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On A Sequence Refining Carleman's Inequality*

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Abstract

In this paper we study the sequence with general term $c_n = \mathrm{e}\sqrt[n]{n!}/(n+1)$, which appears in finite form of Carleman's inequality. We obtain an asymptotic expansion of $\log c_n$ with coefficients that involve Bernoulli numbers, and also we get an asymptotic expansion of c_n. These results lead to some refinements of Carleman's inequality.

1 Introduction and Summary of the Results

For positive real numbers $a_1, \ldots, a_n$, Carleman's inequality [2] in finite form asserts that

$$\sum_{k=1}^{n} (a_1 \cdots a_k)^{\frac{1}{k}} \leqslant \mathrm{e} \sum_{k=1}^{n} a_k. \tag{1}$$

The constant e is the best possible. A proof of this inequality, based on the arithmetic-geometric means (AM-GM) inequality, starts by observing that for each integer $i \geqslant 1$,

$$\sum_{k=i}^{\infty} \frac{1}{k(k+1)} = \frac{1}{i}.$$

Hence

$$\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} i\, a_i \sum_{k=i}^{\infty} \frac{1}{k(k+1)} \geqslant \sum_{i=1}^{n} i\, a_i \sum_{k=i}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{n} \frac{1\, a_1 + 2\, a_2 + \cdots + k\, a_k}{k(k+1)}.$$

The AM-GM inequality over the numbers $a_1, 2\,a_2, \ldots, k\,a_k$ gives

$$\frac{1\, a_1 + 2\, a_2 + \cdots + k\, a_k}{k} \geqslant k!^{\frac{1}{k}} \left(a_1\, a_2 \,\cdots\, a_k\right)^{\frac{1}{k}}.$$

Thus

$$\mathrm{e} \sum_{k=1}^{n} a_k \geqslant \sum_{k=1}^{n} c_k \left(a_1 \cdots a_k\right)^{\frac{1}{k}}, \tag{2}$$

where

$$c_n = \frac{\mathrm{e}\, n!^{\frac{1}{n}}}{n+1}. \tag{3}$$

The function $f(x) = (1+\frac{1}{x})^x$ is strictly increasing for $x > 0$ and admits the limit value $\lim_{x\to\infty} f(x) = \mathrm{e}$. Thus, the inequality $f(x) < \mathrm{e}$ holds for any $x > 0$, from which we get

$$\mathrm{e}^n > \prod_{k=1}^{n} \left(1 + \frac{1}{k}\right)^k = \prod_{k=1}^{n} \frac{(k+1)^k}{k^k} = \frac{(n+1)^n}{\prod_{k=1}^{n} k} = \frac{(n+1)^n}{n!}.$$

This implies that $c_n > 1$ for each $n \geqslant 1$, and by (2) we obtain Carleman's inequality (1). In this note we study the sequence c_n in more detail to obtain the following result, which is a refinement of Carleman's inequality.

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Theorem 1 *Given any integer $r \geqslant 1$, the sequence $(c_n)_{n\geqslant 1}$ defined by (3) is strictly decreasing, and admits the logarithmic asymptotic expansion*

$$\log c_n = \frac{\log n}{2n} + \sum_{j=1}^{2r} \frac{\eta_j}{n^j} + O\Big(\frac{1}{n^{2r+1}}\Big), \tag{4}$$

where $\eta_1 = \log \frac{\sqrt{2\pi}}{\mathrm{e}}$ and for $j > 1$,

$$\eta_j = \frac{B_j}{j(j-1)} + \frac{(-1)^j}{j}, \tag{5}$$

with B_i denoting the i-th Bernoulli number.

Corollary 2 *For given integer $r \geqslant 1$ and for integers j with $1 \leqslant j \leqslant r$ there exist polynomials $P_j(x)$ with degree j such that*

$$c_n = 1 + \sum_{j=1}^{r} \frac{P_j(\log n)}{n^j} + O\left(\Big(\frac{\log n}{n}\Big)^{r+1}\right). \tag{6}$$

The inequality (2) and monotonicity of the sequence $(c_n)_{n\geqslant 1}$ give the following refinements of Carleman's inequality.

