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Three Explicit Non-Algebraic Limit Cycles For A Class Of Polynomial Differential Systems*

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Abstract

The existence of limit cycles is interesting and very important in applications. It is a key to understand the dynamic of polynomial differential systems. The aim of this paper is to investigate a class of planar differential systems of degree $6n+1$ where n is strictly positive integer. Under some suitable conditions, the existence of three non algebraic limit cycles or three algebraic limit cycles. Furthermore, these limit cycles are explicitly given in polar coordinates. Some examples are presented in order to illustrate the applicability of our results.

1 Introduction

We consider two-dimensional polynomial differential systems of the form

$$\begin{cases} \dot{x} = \dfrac{dx}{dt} = P(x,y), \\ \dot{y} = \dfrac{dy}{dt} = Q\left(x,y\right) \end{cases} \tag{1}$$

where P and Q are polynomials of $R[x,y]$ ($R\left[x,y\right]$ denotes the ring of polynomials in the variables x and y with real coefficients). A limit cycle of system (1) is an isolated periodic orbit and it is said to be algebraic if it is contained in the zero set of an algebraic curve, otherwise it is called non-algebraic.

System (1) is integrable on an open set of $\mathbb{R}^2$ if there exists a non constant continuously differentiable function $H:\Omega \to \mathbb{R}$, called a first integral of the system on Ω, which is constant on the trajectories of the system (1) contained in Ω, i.e.

$$\frac{dH(x,y)}{dt} = P\left(x,y\right)\frac{\partial H(x,y)}{\partial x} + Q\left(x,y\right)\frac{\partial H(x,y)}{\partial y} \equiv 0.$$

Moreover, $H=h$ is the general solution of this equation, where h is an arbitrary constant. It is well known that for differential systems defined on the plane $\mathbb{R}^2$ the existence of a first integral determines their phase portrait.

An important problem of the qualitative theory of differential equations is to determine the limit cycles of a system of the form (1). We usually only ask for the number of such limit cycles, but their location as orbits of the system is also an interesting problem. And an even more difficult problem is to give an explicit expression of them. Until recently, the only limit cycles known in an explicit way were algebraic (see, for example, [5] and references therein). After the work of Odani [7], who proved without giving its explicit expression, that the limit cycle appearing in the van der Pol equation is not algebraic, several papers have been published exhibiting polynomial differential systems for which non-algebraic limit cycles are known explicitly (see, [4], [1], [6] and [3]).

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Subsequently, the authors' interests converted to the coexistence of algebraic and non-algebraic limit cycles. (see, [6] for $n = 9$ and [2] for $n = 5$).

In this paper we are interested in studying the integrability and the limit cycles of systems of the form

$$\begin{aligned} \dot{x} &= x + n\left(\alpha x\left(x^2+y^2-\gamma\right) - 2y\left(3x^2+3y^2-\gamma\right)\right)\left(x^2+y^2-\gamma\right)^{2n-1} \\ &\quad\times\left(ax^2+ay^2+bx^2-by^2\right)^n, \\ \dot{y} &= y + n\left(\alpha y\left(x^2+y^2-\gamma\right) + 2x\left(3x^2+3y^2-\gamma\right)\right)\left(x^2+y^2-\gamma\right)^{2n-1} \\ &\quad\times\left(ax^2+ay^2+bx^2-by^2\right)^n \end{aligned} \tag{2}$$

where a, c and γ are real constants and n is strictly positive integer ($n \in \mathbb{N}^*$). Moreover, we determine sufficient conditions for a polynomial differential system to possess an explicit three non-algebraic or three algebraic limit cycles. Concrete examples exhibiting the applicability of our result are introduced.

2 Main Result

As a main result, we shall prove the following theorem.

