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An Efficient Approach For Solution Of Fractional Order Differential-Difference Equations Arising In Nanotechnology*

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Abstract

In the present article for the first time Optimal Homotopy asymptotic method (OHAM) is used for driving approximate solution of time fractional order differential difference equations (FDDEs) arising in physical phenomena such as wave phenomena in fluids, coupled nonlinear optical waveguides and nontechnology fields. The fractional derivatives are defined in the Caputo sense, whose order belongs to the close interval $[0, 1]$. The approximate solution obtained by the proposed method is compared with exact solution. It has been shown that (OHAM) is simple and convergent method for the solution of differential difference equation. The numerical results rendering that the applied method is explicit, effective and easy to use, for handling more general fractional order differential difference equation.

1 Introduction

Differential Difference equations (DDEs) were first studied by Fermi et al. in 1950s [1] and of various significance in science and engineering. In particular, applications of these equations arise in vibration of particles, ladder type electric circuit, collapse of Langmuir waves in plasma physics, currents in electric circuits, wave phenomena in fluids and nanotechnology fields [2, 3, 4]. Similar to (DDEs) the concepts of fractional calculus played a vital role in modeling of different complex phenomena of science and engineering [5, 6, 7]. So it becomes important to introduce some efficient methods for solving fractional order differential equations.

In recent days, an extensive study on fractional order differential equations (FDDEs) is of great interest by the modern researchers to solve them with the help of various modern algorithms.

One of the novel techniques has been introduced by Khan [8] for putting numerical solution of the nonlinear (DDEs) arising in nanotechnology and engineering phenomena. In literature, He et al. [9] used variational iterative method (VIM) for solving (DDEs) with fractional order. The main purpose of our work is to apply Optimal Homotopy Asymptotic Method (OHAM) for the first time for fractional (DDEs) in the form of

$$\frac{d^\alpha \zeta_n}{dt^\alpha} = (\varepsilon + \beta\zeta_n + \gamma\zeta_n^2)(\zeta_{n+1} + \zeta_{n-1}),\ 0 < \alpha \leq 1. \tag{1}$$

Where d_t^α denotes jumariess modified Riemann-Liouville derivative (in time) of order α. Here depended variables of ζ_n are considered to be function of $\zeta(n, t)$ of lattice variables, n represents the n-th lattice and ε, β and γ are arbitrary constants.

Marinca, Herisanu, et al. introduced Optimal Homotopy Asymptotic Method (OHAM) for solving Nonlinear differential equations [10, 11]. Since then, the method was continuously improved. It was shown that in order to improve the accuracy of the results and also in order to ensure a faster convergence, one can use an increased number of convergence-control parameters in the first order of approximation [12, 13]. Recently

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the proposed method has been recently extended by Sarwar et al. for solution of differential equations of fractional order[14, 15].

The whole paper is divided into six sections. The first Section is kept for introduction and literature survey, the second section is devoted to basic definitions from the fractional calculus. The third section contains the fundamental theory of optimal Homotopy asymptotic method for general time fractional order (FDEs). In section four, the proposed method is tested upon time fractional (DDEs). In section five, the numerical results are compared with exact solutions, which show the accuracy of the proposed method. In the last section, a conclusion is provided.

2 Preliminaries

The present section allowed to a description of the purpose of acquainting with sufficient fractional calculus theory, to enable us to follow the solution of fractional order differential-difference equation. Numerous definitions and theories of fractional calculus have been suggested in last two centuries, the well-known definitions suggested by Riemann-Liouville, Welyl, Reize, Compos, and Caputo are given as follow.

Definition 1 *A real valued function $G(x),\ x > 0$ is said to be in space $C_\mu,\ \mu \in R$ if there a real number $p > \mu$ such that $G(x) = x^p G_1(x)$, where $G_1(x) \in C(0, \infty)$, and it is said to be in the space C_μ^m if only if $G^{(m)} \in C_\mu,\ m \in N$.*

Definition 2 *The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$ of a function $G \in C_\mu,\ \mu \geq -1$ is defined as*

$$\begin{cases} J_a^\alpha G(x) = \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_a^x (x-\mu)^{\alpha-1} G(\mu)\ d\mu,\ \alpha x > 0, \\ J_a^0 G(x) = G(x). \end{cases}$$

When we formulate the model of real world problems with fractional calculus, the Riemann-Liouville have certain disadvantages. Caputo proposed a modified fractional differential operator D_a^α in his work on the theory of viscoelasticity.