Corollary 3 *For each $n \geqslant 1$,*

$$\sum_{k=1}^{n} (a_1 \cdots a_k)^{\frac{1}{k}} < c_n \sum_{k=1}^{n} (a_1 \cdots a_k)^{\frac{1}{k}} \leqslant \sum_{k=1}^{n} c_k\, (a_1 \cdots a_k)^{\frac{1}{k}} \leqslant \mathrm{e} \sum_{k=1}^{n} a_k. \tag{7}$$

To prove Theorem 1 we need the following asymptomatic formula for $\log n!$. The following result provides such expansion.

Proposition 4 *Given any positive integer r, as $n \to \infty$ we have*

$$\log n! = n \log n - n + \frac{1}{2}\log n + \log\sqrt{2\pi} + \sum_{j=1}^{r} \frac{B_{2j}}{(2j)(2j-1)n^{2j-1}} + O\Big(\frac{1}{n^{2r+1}}\Big). \tag{8}$$

Meanwhile, to obtain monotonicity results of some sequences appearing in the proofs, the following lemma is very useful.

Lemma 5 *For given $\theta \in \mathbb{R}$ and for $x > 0$, let*

$$f_\theta(x) = \Big(1 + \frac{1}{x}\Big)^{x+\theta}.$$

Then, for $x \in (0, \infty)$, the function $f_0(x)$ is strictly increasing and the function $f_{\frac{1}{2}}(x)$ is strictly decreasing. Moreover, $f_0(x) < \mathrm{e}$ and $f_{\frac{1}{2}}(x) > \mathrm{e}$ for each $x > 0$.

2 Proofs

Proof of Lemma 5. We have

$$f'_\theta(x) := \frac{d}{dx} f_\theta(x) = f_\theta(x) g_\theta(x),$$

where

$$g_\theta(x) = \log\Big(1 + \frac{1}{x}\Big) - \frac{x+\theta}{x(x+1)}.$$

Hence

$$g_\theta'(x) := \frac{d}{dx} g_\theta(x) = \frac{2tx - x + t}{x^2(x+1)^2}.$$

For each fixed θ, we observe that $\lim_{x\to\infty} f_\theta(x) = \mathrm{e}$ and $\lim_{x\to\infty} g_\theta(x) = 0$. Since $g_0'(x) < 0$ it follows that $g_0(x) > 0$. Thus $f_0'(x) > 0$ and f_0 is strictly increasing, and this implies that $f_0(x) < \lim_{x\to\infty} f_{\frac{1}{2}}(x) = \mathrm{e}$. Also, since $g_{\frac{1}{2}}'(x) > 0$ we obtain $g_{\frac{1}{2}}(x) < 0$, and $f_{\frac{1}{2}}'(x) < 0$. Hence, $f_{\frac{1}{2}}$ is strictly decreasing. Moreover, $f_{\frac{1}{2}}(x) > \lim_{x\to\infty} f_{\frac{1}{2}}(x) = \mathrm{e}$. This completes the proof. ■

Proof of Proposition 4. By using Euler–Maclaurin summation formula (see [3]), for any integer $r \geqslant 1$ and for any integer $n \geqslant 1$ we obtain

$$\sum_{k=1}^{n} \log k = n \log n - n + \frac{1}{2} \log n + s_r + \mathcal{J} - \mathcal{I},$$

with

$$s_r = 1 + \int_1^\infty \frac{B_{2r}(\{x\})}{2rx^{2r}} dx - \sum_{j=1}^{r} \frac{B_{2j}}{(2j)(2j-1)},$$

which is a constant depending, at most on r, and

$$\mathcal{J} = \sum_{j=1}^{r+1} \frac{B_{2j}}{(2j)(2j-1)n^{2j-1}}, \qquad \mathcal{I} = \int_n^\infty \frac{B_{2(r+1)}(\{x\})}{2(r+1)x^{2(r+1)}} dx.$$

