, from the proof of Lemma 1 that a necessary condition for the existence of a positive solution of problem (9)–(10) in a left neighborhood of $\eta=\lambda$, is either $\beta>0$ for $m>1$, $\beta<0$ for $m\in(0,1)$ or $\beta=0$ and $\alpha>0$.

Then the value $\frac{\beta}{m-1}$ is always positive. In this case the equation (41) is equivalent to

$$-g'(\eta)\,g^{m-1}(\eta) = \left(\frac{\beta}{m-1}\right)^{1-m}\eta^{1-m}. \tag{42}$$

The integration of (42) starting from η to λ gives:

$$\frac{1}{m}g^{m}(\eta) = \frac{1}{2-m}\left(\frac{\beta}{m-1}\right)^{1-m}\left(\lambda^{2-m}-\eta^{2-m}\right).$$

From (41), we have after an elementary computation that:

$$f(\eta;\lambda) = \left(\frac{\beta m}{(m-1)(2-m)}\eta^{m}\left(\lambda^{2-m}-\eta^{2-m}\right)\right)^{\frac{1}{m}},\ 0\le\eta<\lambda, \tag{43}$$

is a continuous function, for any choice of $\lambda>0$, with compact support and satisfies:

$$f(0;\lambda) = f(\lambda;\lambda) = 0.$$

Theorem 8 *The following statements (i) and (ii) hold.*

(i) Assume that $U=0$. Then for every $\lambda>0$ there exists a solution $f(\eta;\lambda)$ of the boundary value problem (9)–(10) which is positive in $(0,\lambda)$ if and only if $\alpha(m-1)+2\beta=0$ and $\beta<0$ for $m\in(0,1)$, or if $\alpha(m-1)+2\beta=0$ and $\beta>0$ for $m>1$.

(ii) Assume that $U>0$. Then the boundary value problem (9)–(10) admits a unique solution and there exists a unique $\lambda(U)>0$ such that $f(\eta;\lambda(U))$ is positive on $(0,\lambda)$ if and only if $\alpha(m-1)+2\beta<0$ and $\beta\le0$ for $m\in(0,1)$, or if $\alpha(m-1)+2\beta>0$ and $\beta\ge0$ for $m>1$.

Proof. By Lemma 1 a necessary condition for the existence of such a solution is that $\beta\le0$ for $m\in(0,1)$ or $\beta\ge0$ for $m>1$.

For $\beta=0$, $\alpha>0$, we already gave the solution (15) of (19) for $m\in(0,1)\cup(1,2)$ which is:

$$f(\eta;\lambda) = \left(\frac{\alpha m^{2}}{2(2-m)}(\lambda-\eta)^{2}\right)^{\frac{1}{m}},\ 0\le\eta<\lambda.$$

This solution is continuous and satisfies for every $U > 0$, that $f(0;\lambda) = U$ and $f(\lambda;\lambda) = 0$, where

$$\lambda(U) = \sqrt{\frac{2(2-m)}{\alpha m^2} U^m}.$$

In the following, for the case $\beta \neq 0$, we put $\beta < 0$ for $m \in (0,1)$ and $\beta > 0$ for $m > 1$.

We already proved in Lemma 3, the local existence of positive solution in the left neighborhood of $\eta = \lambda$ of problem (9)–(10). This local solution is unique and can be continued back to $\eta = 0$ as a positive solution with $f(0) > 0$ if and only if $\beta < 0$ and $\alpha(m-1)+2\beta < 0$ for $m \in (0,1)$, also if $\beta > 0$ and $\alpha(m-1)+2\beta > 0$ for $m > 1$ (by Lemmas 6 and 7).