Definition 3 *The fractional derivative of $G(x)$ in the Caputo sense is defined as*

$$D_a^\alpha G(x) = J_a^{m-\alpha} D^m G(x) = \frac{1}{\Gamma(m-a)} \int_a^x (x-\eta)^{m-\alpha-1} G^{(m)}(\eta) d\eta,$$

where $m-1 < \alpha \leq m,\ m \in N,\ x > 0,\ G \in C_{-1}^m$.

Definition 4 *If $m-1 < \alpha \leq m,\ m \in N$ and $G \in C_\mu^m,\ \mu \geq -1$, then*

$$D_a^\alpha J_a^\alpha G(x) = G(x) \quad \text{and} \quad J_a^\alpha D_a^\alpha G(x) = G(x) - \sum_{k=0}^{m-1} G^{(k)}(x) \frac{(x-a)^k}{k!},\ x > 0.$$

Properties of operator J^α can be found in [3, 4, 5, 6]. We mention the following: For $G \in C_\mu^m, \alpha, \beta > 0,\ \mu \geq -1,\ \gamma \geq -1$,

$$\begin{cases} J_a^\alpha J_a^\beta G(x) = J_a^{\alpha+\beta} G(x), \\ J_a^\alpha J_a^\beta G(x) = J_a^\beta J_a^\alpha G(x), \\ J_a^\alpha (x-a)^\gamma = \dfrac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} (x-a)^{\alpha+\gamma}. \end{cases}$$

3 Analysis of (OHAM) for Fractional Order Differential Difference Equations (FDDEs)

To clarify the fundamental theory of Optimal Homotopy asymptotic method for time fractional order differential difference equation, we consider the following general fractional order (FDDE), given by eq. (2):

$$\frac{\partial^\alpha \zeta(r,t)}{\partial t^\alpha} = A(\zeta(r,t)) + F(r,t), \quad \alpha > 0, \tag{2}$$

subject to the conditions

$$\begin{cases} \zeta(r,0) = \varphi_0(r), \ 0 \le r \le L, \\ \zeta(a,t) = \varphi_1(r), \ \zeta(b,t) = \varphi_2(r), \ 0 \le t \le T. \end{cases}$$

In above equations $\frac{\partial^\alpha}{\partial t^\alpha}$ is the Caputo or Riemann-Liouville fraction derivative operator, A represents the differential operator, and $\zeta(r,t)$ is unknown function. The function $F(r,t)$ represents an analytic function. r and t are spatial and temporal variables.

(a) Construct an optimal Homotopy for eq. (2), $\phi(r,t;p) : \Omega \times [0,1] \longrightarrow R$ which satisfies:

$$(1-p)\left(\frac{\partial^\alpha \phi(r,t)}{\partial t^\alpha} - F(r,t)\right) - H(r,p;C)\left(\frac{\partial^\alpha \phi(r,t)}{\partial t^\alpha} - A(\phi(r,t)) + F(r,t)\right) = 0. \tag{3}$$

In eq. (3), $p \in [0,1]$ is the embedding parameter and $H(r,p;C)$ is auxiliary function. For $p \neq 0, \ H(r,0) = 0$. When p increases in the interval $[0,1]$, the solution $\phi(r,t)$ converges rapidly to the exact solution.

Remark 1 *The efficiency of OHAM, which does not need a small and large parameter in equation, is based on the construction and determination of the auxiliary functions combined with a convenient way to optimally control the convergence of the solution. The auxiliary function $H(r,p;C)$ provides us the simple way to adjust and control the convergence and also increases the accuracy of the results and effectiveness of the method. The auxiliary function $H(r,p)$ takes the following form*

$$H(r,p) = pk_1(r,C_i) + p^2 k_2(r,C_i) + p^3 k_3(r,C_i) + \ldots + p^m k_m(r,C_i).$$

In above equation $C_i, \ i = 1,2,\ldots$ are the convergence control parameters and each $k_i(r), \ i = 1,2,\ldots$ is a function of r.