Theorem 1 *Consider polynomial differential system (2). Then the following statements hold.*

(1) *If $a > |b|, b \neq 0, \alpha < 0$, and $4^n n\alpha\gamma^{3n}(a-|b|)^n + 27^n < 0$, then system (2) has three explicit non algebraic limit cycles, given in polar coordinates (r, θ) by*

$$r_1(\theta) = \sqrt{\tfrac{2}{3}\gamma\left(1+\cos\left(\tfrac{1}{3}\arccos\left(\tfrac{27}{2\gamma^3}\left(\rho(\theta)-\tfrac{2}{27}\gamma^3\right)\right)\right)\right)},$$

$$r_2(\theta) = \sqrt{\tfrac{2}{3}\gamma\left(1+\cos\left(\tfrac{1}{3}\arccos\left(\tfrac{27}{2\gamma^3}\left(\rho(\theta)-\tfrac{2}{27}\gamma^3\right)\right)-\tfrac{2\pi}{3}\right)\right)},$$

$$r_3(\theta) = \sqrt{\tfrac{2}{3}\gamma\left(1+\cos\left(\tfrac{1}{3}\arccos\left(\tfrac{27}{2\gamma^3}\left(\rho(\theta)-\frac{2}{27}\gamma^3\right)\right)-\tfrac{4\pi}{3}\right)\right)}$$

where

$$\rho(\theta) = e^{\alpha n\theta}\left(\frac{e^{2\alpha n\pi}f(2\pi)}{1-e^{2\alpha n\pi}} + f(\theta)\right) \quad \text{and} \quad f(\theta) = \int_0^\theta \tfrac{e^{-\alpha n s}}{(a+b\cos 2s)^n}ds.$$

(2) *If $a > 0, b = 0, \alpha < 0$, and $4^n n\alpha\gamma^{3n}a^n + 27^n < 0$, then system (2) has three explicit algebraic limit cycles, given in Cartesian coordinates (x, y), by*

$$\left(x^2+y^2\right)^n\left(x^2+y^2-\gamma\right)^{2n} + \frac{1}{a^n n\alpha} = 0.$$

Proof. Firstly, we have

$$x\dot{y} - y\dot{x} = 2n\left(x^2+y^2\right)\left(3x^2+3y^2-\gamma\right)\left(x^2+y^2-\gamma\right)^{2n-1}\left(ax^2+ay^2+bx^2-by^2\right)^n.$$

Thus, since $a > |b|$ then, the equilibrium points of system (2) are present on the curve

$$\left(3x^2+3y^2-\gamma\right)\left(x^2+y^2-\gamma\right)^{2n-1} = 0. \tag{3}$$

Proof of statement (1) of Theorem 1. The polynomial differential system (2) in polar coordinates becomes

$$\dot{r} = r + nr\alpha r^{2n}\left(r^2-\gamma\right)^{2n}(a+b\cos 2\theta)^n, \tag{4}$$

$$\dot{\theta} = 2nr^{2n}\left(r^2-\gamma\right)^{2n-1}\left(3r^2-\gamma\right)(a+b\cos 2\theta)^n.$$

The differential system (4) where $2nr^{2n}\left(r^2-\gamma\right)^{2n-1}\left(3r^2-\gamma\right)(a+b\cos 2\theta)^n \neq 0$ can be written as the equivalent differential equation

$$\frac{dr}{d\theta} = \frac{r + n\alpha r^{2n-1}\left(-\gamma+r^2\right)^{2n}(a+b\cos 2\theta)^n}{2nr^{2n}\left(r^2-\gamma\right)^{2n-1}\left(3r^2-\gamma\right)(a+b\cos 2\theta)^n}. \tag{5}$$

Via the change of variables $\rho = \left(r^2\left(r^2-\gamma\right)^2\right)^n$, the differential equation (5) becomes the linear differential equation

$$\frac{d\rho}{d\theta} = n\alpha\rho + \frac{1}{(a+b\cos 2\theta)^n}. \tag{6}$$

The general solution of linear equation (6) is

$$\rho(\theta,h) = e^{\alpha n\theta}(h+f(\theta)),$$

where $h \in \mathbb{R}$, $f(\theta) = \int_0^\theta \frac{e^{-\alpha ns}}{(a+b\cos 2s)^n}ds$. Consequently, the implicit form of the solution of the differential equation (5) is

$$F(r,\theta) = r^{2n}(\theta)\left(r^2(\theta)-\gamma\right)^{2n} - e^{n\alpha\theta}(h+f(\theta)) = 0.$$

Notice that system (2) has a periodic orbit if and only if equation (5) has a strictly positive 2π-periodic solution. This, moreover, is equivalent to the existence of a solution of (5) that satisfies $r(0,r_*) = r(2\pi,r_*)$ and $r(\theta,r_*) > 0$ for any θ in $[0,2\pi]$.