Now, the boundary condition (10) at $\eta = 0$ is satisfied if we can find a $\lambda(U)$ such that

$$f(0;\lambda) = U. \tag{44}$$

If only one such a λ exists, the solution is unique. We distinguish two cases:

(i) $U = 0$. By Lemmas 6 and 7, the equation (44) can only be satisfied if $\alpha(m-1) + 2\beta = 0$. Moreover (44) is then satisfied for any $\lambda > 0$. Then there exists a nontrivial weak solution f with compact support of the problem (9)–(10) with the property:

$$\begin{cases} f(\eta) > 0 & \text{on} \quad (0,\lambda), \\ f(\eta) = 0 & \text{on} \quad \{0\} \cup [\lambda,\infty). \end{cases}$$

(ii) $U > 0$. It follows from Lemmas 6 and 7 that now a necessary condition for (44) to have a solution is that $\beta < 0$ and $\alpha(m-1) + 2\beta < 0$ for $m \in (0,1)$ or $\beta > 0$ and $\alpha(m-1) + 2\beta > 0$ for $m > 1$.

With this intention, let us suppose that $f(\eta;\lambda)$ is a solution of the problem (9)–(10) on $(0,\lambda)$, then for any $\mu > 0$ the function $\mu^{-\frac{2}{m}} f(\mu\eta;\mu\lambda)$ is a solution of (9)–(10) on $(0,\mu\lambda)$. If $\mu = \lambda^{-1}$, then:

$$\lambda^{\frac{2}{m}} f(0;1) = U. \tag{45}$$

Because $f(0;1) > 0$ in the cases $\alpha(m-1) + 2\beta < 0$ for $m \in (0,1)$ or $\alpha(m-1) + 2\beta > 0$ for $m > 1$. We obtain a unique solution $\lambda = \lambda(U)$ of (45).

Thus, the function $f(\eta;\lambda(U))$ is the unique solution of (9)–(10), with the property:

$$\begin{cases} f(\eta) > 0 & \text{on} \quad [0,\lambda), \\ f(\eta) = 0 & \text{on} \quad [\lambda,\infty). \end{cases}$$

Hence the theorem. ■

We denote by $(z)_+$ the positive part of z, which is z if $z > 0$ and else is zero.

Now, we give the principal theorem of this work.

Theorem 9 (Global existence and blow-up of self-similar solutions) *Let $a(t)$, $c(t)$ be positive real functions of t, which satisfy:*

$$a(0) = c(0) = 1, \ \dot{a}(0) = -\beta, \ \dot{c}(0) = \alpha. \tag{46}$$

Then, for $f \in C([0,\lambda], \mathbb{R}^+)$, the problem (7) admits an exact solution in the form:

$$u(x,t) = c(t) f(\eta), \ \text{with } \eta = \frac{x}{a(t)}, \ x \in \mathbb{R}, \ t > 0, \tag{47}$$

if the basic profile f is a solution of the problem (9) on $[0,\lambda]$ and satisfies in each point:

$$f^m(\eta) f''(\eta) = \alpha f(\eta) + \beta\eta f'_\eta(\eta).$$

Thereupon, we separate the following cases:

(1) If $\alpha m + 2\beta < 0$, the functions $a(t)$, $c(t)$ are given by:

$$\begin{cases} a(t) = (1 - (\alpha m + 2\beta) t)^{\frac{\beta}{\alpha m + 2\beta}}, \\ c(t) = (1 - (\alpha m + 2\beta) t)^{-\frac{\alpha}{\alpha m + 2\beta}}, \end{cases} \quad \forall t > 0. \tag{48}$$

(2) If $\alpha m + 2\beta = 0$, the functions $a(t)$, $c(t)$ in this case are given by:

$$\begin{cases} a(t) = e^{-\beta t}, \\ c(t) = e^{\alpha t}, \end{cases} \quad \forall t > 0. \tag{49}$$

In each case (1) and (2). The problem (7) admits a global solution in time under the generalized self-similar form, this solution defined for all $t > 0$. Moreover, if $\alpha < 0$, we have

$$\lim_{t \to +\infty} u(x,t) = 0 \quad \text{for all } x \in \mathbb{R}.$$

(3) If $\alpha m + 2\beta > 0$, the functions $a(t)$, $c(t)$ are given by:

$$\begin{cases} a(t) = (1 - (\alpha m + 2\beta) t)_+^{\frac{\beta}{\alpha m + 2\beta}}, \\ c(t) = (1 - (\alpha m + 2\beta) t)_+^{-\frac{\alpha}{\alpha m + 2\beta}}, \end{cases} \quad 0 < t < T. \tag{50}$$