(b) By expanding $\phi(r,t,p;C_i)$ in Taylor's series about p, we obtain the approximate solution

$$\phi(r,t;C_i) = \zeta_0(r,t) + \sum_{k=1}^{m} \zeta_k(r,t;C_i)p^k, \quad i = 1,2,\ldots. \tag{4}$$

Remark 2 *It is clear that the convergence of the series equation (4) depends upon the auxiliary convergence control parameters. If it converges at $p = 1$, then we have*

$$\zeta(r,t;C_i) = \zeta_0(r,t) + \sum_{k=1}^{\infty} \zeta_k(r,t;C_i), \quad i = 1,2,\ldots. \tag{5}$$

(c) Putting eq. (5) in eq. (3) and comparing the coefficients of same power of p, we get zero order, 1st order, and the 2nd order problems as follow.

$$p^0 : \frac{\partial^\alpha \zeta_0(r,t)}{\partial t^\alpha} - F(r,t) = 0,$$

$$p^1 : \frac{\partial^\alpha \zeta_1(r,t,C_1)}{\partial t^\alpha} - (1+C_1)\frac{\partial^\alpha \zeta_0(r,t)}{\partial t^\alpha} + (1+C_1)F(r,t) + C_1 A(\zeta_0(r,t)) = 0,$$

$$p^2 : \frac{\partial^\alpha \zeta_2(r,t,C_1,C_2)}{\partial t^\alpha} - (1+C_1)\frac{\partial^\alpha \zeta_1(r,t,C_1)}{\partial t^\alpha} - C_2\frac{\partial^\alpha \zeta_(r,t)}{\partial t^\alpha} + C_1 A(\zeta_1(r,t,C_1)) + C_2(F(r,t)) + A(\zeta_0(r,t)) = 0.$$

Remark 3 *The time fractional derivatives are involved in the above zeorth order, first order and second order problems. We also know that there does not exist the Caputo fractional integral order operator.*

(d) By applying theJ^α operator on the above fractional order problems, we obtain a series of solutions as follow

$$\zeta_0(r,t) = J^\alpha(F(r,t)) = 0,$$

$$\zeta_1(r,t;C_1) = J^\alpha\left((1+C_1)\frac{\partial^\alpha \zeta_0(r,t)}{\partial t^\alpha} - C_1 A(\zeta_0(r,t))\right),$$

$$\begin{aligned}\zeta_2(r,t;C_1,C_2) &= J^\alpha\Bigg((1+C_1)\frac{\partial^\alpha \zeta_1(r,t;C_1)}{\partial t^\alpha} + C_2\frac{\partial^\alpha \zeta_0(r,t)}{\partial t^\alpha} \\ &\quad - C_1 A(\zeta_1(r,t;C_1)) - C_2(F(r,t) + A(\zeta_0(r,t)))\Bigg).\end{aligned}$$

By putting the above solutions in (5), one can get the approximate solution

$$\tilde{\zeta}_n(n,t;C_i) = \zeta_{n0}(n,t) + \zeta_{n1}(n,t;C_1) + \zeta_{n2}(n,t;C_1,C_2) + ... \tag{6}$$

The residual $R(r,t;C_i)$ is obtained by substituting (6) into (2).

(e) The convergence control parameters C_1, C_2, ... can be found by using either: the Least Squares Homotopy Perturbation Method (LSHPM) [16], collocation method, or the Optimal Auxiliary Functions Method (OAFM) [17].

To find the optimal values of auxiliary convergence control parameters in which we first construct the functional

$$\chi(C_i) = \int_0^t \int_\Omega R^2(r,t;C_i)\ drdt$$

and then calculate the optimum values of auxiliary convergence control parameters C_i by solving the following system of equations.

$$\frac{\partial \chi}{\partial C_1} = \frac{\partial \chi}{\partial C_2} = ... = \frac{\partial \chi}{\partial C_m} = 0.$$

The approximate solution is obtained by putting the optimum values of the auxiliary convergence control parameters in (6).

(f) **Collocation method:** in this paper we used collocation method to find the auxiliary constants C_i. This is done by taking points S_i in the problem domain then solving the system

$$R(S_i, C_1, C_2, ..., C_m) = 0, \quad i = 1,2,...,m.$$

4 Governing Equations

In this section, to illustrate the efficiency and accuracy of the proposed method, we have achieved approximate solution of fractional order differential difference equations. All computational work is done with the help of Mathematica 10.

Example 1 *If we put $\varepsilon = \beta = 0$ and $\gamma = 1$, then (1) becomes with appropriate initial condition as,*

$$\begin{cases} \dfrac{\partial \zeta_n(n,t)}{\partial t^\alpha} = (\zeta_n(n,t))^2\,(\zeta_{n+1}(n,t) - \zeta_{n-1}(n,t)) = 0, \quad 0 < \alpha \leq 1, \\ \zeta_n(n,0) = 1 - \dfrac{1}{n^2}. \end{cases} \tag{7}$$

For special case $\alpha = 1$, exact solution for (7) is found by [18] as

$$\zeta_n(n,t) = \frac{n}{1-2t}.$$

Using the (OHAM) formulation discussed in section 3, we get the following zeroth order, first order, second order and third order problems with initial conditions as:

Zeroth order problem:

$$\frac{\partial^\alpha \zeta_{n0}(n,t)}{\partial t^\alpha} = 0,\ \zeta_{n0}(n,0) = 1 - \frac{1}{n^2}. \tag{8}$$

First order problem:

$$\begin{cases} \dfrac{\partial^\alpha \zeta_{n1}(n,t;C_1)}{\partial t^\alpha} = C_1\zeta_{(n-1)0}(n,t)\zeta_n^2(n,t) - C_1\zeta_{(n+1)0}(n,t)\zeta_{n0}^2(n,t) \\ \qquad\qquad + \dfrac{\partial^\alpha \zeta_{n0}(n,t)}{\partial t^\alpha} + \dfrac{\partial^\alpha \zeta_{n0}(n,t)}{\partial t^\alpha}, \\ \zeta_{n1}(n,0) = 0. \end{cases} \tag{9}$$

Second order problem:

$$\begin{cases} \dfrac{\partial^\alpha \zeta_{n2}(n,t;C_1,C_2)}{\partial t^\alpha} = C_1\zeta_{(n-1)1}(n,t)\zeta_{n0}^2(n,t) + C_2\zeta_{(n-1)0}(n,t)\zeta_n^2(n,t) \\ \qquad - C_2\zeta_{(n+1)0}(n,t)\zeta_n^2(n,t) - C_1\zeta_{(n+1)1}(n,t)\zeta_{n0}^2(n,t) \\ \qquad + 2C_1\zeta_{(n-1)0}(n,t)\zeta_0(n,t)\zeta_1(n,t) - 2C_1\zeta_{(n+1)0}(n,t)\zeta_0(n,t)\zeta_0(n,t) \\ \qquad + C_2\dfrac{\partial^\alpha \zeta_{n0}(n,t)}{\partial t^\alpha} + \dfrac{\partial^\alpha \zeta_{n1}(n,t)}{\partial t^\alpha} + C_1\dfrac{\partial^\alpha \zeta_{n1}(n,t)}{\partial t^\alpha}, \\ \zeta_{n2}(n,0) = 0. \end{cases} \tag{10}$$

Third order problem:

$$\begin{cases} \dfrac{\partial^\alpha \zeta_{n3}(n,t,C_1,C_2,C_3)}{\partial t^\alpha} = C_1\zeta_{(n-1)2}(n,t)\zeta_{n0}^2(n,t) + C_2\zeta_{(n-1)1}(n,t)\zeta_{n0}^2(n,t) \\ \qquad + C_3\zeta_{(n-1)0}(n,t)\zeta_{n0}^2(n,t) - C_3\zeta_{(n-1)0}(n,t)\zeta_{(n+1)0}^2(n,t) \\ \qquad + C_3\zeta_{(n+1)0}(n,t)\zeta_0(n,t)\zeta_{n0}^2(n,t) - C_2\zeta_{(n+1)1}(n,t)\zeta_{n0}^2(n,t) \\ \qquad + C_1\zeta_{(n+1)2}(n,t)\zeta_{n0}^2(n,t) + 2C_1\zeta_{(n-1)1}(n,t)\zeta_{n0}(n,t)\zeta_{n1}(n,t) \\ \qquad + 2C_1\zeta_{(n-1)0}(n,t)\zeta_{n0}(n,t)\zeta_{n1}(n,t) - 2C_1\zeta_{(n+1)0}(n,t)\zeta_{n1}(n,t)\zeta_{n0}(n,t) \\ \qquad + 2C_1\zeta_{(n+1)1}(n,t)\zeta_{n0}(n,t)\zeta_{n1}(n,t) \\ \qquad + c_1\zeta_{(n-1)0}(n,t)\zeta_{n1}^2(n,t)C_1\zeta_{(n+1)0}(n,t)\zeta_{n1}^2(n,t) \\ \qquad + 2C_1\zeta_{(n-1)0}(n,t)\zeta_{n0}(n,t)\zeta_{n2}(n,t) - 2C_1\zeta_{(n+1)0}(n,t)\zeta_{n0}(n,t)\zeta_{n2}(n,t) \\ \qquad + C_3\dfrac{\partial^\alpha \zeta_{n0}(n,t)}{\partial t^\alpha} + C_2\dfrac{\partial^\alpha \zeta_{n1}(n,t)}{\partial t^\alpha} + \dfrac{\partial^\alpha \zeta_{n2}(n,t)}{\partial t^\alpha} = 0, \\ \zeta_{n3}(n,0) = 0. \end{cases} \tag{11}$$