Note that $B_i(\{x\})$ denotes the i-th Bernoulli function, which is bounded. Hence $\mathcal{J} \ll \frac{1}{n}$ and

$$|\mathcal{I}| \leqslant \int_n^\infty \frac{|B_{2(r+1)}(\{x\})|}{2(r+1)x^{2(r+1)}} dx \ll \int_n^\infty \frac{dx}{x^{2(r+1)}} \ll \frac{1}{n^{2r+1}}.$$

Thus, we obtain

$$\mathcal{J} - \mathcal{I} = \sum_{j=1}^{r} \frac{B_{2j}}{(2j)(2j-1)n^{2j-1}} + O\Big(\frac{1}{n^{2r+1}}\Big).$$

To conclude the proof we show $s_r = \log\sqrt{2\pi}$, which asserts that s_r is an absolute constant. We define the sequence $(\delta_n)_{n\geqslant 1}$ by

$$n! = \Big(\frac{n}{\mathrm{e}}\Big)^n \sqrt{n}\,\delta_n.$$

The inequality $\delta_{n+1} < \delta_n$ is equivalent with the assertion that $(1+\frac{1}{n})^{n+\frac{1}{2}} > \mathrm{e}$, which holds due to Lemma 5 for the case $\theta = \frac{1}{2}$. Hence, δ_n admits a limit values as $n \to \infty$. Let $\delta = \lim_{n\to\infty} \delta_n$. Since $\mathcal{J} - \mathcal{I} \ll \frac{1}{n}$, we obtain

$$s_r = \lim_{n\to\infty} \Big(\log n! - \Big(n\log n - n + \frac{1}{2}\log n\Big)\Big) = \lim_{n\to\infty} \log \delta_n = \log \delta.$$

Let $w_n = \prod_{k=1}^{n} \big(\frac{2k}{2k-1}\,\frac{2k}{2k+1}\big)$ denote the truncated Wallis' product. We have

$$w_n = \Big(\frac{n!^2 2^{2n}}{(2n)!}\Big)^2 \frac{1}{2n+1},$$

and

$$\frac{n!^2 2^{2n}}{(2n)!}\sqrt{\frac{2}{n}} = \frac{(\delta_n)^2}{\delta_{2n}}.$$

Hence

$$w_n = \Big(\frac{(\delta_n)^2}{\delta_{2n}}\Big)^2 \frac{n}{2(2n+1)}. \tag{9}$$

By using Wallis' product formula which asserts that $\lim_{n\to\infty} w_n = \frac{\pi}{2}$ (see [1] for an elementary proof), and taking limits from both sides of (9) we obtain

$$\frac{\pi}{2} = \frac{\delta^2}{4}.$$

Thus, $\delta = \sqrt{2\pi}$ and consequently $s_r = \log\sqrt{2\pi}$. This completes the proof. ■

Proof of Theorem 1. The inequality $c_{n+1} < c_n$ is equivalent with $t_n > 1$, where

$$t_n = \frac{n!\,(n+2)^{n(n+1)}}{(n+1)^{n(n+2)}}.$$

Lemma 5 for the case $\theta = 0$ implies that $f_0(n+2) > f_0(n+1)$, and this is equivalent with $(\frac{t_{n+1}}{t_n})^{\frac{1}{n+1}} > 1$. Hence, $t_{n+1} > t_n \geqslant t_1 = \frac{9}{8} > 1$. This implies that c_n is strictly decreasing. To show (4) we use the expansion (8) as follows

$$\begin{aligned}\log c_n &= 1 - \log(n+1) + \frac{1}{n}\log n! \\ &= \frac{\log n}{2n} + \frac{\log\sqrt{2\pi}}{n} + \sum_{j=1}^{r} \frac{B_{2j}}{(2j)(2j-1)n^{2j}} - \log\Big(1+\frac{1}{n}\Big) + O\Big(\frac{1}{n^{2r+1}}\Big).\end{aligned}$$