The moment $T = \frac{1}{\alpha m + 2\beta}$ represents the maximal existence value of the functions $a(t)$, $c(t)$. Moreover; if $\alpha > 0$, the problem (7) admits a solution under the generalized self-similar form, which blows up in a finite time. The solution is defined for all $t \in (0, T)$, where T represents the blow-up time of the solution such that:

$$\text{for all } x \in \mathbb{R}, \quad \lim_{t \to T^-} u(x,t) = +\infty, \quad \text{with } T = \frac{1}{\alpha m + 2\beta} > 0.$$

Proof. We have already proved in Theorem 8 the existence of solution f of (9), which is with compact support $[0, \lambda]$ if and only if

1. $\alpha(m-1) + 2\beta = 0$ and $[\beta < 0 \text{ for } m \in (0,1) \text{ or } \beta > 0 \text{ for } m > 1]$,
2. $\alpha(m-1) + 2\beta < 0$ and $\beta \le 0$ for $m \in (0,1)$,
3. $\alpha(m-1) + 2\beta > 0$ and $\beta \ge 0$ for $m > 1$.

Now, to determine the functions $a(t)$, $c(t)$, just solve the system (6) which is

$$\begin{cases} \frac{\dot{c}}{c} = k_1 \frac{c^m}{a^2}, \\ \frac{\dot{a}}{a} = k_2 \frac{c^m}{a^2}. \end{cases} \tag{51}$$

The conditions (46), imply $k_1 = \alpha$ and $k_2 = -\beta$, then the system (51), can be resolved as follows

$$\begin{cases} \frac{\dot{c}}{c} = \alpha \frac{c^m}{a^2}, \\ \frac{\dot{a}}{a} = -\beta \frac{c^m}{a^2}. \end{cases} \Rightarrow \frac{\dot{a}}{a} = -\frac{\beta}{\alpha} \frac{\dot{c}}{c},$$

then we deduce after an integration from 0 to t that:

$$a(t) = c^{-\frac{\beta}{\alpha}}(t). \tag{52}$$

If we replace (52) in (51), we obtain

$$c^{-\frac{\alpha m+2\beta}{\alpha}-1}dc=\alpha dt. \tag{53}$$

If $\alpha m+2\beta\neq 0$, we obtain easily the solution of (53) as follows

$$c\left(t\right)=\left(1-\left(\alpha m+2\beta\right)t\right)_{+}^{-\frac{\alpha}{\alpha m+2\beta}}$$

in the same way we find:

$$a\left(t\right)=\left(1-\left(\alpha m+2\beta\right)t\right)_{+}^{\frac{\beta}{\alpha m+2\beta}}.$$

We deduce that the functions $a\left(t\right),\ c\left(t\right)$ are globally defined if $\alpha m+2\beta<0$, and $a\left(t\right),\ c\left(t\right)$ are maximal functions if $\alpha m+2\beta>0$, and well defined if and only if

$$0<t<T=\frac{1}{\alpha m+2\beta}.$$

If $\alpha m+2\beta=0$, the functions $a\left(t\right),\ c\left(t\right)$ are defined globally, we obtain in this case:

$$\begin{cases} a\left(t\right)=e^{-\beta t}, \\ c\left(t\right)=e^{\alpha t}, \end{cases} \quad \forall t>0.$$

We notice from this theorem that we have two time behaviors of functions $a\left(t\right),\ c\left(t\right)$, their behaviors depend on parameters of similarity α,β.