Using the inverse J^α operator both sides of the above problems (8)–(11), we get the zeroth order, first order, second order and third order problems solutions as

$$\zeta_{n0}(n,t) = 1 - \frac{1}{n^2},$$

$$\zeta_{n1}(n,t;C_1) = -\frac{4C_1 t^\alpha}{n^3\Gamma(1+\alpha)},$$

$$\zeta_{n2}(n,t;C_1,C_2) = -\frac{4t^\alpha}{n^4}\left(-\frac{(C_1+C_1^2+C_2)(n-1)}{\Gamma(1+\alpha)} - \frac{6c_1^2 t^\alpha}{\Gamma(1+2\alpha)}\right),$$

$$\begin{aligned}\zeta_{n3}(n,t;C_1,C_2,C_3) \quad = \quad & -\frac{4t^\alpha}{n^5\Gamma(1+\alpha)^2}\Bigg(-C_2 - C_1(1+C_1)^2 - 2C_2 + C_3)n^2\Gamma(1+\alpha) \\ & -\frac{3\times 4^{1-\alpha}C_1(C_1+C_1^2+C_2)n\sqrt{\pi}t^\alpha\Gamma(1+\alpha)}{\Gamma(0.5+\alpha)} \\ & -\frac{48C_1^2t^{2\alpha}((1+n^2)\Gamma(1+\alpha))^2 - n^2\Gamma(1+2\alpha)}{(n^2-1)^2\Gamma(1+3\alpha)}\Bigg).\end{aligned}$$

The third order approximate solution obtained by (OHAM) is given by the following expression:

$$\zeta_n(n,t;C_i) = \zeta_{n0}(n,t) + \zeta_{n1}(n,t;C_1) + \zeta_{n2}(n,t;C_1,C_2) + \zeta_{n3}(n,t;C_1,C_2,C_3).$$

For residuals, we take the following form

Table 1: Auxiliary convergence-control parameters for Example 4.1 for different value of α.

α	C_1	C_2	C_3
0.5	-0.90262240486192488	-0.06573503341682661	0.011509204699581811
1.0	-0.10545358851366554	-0.24335683749370854	-0.28329952908692363

$$R(n,t;C_i) = D_t^\alpha\zeta_n(n,t) - \zeta_n^2(n,t;C_1,C_2,C_3)\Big(\zeta_{n+1}(n,t;C_1,C_2,C_3) - \zeta_{n-1}(n,t;C_1,C_2,C_3)\Big). \qquad (12)$$

Where $\zeta_{n+1}(n,t;C_1,C_2,C_3)$ and $\zeta_{n-1}(n,t;C_1,C_2,C_3)$ are obtained by replacing $n+1$ and $n-1$ in equation (12). To find the values of constants C_1, C_2 and C_3 we used collocation method given in section 3 shown in equation (12), for different values of α . Using the auxiliary constants given in table 1, we obtain the third order approximate solution.

Example 2 *If we put $\varepsilon = 1$, $\beta = 0$, then (1) becomes with appropriate initial condition as,*

$$\begin{cases} \dfrac{\partial\zeta_n(n,t)}{\partial t^\alpha} = (1-\zeta_n(n,t))^2\left(\zeta_{n+1}(n,t) - \zeta_{n-1}(n,t)\right) = 0, \;\; 0<\alpha\le 1, \\ \zeta_n(n,0) = \tanh(k)\tanh(kn). \end{cases} \qquad (13)$$

For $\alpha = 1$, exact solution for (13) is found by [19] as

$$\zeta_n(n,t) = \Big(\tanh(k)\tanh(kn + 2\tanh k)t)\Big).$$

Using the same procedure as above, we get the following zero order, first order and second order problems with initial conditions as:

Zeroth order problem:

$$\frac{\partial^\alpha\zeta_{n0}(n,t)}{\partial t^\alpha} = 0,\;\; \zeta_{n0}(n,0) = \tanh(k)\tanh(kn).$$

First order problem:

$$\begin{cases} C_1\dfrac{\partial^\alpha\zeta_{(n-1)0}(n,t)}{-}C_1\zeta_{(n+1)0}(n,t) + C_1\zeta_{(n-1)0}(n,t)\zeta_{n0}^2(n,t) + C_1\zeta_{(n+1)0}(n,t)\zeta_{n0}^2(n,t) \\ +\dfrac{\partial^\alpha\zeta_{n0}(n,t)}{\partial t^\alpha} + \dfrac{\partial^\alpha\zeta_{n1}(n,t)}{\partial t^\alpha}, \\ \zeta_{n1}(n,0) = 0. \end{cases}$$