We remark that the solution $r(\theta,r_0)$ of the differential equation (5) such as $r(0,r_0) = r_0 > 0$, corresponds to the value $h = r_0^{2n}\left(r_0^2-\gamma\right)^{2n}$, provided a rewriting of the implicit form of the solution of the differential equation (5) as

$$F(r,\theta) = r^{2n}(\theta)\left(r^2(\theta)-\gamma\right)^{2n} - \rho(\theta,r_0) = 0.$$

where

$$\rho(\theta,r_0) = e^{\alpha n\theta}\left(r_0^{2n}\left(r_0^2-\gamma\right)^{2n} + f(\theta)\right).$$

A periodic solution of system (4) must satisfy the condition $r(2\pi,r_0) = r(0,r_0)$.

Since $r^{2n}(\theta,r_0)\left(r^2(\theta,r_0)-\gamma\right)^{2n} = \rho(\theta,r_0)$, we see that $r(2\pi,r_0) = r(0,r_0)$ if and only if $\rho(2\pi,r_0) = \rho(0,r_0)$ and we have

$$\rho_0 = \rho(0,r_0) = r_0^{2n}\left(r_0^2-\gamma\right)^{2n} \quad \text{and} \quad \rho(2\pi,r_0) = e^{2\alpha n\pi}\left(r_0^{2n}\left(r_0^2-\gamma\right)^{2n} + f(2\pi)\right),$$

then, the condition $\rho(2\pi,r_0) = \rho(0,r_0)$ implies that

$$\rho_0 = r_0^{2n}\left(r_0^2-\gamma\right)^{2n} = \frac{e^{2\alpha n\pi}}{1-e^{2\alpha n\pi}}f(2\pi), \tag{7}$$

thus

$$\rho(\theta) = e^{\alpha n\theta}\left(\frac{e^{2\alpha n\pi}}{1-e^{2\alpha n\pi}}f(2\pi) + f(\theta)\right) \tag{8}$$

and the implicit form of the solution of (5) such that $r(2\pi,r_0) = r(0,r_0)$ can be written as

$$F(r,\theta) = r^{2n}(\theta)\left(r^2(\theta)-\gamma\right)^{2n} - \rho(\theta) = 0. \tag{9}$$

where

$$\rho(\theta) = e^{\alpha n\theta}\left(\frac{e^{2\alpha n\pi}f(2\pi)}{1-e^{2\alpha n\pi}} + f(\theta)\right) \quad \text{and} \quad f(\theta) = \int_0^\theta \frac{e^{-\alpha ns}}{(a+b\cos 2s)^n}ds.$$

Next we prove that $0 < \rho^{\frac{1}{n}}(\theta) < \dfrac{2\gamma^3}{27}$ for all $\theta \in \mathbb{R}$. Let $\varphi(\theta) = \left(\frac{4\gamma^3}{27}\right)^n e^{-\alpha n\theta} - f(\theta)$. We have

$$\frac{d\varphi(\theta)}{d\theta} = -\frac{1}{27^n}\frac{e^{-\alpha n\theta}}{(a+b\cos 2\theta)^n}\left(4^n n\alpha\gamma^{3n}(a+b\cos 2\theta)^n + 27^n\right).$$

Since $a > |b|$, $\alpha < 0$ and $\gamma > 0$, we see that

$$0 < a - |b| < a + b\cos 2\theta < a + |b|$$

and

$$4^n n\alpha\gamma^{3n}(a+|b|)^n + 27^n < 4^n n\alpha\gamma^{3n}(a+b\cos 2\theta)^n + 27^n < 4^n n\alpha\gamma^{3n}(a-|b|)^n + 27^n.$$