In (1) and (2) i.e $\alpha m+2\beta\leq 0$, the functions $a\left(t\right),\ c\left(t\right)$ are defined globally in time. Now, the profile f is a bounded positive function for all $x\in\mathbb{R}$. If $\alpha<0$ we have:

$$\lim_{t\to+\infty}c\left(t\right)=0.$$

In each case (1) and (2). Thus:

$$\lim_{t\to+\infty}u\left(x,t\right)=\lim_{t\to+\infty}c\left(t\right)f\left(\frac{x}{a\left(t\right)}\right)=0.$$

(3) In the case $\alpha m+2\beta>0$, we have $a\left(t\right),\ c\left(t\right)$ given in (50), is well defined if and only if:

$$0<t<T=\frac{1}{\alpha m+2\beta}.$$

We recall that the solution blows up in finite time if there exists a time $T<+\infty$, which we call it the blow-up time, such that the solution is well defined for all $0<t<T$, while

$$\sup_{x\in\mathbb{R}}\left|u\left(x,t\right)\right|\to+\infty,\ \text{ when } t\to T^{-}.$$

If $\alpha>0$, the value T represents the blow-up time of the solution, thus $\lim\limits_{t\to T^{-}}c\left(t\right)=+\infty$ and

$$\lim_{t\to T^{-}}u\left(x,t\right)=\lim_{t\to T^{-}}c\left(t\right)f\left(\frac{x}{a\left(t\right)}\right)=+\infty.$$

■

4 Examples

Example 1 (Blow-up solution) *For $\beta = 0$ and $\alpha > 0$, we have from Theorem 9 that $\alpha m + 2\beta = \alpha m > 0$ and*

$$a(t) = 1,\ c(t) = (1 - \alpha m t)_+^{-\frac{1}{m}},\ \text{where } 0 < t < T = \frac{1}{\alpha m}.$$

In this case, the self-similar form is a solution of separation of variables $u(x,t) = c(t) f(x)$, where f is given in (15), by:

$$f(x) = \left(\frac{\alpha m^2}{2(2-m)} (\lambda - x)^2\right)^{\frac{1}{m}},\ m \in (0,1) \cup (1,2),\ 0 \leq x < \lambda.$$

Then, the solution of (1) in the generalized self-similar form (47) for $\beta = 0$, $\alpha > 0$ is given by:

$$u(x,t) = \begin{cases} \left(\frac{\alpha m^2}{2(2-m)(1-\alpha m t)} (\lambda - x)^2\right)^{\frac{1}{m}}, & 0 \leq x < \lambda,\ m \in (0,1) \cup (1,2). \\ 0, & \text{otherwise.} \end{cases}$$

This solution blows up when $t \to \frac{1}{\alpha m}$. In this case

$$\lambda(U) = \sqrt{\frac{2(2-m)}{\alpha m^2} U^m},\ \text{for every } U > 0.$$

Example 2 (Global and blow-up solutions) *Now, we present the new explicit solutions on the generalized self-similar form of the equation (1). Because the profile f is bounded with compact support, then it is an integrable function on $\mathbb{R}$, where*

$$\int_{\mathbb{R}} f(\xi)\, d\xi = M,\ \text{for some } M > 0.$$

Therefore, for $m \in (0,1) \cup (1,\infty)$ and

$$\int_{\mathbb{R}} u(s,t)\, ds = c^m(t).$$

We can find explicitly a new exact solution of (1). Indeed

$$\int_{\mathbb{R}} u(s,t)\, ds = \int_{\mathbb{R}} c(t) f\left(\frac{s}{a(t)}\right) ds = a(t) c(t) \int_{\mathbb{R}} f(\xi)\, d\xi = c^m(t),$$

this implies that

$$\frac{c^{m-1}(t)}{a(t)} = M. \tag{54}$$

According to the formulas $a(t)$ and $c(t)$ in (48) and (49), the equality (54) implies that:

$$\beta = \alpha(1 - m).$$

Because $\beta < 0$ for $m \in (0,1)$ and $\beta > 0$ for $m > 1$, then $\alpha < 0$.