Second order problem:

$$\begin{cases} \dfrac{\partial^\alpha \zeta_{n2}(n,t;C_1,C_2)}{\partial t^\alpha} = C_1\zeta_{(n-1)1}(n,t) + C_2\zeta_{(n-1)0}(n,t) - C_2\zeta_{(n+1)0}(n,t) - C_1\zeta_{(n+1)0}(n,t) \\ \qquad - C_1\zeta_{(n-1)1}(n,t)\zeta_0^2(n,t) - C_2\zeta_{(n-1)0}(n,t)\zeta_{n0}^2(n,t) + C_2\zeta_{(n+1)0}(n,t)\zeta_{n0}^2(n,t) \\ \qquad + C_1\zeta_{(n+1)1}(n,t)\zeta_{n0}^2(n,t) + 2C_1\zeta_{(n-1)0}(n,t)\zeta_{n0}(n,t)\zeta_{n1}(n,t) \\ \qquad +2C_1\zeta_{(n+1)0}(n,t)\zeta_{n1}(n,t) + C_2\dfrac{\partial^\alpha \zeta_{n0}(n,t)}{\partial t^\alpha} + \dfrac{\partial^\alpha \zeta_{n1}(n,t)}{\partial t^\alpha} + C_1\dfrac{\partial^\alpha \zeta_{n0}(n,t)}{\partial t^\alpha} = 0\,, \\ \zeta_{n2}(n,0) = 0. \end{cases}$$

Solution of the above problems are given as follows:

$$\zeta_{n0}(n,t) = \tanh(k)\tanh(kn),$$

$$\zeta_{n1}(n,t;C_1) = \frac{-2C_1 t^\alpha sech^2(kn)\tanh^2(k)}{\Gamma(1+\alpha)},$$

$$\begin{aligned} \zeta_{n2}(n,t;C_1,C_2) \quad = \quad & \frac{4^{-\alpha}t^\alpha\tanh(k)}{\Gamma(1+\alpha)\Gamma(0.5+\alpha)}\Bigg(0.5C_1^2\sqrt{\pi}t^\alpha\cosh k(4-3n) \\ & -\cosh k(-4+n) + 2\cosh k(-2+n) + \cosh(kn) - \cosh(3kn) \\ & -2\cosh k(2+n)\sec h^2(k)\sec h^2 k(-1+n)\sec h^3(kn)\tanh(k) \\ & -4^\alpha(C_1+C_1^2+C_2)\Gamma(0.5+\alpha) - 1 + \tanh^2(k)\tanh^2(kn)\Big(\tanh k(-1+n) \\ & -\tanh k(1+n)\Big)\Bigg)\,. \end{aligned}$$

The second order approximate solution obtained by (OHAM) is given by the following expression:

$$\zeta_n(n,t;C_i) = \zeta_{n0}(n,t) + \zeta_{n1}(n,t;C_1) + \zeta_{n2}(n,t;C_1,C_2). \tag{14}$$

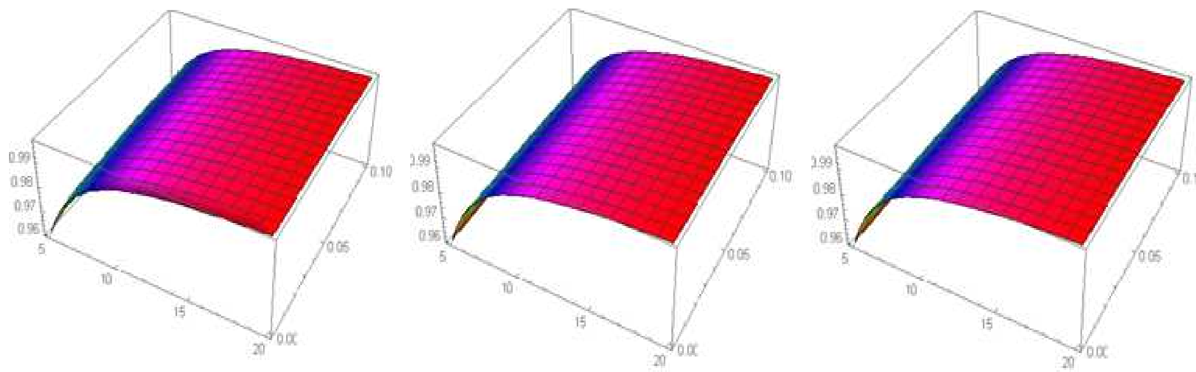
To find the values of constants C_1 and C_2 we used collocation method given in section 3, shown in equation (14), for different values of α. Using the auxiliary constants given in table 2, we obtain the second order approximate solution.