Expanding $-\log(1+t)$ as $t \to 0$, and letting $t = \frac{1}{n}$ we obtain

$$-\log\Big(1+\frac{1}{n}\Big) = \sum_{j=1}^{2r} \frac{(-1)^j}{jn^j} + O\Big(\frac{1}{n^{2r+1}}\Big).$$

We write

$$\sum_{j=1}^{r} \frac{B_{2j}}{(2j)(2j-1)n^{2j}} = \sum_{j=2}^{2r} \frac{\bar{B}_j}{n^j},$$

where $\bar{B}_j = \frac{B_j}{j(j-1)}$ when j is even and $\bar{B}_j = 0$ when j is odd. Thus

$$\log c_n = \frac{\log n}{2n} + \frac{\log\sqrt{2\pi} - 1}{n} + \sum_{j=2}^{2r} \Big(\frac{\bar{B}_j}{n^j} + \frac{(-1)^j}{jn^j}\Big) + O\Big(\frac{1}{n^{2r+1}}\Big),$$

which is (4) with

$$\eta_j = \begin{cases} \log\frac{\sqrt{2\pi}}{\mathrm{e}} & \text{for } j = 1, \\ \frac{B_j}{j(j-1)} + \frac{1}{j} & \text{for } j > 1 \text{ and } j \text{ even}, \\ -\frac{1}{j} & \text{for } j > 1 \text{ and } j \text{ odd}. \end{cases}$$

Since $B_j = 0$ for odd values of $j > 1$, we obtain (5). ■

Proof of Corollary 2. Given any integer $r \geqslant 1$, applying the exponential function to both sides of (4) we get

$$c_n = n^{\frac{1}{2n}} \exp\left(\sum_{j=1}^{2r} \frac{\eta_j}{n^j} + O\Big(\frac{1}{n^{2r+1}}\Big)\right) = n^{\frac{1}{2n}} \exp\Big(\sum_{j=1}^{2r} \frac{\eta_j}{n^j}\Big)\Big(1 + O\Big(\frac{1}{n^{2r+1}}\Big)\Big).$$

Note that

$$n^{\frac{1}{2n}} = \exp\Big(\frac{\log n}{2n}\Big) = \sum_{j=0}^{2r} \frac{1}{j!}\Big(\frac{\log n}{2n}\Big)^j + O\Big(\Big(\frac{\log n}{n}\Big)^{2r+1}\Big).$$

Also, we write

$$\exp\Big(\sum_{j=1}^{2r}\frac{\eta_j}{n^j}\Big)=\sum_{i=0}^{2r}\frac{1}{i!}\Big(\sum_{j=1}^{2r}\frac{\eta_j}{n^j}\Big)^i+O\Big(\frac{1}{n^{2r+1}}\Big).$$

Thus,

$$c_n=\left(\sum_{j=0}^{2r}\frac{1}{2^j j!}\Big(\frac{\log n}{n}\Big)^j\right)\left(\sum_{i=0}^{2r}\frac{1}{i!}\Big(\sum_{j=1}^{2r}\frac{\eta_j}{n^j}\Big)^i\right)+O\left(\Big(\frac{\log n}{n}\Big)^{2r+1}\right).$$

Although, multiplying product of sums gives a number of terms weaker than Oh term, after simplifying we get (6). ■

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References

[1] J. Wästlund, An elementary proof of the Wallis product formula for pi, Amer. Math. Monthly, 114(2007), 914–917.

[2] T. Carleman, Sur les fonctions quasi-analytiques, Conférences faites au cinquième congres des mathématiciens Scandinaves, Helsinki, (1923), 181–196.

[3] A. M. Odlyzko, Asymptotic Enumeration Methods, Handbook of Combinatorics, Volumes 1, 2, Elsevier, Amsterdam, 1995.