For $\alpha(m-1) + \beta = 0$, we already gave the solution (18) of (9) for $m \in (0,1) \cup (1,\infty)$ which is:

$$f(\eta; \lambda) = \left(\frac{\beta m}{2(m-1)} (\lambda^2 - \eta^2)\right)^{\frac{1}{m}},\ 0 \leq \eta < \lambda.$$

Now, we determine the functions $a(t)$ and $c(t)$. We have from Theorem 9

$$\alpha m + 2\beta = \alpha m + 2\alpha(1 - m) = \alpha(2 - m).$$

If

$$\alpha (2-m) \neq 0, \ \textit{i.e. } m \neq 2,$$

the functions $a(t)$ *and* $c(t)$ *are given by*

$$\begin{cases} a(t) = (1-\alpha(2-m)t)_+^{\frac{m-1}{m-2}}, \\ c(t) = (1-\alpha(2-m)t)_+^{\frac{1}{m-2}}, \end{cases} \quad m \neq 2, \ 0 < t < T.$$

Where T *represents the maximal existence value*

$$\begin{cases} T = \frac{1}{\alpha(2-m)}, & \textit{if } m > 2, \\ T = +\infty, & \textit{if } m < 2. \end{cases}$$

We obtain the exact solution of (1), for $m \in (0,1) \cup (1,2) \cup (2,\infty)$ *as follows*

$$u(x,t) = \begin{cases} c(t) \left(\frac{\beta m}{2(m-1)} \left(\lambda^2 - x^2 (1-\alpha(2-m)t)_+^{\frac{2(1-m)}{m-2}} \right) \right)^{\frac{1}{m}}, & |x| < r_1, \\ 0, & \textit{otherwise}, \end{cases} \tag{55}$$

where

$$r_1 = \lambda (1-\alpha(2-m)t)_+^{\frac{m-1}{m-2}}.$$

This solution is defined globally in time if $m \in (0,1) \cup (1,2)$, *and blows up when*

$$t \to \frac{1}{\alpha(2-m)}, \ \textit{for } m \in (2,\infty).$$

In this case

$$\lambda(U) = \sqrt{\frac{2(m-1)}{\beta m} U^m}, \ \textit{for every } U > 0.$$

If $m = 2$, *the functions* $a(t)$ *and* $c(t)$ *are written as (49), in this case we obtain* $\beta = -\alpha$ *and*

$$u(x,t) = \begin{cases} e^{\alpha t} \sqrt{\beta \left(\lambda^2 - x^2 e^{2\beta t} \right)}, & |x| < \lambda e^{-\beta t}, \\ 0, & \textit{otherwise}. \end{cases}$$

This solution is defined globally in time. In this case

$$\lambda(U) = \frac{U}{\sqrt{\beta}}, \ \textit{for every } U > 0.$$

For the functions $a(t)$ *and* $c(t)$ *that satisfy the parameters of the classical self-similar form, i.e.*

$$c(t) = t^{\alpha}, \ a(t) = t^{-\beta}.$$

The self-similar solution (8) of the equation (1) is written as:

$$u(x,t) = t^{\alpha} f(\eta), \ \textit{with } \eta = x t^{\beta},$$

where α *and* β *are exponents which satisfy the similarity condition ([11], [12], [13])*

$$\alpha m + 2\beta + 1 = 0, \tag{56}$$

and the function f *is the self similar profile which to be determined by the solution of the following differential equation:*

$$f^m(\eta) f''(\eta) = \alpha f(\eta) + \beta \eta f(\eta).$$

If $\beta = \alpha(1-m)$, *the similarity condition (56) is equivalent to:*

$$\alpha m + 2\beta + 1 = \alpha(2-m) + 1 = 0,$$

this implies that:

$$\alpha = \frac{1}{m-2} \text{ and } \beta = \frac{1-m}{m-2}.$$

We get for $m \in (0,1)$ *that* $\alpha, \beta < 0$. *So the solution (55) for* $c(t) = t^{\alpha}$, $a(t) = t^{-\beta}$, *is written as:*

$$\mathcal{U}(x,t) = t^{\frac{1}{m-2}} \left(\frac{m}{2(2-m)} \left(\lambda^2 - x^2 t^{\frac{2(1-m)}{m-2}} \right) \right)^{\frac{1}{m}}, \text{ for } |x| < \lambda t^{\frac{m-1}{m-2}}, \ m \in (0,1).$$