Table 2: Auxiliary convergence-control parameters for Example 4.2 for different value of α.

α	C_1	C_2
0.5	-0.7312545307427631	0.009962161348225711
1.0	-0.851050312175699	0.008851033836105394

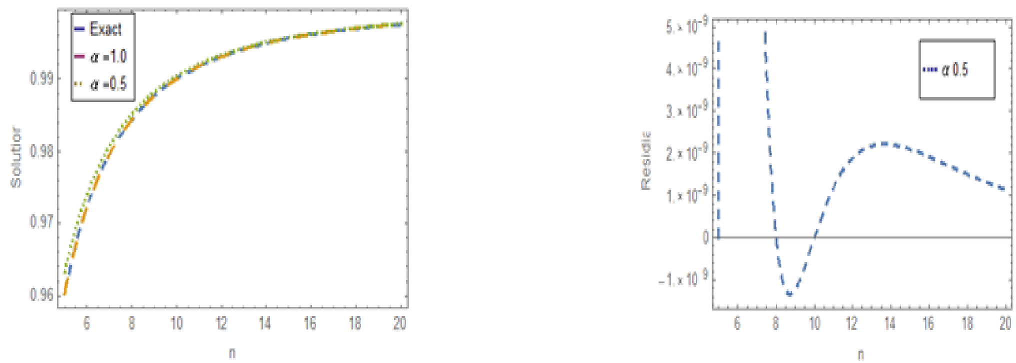
Table 3: Third order approximate solution obtained by (OHAM) for Example 4.1 at $t = 0.01$.

n	Solution $\alpha = 0.5$	Solution $\alpha = 1$	Solution Exact	Residual $\alpha = 0.5$	Absolute error for $\alpha = 1$
-20	0.997442	0.997495	0.997495	1.54298×10^{-8}	7.40546×10^{-9}
-10	0.989521	0.98996	0.98996	2.05193×10^{-7}	3.94379×10^{-9}
10	0.990426	0.99004	0.99004	-1.34441×10^{-17}	2.24183×10^{-7}
20	0.99755	0.997505	0.997505	1.13003×10^{-9}	2.11704×10^{-8}



(a) 3D plot of (OHAM) solution for fractional Example 1 equation at $\alpha = 0.5$ (b) 3D plot of (OHAM) solution for fractional Example 1 equation at $\alpha = 1$ (c) 3D plot of exact solution for fractional Example 1 equation at $\alpha = 1$

Figure 1: Numerical Results at different α for Example 1.

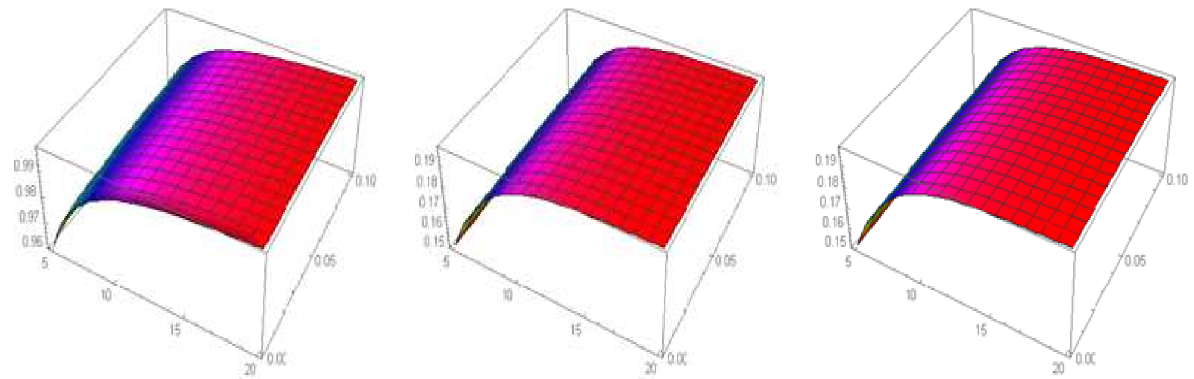


(a) The curves show the comparison between exact solution and approximate solution for different values of α at $t = 0.5$ for Example 1 (b) Residual obtained by OHAM for at $t = 0.5$, for Example 1

Figure 2: Numerical Results for Example 1.