Also

$$\mathcal{U}(x,t) = \begin{cases} \left(\frac{1}{1-m}\right)^{\frac{1}{m}} \left(t^{\frac{1-m}{m-2}} \left(\frac{m(1-m)}{2(2-m)} \left(\lambda^2 - x^2 t^{\frac{2(1-m)}{m-2}} \right) \right)^{\frac{1-m}{m}} \right)^{\frac{1}{1-m}}, & |x| < \lambda t^{\frac{m-1}{m-2}}, \\ 0, & \text{otherwise.} \end{cases} \tag{57}$$

If we put for $m \in (0,1)$ *the parameter* $p = \frac{1}{1-m} > 1$, *then the solution (57) of the equation (1) in the classical self-similar form is written as:*

$$\mathcal{U}(x,t) = p^{\frac{p}{p-1}} \mathcal{B}^p_{\lambda}(x,t), \ p > 1.$$

Where $\mathcal{B}_{(\cdot)}$ *is the Barenblatt solutions of the porous medium equation (2), which are given in (3) under the following form:*

$$\mathcal{B}_{\lambda}(x,t) = \begin{cases} t^{-\frac{1}{p+1}} \left(\frac{p-1}{2p(p+1)} \left(\lambda^2 - x^2 t^{-\frac{2}{p+1}} \right) \right)^{\frac{1}{p-1}}, & \text{for } |x| < \lambda t^{\frac{1}{p+1}}, \ p > 1, \\ 0, & \text{otherwise.} \end{cases}$$

Example 3 (Global solution) *We present the second new explicit solutions on the generalized self-similar form of the equation (1). For* $0 < m \neq 1$ *and*

$$\int_{\mathbb{R}} u(s,t)\, ds = \frac{c^m(t)}{a(t)},$$

we can find explicitly a new exact solution of (1). In fact

$$\int_{\mathbb{R}} u(s,t)\, ds = \int_{\mathbb{R}} c(t) f\left(\frac{s}{a(t)}\right) ds = a(t)\, c(t) \int_{\mathbb{R}} f(\xi)\, d\xi = \frac{c^m(t)}{a(t)},$$

this implies that

$$\frac{c^{m-1}(t)}{a^2(t)} = M. \tag{58}$$

According to the formulas $a(t)$ *and* $c(t)$ *in (48) and (49), the equality (58) implies that:*

$$\beta = \frac{\alpha(1-m)}{2}, \text{ then } \alpha < 0.$$

For $\alpha(m-1) + 2\beta = 0$, *we gave the solution (43) of (9) for* $m \in (0,1) \cup (1,2)$ *which is:*

$$f(\eta;\lambda) = \left(\frac{\beta m}{(m-1)(2-m)} \eta^m \left(\lambda^{2-m} - \eta^{2-m} \right) \right)^{\frac{1}{m}}, \ 0 \le \eta < \lambda.$$

Now, we determine the functions $a(t)$ *and* $c(t)$. *We have from Theorem 9*

$$\alpha m + 2\beta = \alpha m + \alpha(1-m) = \alpha < 0.$$

The functions $a(t)$ and $c(t)$ are given by:

$$\begin{cases} a(t) = (1-\alpha t)^{\frac{1-m}{2}}, \\ c(t) = \frac{1}{1-\alpha t}, \end{cases} \quad \forall t > 0.$$

We obtain the exact solution of (1), for $m \in (0,1) \cup (1,2)$ as follows:

$$u(x,t) = \begin{cases} \frac{1}{1-\alpha t}\left(\frac{\beta m x^m}{(m-1)(2-m)a^m(t)}\left(\lambda^{2-m} - x^{2-m} c^{\frac{(1-m)(2-m)}{2}}(t)\right)\right)^{\frac{1}{m}}, & x \in r_2, \\ 0, & \textit{otherwise}, \end{cases}$$

where

$$r_2 = \left(0, \lambda(1-\alpha t)^{\frac{1-m}{2}}\right).$$

This solution is defined globally in time for any $m \in (0,1) \cup (1,2)$. In this case $\lambda > 0$.