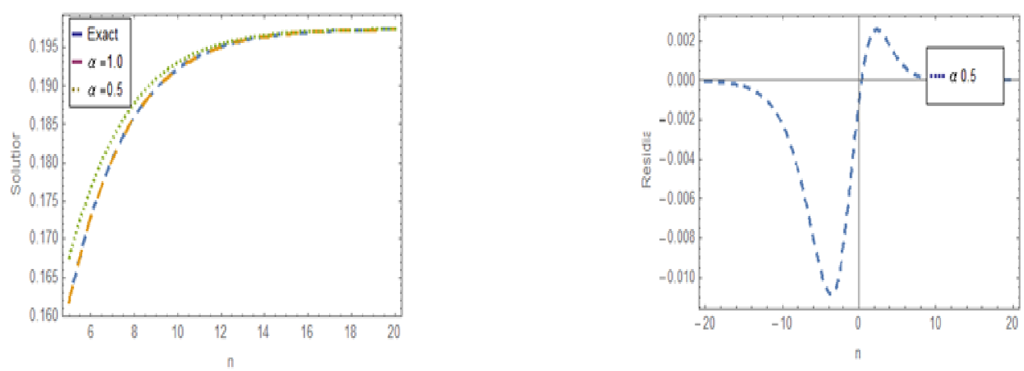
5 Results and Discussions

The extension of (OHAM) scheme for (DDEs) presented in section 3 and numerical examples of the formulation in the examples given in section 4, provides highly accurate solution without any small parameter assumption and discretization for the problems. For finding the numerical approximate solutions of fractional order differential difference equations, Tables (1-2) show the values of Auxiliary convergence-control parameters for fractional order DE equations for different value of α for Examples 4.1 and 4.2. Table 3 shows the approximate of 3rd order OHAM solution with exact solution equation for different values of α for Example 1. Table 4 shows the approximate of 2nd order (OHAM) solution with exact solution equation for different values of α for example at $t = 0.01$. Figures 1(a,b,c) show the 3D plots for 3rd order (OHAM) and exact solutions for Example 1 at $\alpha = 0.5$ and 1. Figure 2(a) shows the 2D plot of 3rd order (OHAM) solution for different values of α. Figure 2(b) shows the residual obtained by (OHAM) for Example 1 at $\alpha = 0.5$. Figures 3(a,b,c) show the 3D plots for 2nd order (OHAM) and exact solutions Example 2 at $\alpha = 0.5$ and 1. Figure 4(a) shows the 2D plot of 2nd order (OHAM) solution for different values of α for Example 2. Figure 4(b) shows the residual obtained by (OHAM) for Example 2.



(a) 3D plot of (OHAM) solution for Example 4.2 at $\alpha = 0.5$ (b) 3D plot of (OHAM) solution for Example 4.2 at $\alpha = 1$ (c) 3D plot of exact solution for Example 4.2 at $\alpha = 1$

Figure 3: Numerical Results at different α for Example 2.



(a) The curves show the comparison between exact solution and approximate solution for different values of α at $t = 0.4$ for Example 2 (b) Residual obtained by (OHAM) for at $t = 0.4$, for Example 2

Figure 4: Numerical Results for Example 2.

The results obtained by 3rd order and 2nd order (OHAM) solution for time fractional DDEs are in good agreement to the exact solution even if we take less terms.

6 Conclusion

From the numerical results, we concluded that (OHAM) converges rapidly to the exact solution at lower order of approximations for (DDEs). The results obtained by proposed method are very encouraging in comparison with exact solution. The proposed technique is capable and delivers very high accuracy at lower order of approximations. This technique does not require discretization like other numerical and approximate methods and also free from small parameter assumption. The convergence of this method do not depend upon the initial approximation. The accuracy of proposed method can further be increased by taking higher order approximations, and as a result it will be more appealing for researchers to apply this method for solving fractional order physical models arising in science and engineering.

Table 4: Second order approximate solution obtained by (OHAM) for Example 4.2 at $t = 0.4$ and $k = 0.2$.

n	Solution $\alpha = 0.5$	Solution $\alpha = 1$	Solution Exact	Residual $\alpha = 0.5$	Absolute error for $\alpha = 1$
-20	-0.197157	-0.197198	-0.197194	-4.50088×10^{-5}	3.86672×10^{-6}
-10	-0.185772	-0.187899	-0.187703	-2.25508×10^{-3}	1.95759×10^{-4}
10	0.192986	0.192166	0.192173	-4.1345×10^{-5}	6.54582×10^{-6}
20	0.197294	0.197279	0.197279	1.03601×10^{-7}	1.20823×10^{-7}

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