5 Conclusion

In this paper we have discussed the existence and uniqueness of positive solutions for a class of nonlinear diffusion equation not in divergence form, under the generalized self-similar form. Also we have found new solutions; the behavior of these solutions depends on some parameters (that satisfy some conditions), which make their existence global or local, and we generalized the families of self-similar solutions of the porous medium equation, which is formed by the Barenblatt solutions.

Acknowledgments. The authors are deeply grateful to the reviewers and editors for their insightful comments that have helped to improve the quality of this research work that is supported by the General Direction of Scientific Research and Technological Development (DGRSTD)- Algeria.

References

[1] W. Arendt and R. Chill, Global existence for quasilinear diffusion equations in isotropic nondivergence form, Vol. IX(2010), 523–539.

[2] G. I. Barenblatt, On some unsteady motions of a liquid or a gas in a porous medium, Akad. Nauk SSSR. Prikl. Mat. Meh., 16(1952). 67–78.

[3] N. Benhamidouche, Exact solutions to some nonlinear PDEs, traveling profiles method, Electron. J. Qual. Theory Differ. Equ., 2008, 7 pp.

[4] N. Benhamidouche and Y. Arioua, New method for constructing exact solutions to non-linear PDEs, Int. J. Nonlinear Sci., 7(2009), 395–398.

[5] L. Caffarelli and J. L. Vàzquez, Viscosity solutions for the porous medium equation, Proc. Sympos. Pure Math., 65(1999), 13–26.

[6] R. Ferreira, A. D. Pablo and J. D. Rossi, Blow-up for a degenerate diffusion problem not in divergence form, Indiana Univ. Math. J., 55(2006), 955–974.

[7] B. E. Gilding and L. A. Peletier, On a class of similarity solution of the porous medium equation I, J. Math. Anal. Appl., 55(1976), 351–364.

[8] A. Granas and J. Dugundji, Fixed Point Theory, Springer-Verlag, New York 2003.

[9] J. Hulshof and J. L. Vàzquez, Self-similar solutions of the second kind for the modified porous medium equation, European J. Appl. Math., 5(1994), 391–403.

[10] J. Hulshof and J. L. Vàzquez, Maximal viscosity solutions of the modified porous medium equation and their asymptotic behaviour, European J. Appl. Math., 7(1996), 453–471.

[11] N. H. Ibragimov, CRC Handbook of Lie Group Analysis of Differential Equations, Vol 1 Symmetries, Exact solutions and Conservation Laws, CRS Press , Boca Raton, 1994.

[12] P. J. Olver, Application of Lie Groups to Differential Equations, Springer-Verlag, NewYork, 1986.

[13] A. D. Polyanin and V. F. Zaitsev, Handbook of Nonlinear Partial Equation, Chapman&Hall/CRC, Boca Raton, 2004.

[14] J. L. Vàzquez, The Porous Medium Equation-Mathematical Theory, Oxford, 2007.

[15] C. Wang and J. Yin, Traveling wave fronts of a degenerate parabolic equation equation with non-divergence form, J. Partial Diff. Eqs., 16(2003), 62–74.

[16] C. Wang and J. Yin, Shrinking self-similar solution of a nonlinear diffusion equation with nondivergence form, J. Math. Anal. Appl., 289(2004), 387–404.

[17] W. Zhou and Z. Yao, Cauchy problem for a degenerate parabolic equation with non-divergence form, Acta Math. Sci. Ser. B, 30(2010), 1679–1686.

[18] Y. B. Zeldovich and A. S. Kompaneets, Towards a theory of heat conduction with thermal conductivity depending on the temperature, Collection of Papers Dedicated to the 70th Anniversary of A.F. Ioffe, Izd. Akad. Nauk SSMR, Moscow (1950), 61–72.