Since $4^n n\alpha\gamma^{3n}(a-|b|)^n + 27^n < 0$, it follows that $4^n n\alpha\gamma^{3n}(a+b\cos 2\theta)^n + 27^n < 0$ and the function $\theta \mapsto \varphi(\theta)$ is strictly increasing with

$$\varphi(0) = \left(\frac{4\gamma^3}{27}\right)^n < \left(\frac{4\gamma^3}{27}\right)^n e^{-\alpha n\theta} - f(\theta) < \varphi(2\pi) = \left(\frac{4\gamma^3}{27}\right)^n e^{-2\alpha n\pi} - f(2\pi). \tag{10}$$

Since $\left(\frac{4\gamma^3}{27}\right)^n < \left(\frac{4\gamma^3}{27}\right)^n e^{-2\alpha n\pi} - f(2\pi)$, we see that

$$f(2\pi) < \left(e^{-2\alpha n\pi} - 1\right)\left(\frac{4\gamma^3}{27}\right)^n = \frac{1-e^{2\alpha n\pi}}{e^{2\alpha n\pi}}\left(\frac{4\gamma^3}{27}\right)^n.$$

Taking into account (10) and since $\alpha < 0$, it follows that

$$\frac{e^{2\alpha n\pi}f(2\pi)}{1-e^{2\alpha n\pi}} < \left(\frac{4\gamma^3}{27}\right)^n < \left(\frac{4\gamma^3}{27}\right)^n e^{-\alpha n\theta} - f(\theta),$$

thus

$$\frac{e^{2\alpha n\pi}f(2\pi)}{1-e^{2\alpha n\pi}} < \left(\frac{4\gamma^3}{27}\right)^n e^{-\alpha n\theta} - f(\theta)$$

and

$$\begin{aligned} \rho(\theta) &= e^{\alpha n\theta}\left(\frac{e^{2\alpha n\pi}}{1-e^{2\alpha n\pi}}f(2\pi) + f(\theta)\right) \\ &< e^{\alpha n\theta}\left(\left(\frac{4\gamma^3}{27}\right)^n e^{-\alpha n\theta} - f(\theta) + f(\theta)\right) = \left(\frac{4\gamma^3}{27}\right)^n. \end{aligned}$$

Then $\rho^{\frac{1}{n}}(\theta) < \frac{4\gamma^3}{27}$ for all $\theta \in [0, 2\pi[$. On the other hand, because $f(\theta) > 0$ for all $\theta \in \mathbb{R}$ and $\alpha < 0$, we have

$$\rho(\theta) = e^{\alpha n\theta}\left(\frac{e^{2\alpha n\pi}}{1-e^{2\alpha n\pi}}f(2\pi) + f(\theta)\right) > 0.$$

From (9), we have $r^{2n}(\theta)\left(r^2(\theta)-\gamma\right)^{2n} = \rho(\theta)$, this is equivalent to $r^2(\theta)\left(r^2(\theta)-\gamma\right)^2 = \rho^{\frac{1}{n}}(\theta)$, then we have

$$\left(r^2\right)^3 - 2\gamma\left(r^2\right)^2 + \gamma^2 r^2 - \rho^{\frac{1}{n}} = 0. \tag{11}$$

This equation can be reduced by the transformation $R + \frac{2\gamma}{3} = r^2$ to the cubic equation

$$R^3 - \frac{1}{3}\gamma^2 R + \left(\frac{2}{27}\gamma^3 - \rho^{\frac{1}{n}}\right) = 0. \tag{12}$$

The discriminant of (12) is $\triangle = -\rho^{\frac{1}{n}}(\theta)\left(27\rho^{\frac{1}{n}}(\theta) - 4\gamma^3\right)$. Since $0 < \rho^{\frac{1}{n}}(\theta) < \frac{4\gamma^3}{27}$ for all $\theta \in [0, 2\pi[$, then $\triangle > 0$ for all $\theta \in \mathbb{R}$ and there are three different solutions of the equation (12), and thus the roots are given by (for more details see [9]):

$$R_1(\theta) = \tfrac{2}{3}\gamma\left(\cos\left(\tfrac{1}{3}\arccos\left(\tfrac{27}{2\gamma^3}\left(\rho^{\frac{1}{n}}(\theta) - \tfrac{2}{27}\gamma^3\right)\right)\right)\right),$$

$$R_2(\theta) = \tfrac{2}{3}\gamma\left(\cos\left(\tfrac{1}{3}\arccos\left(\tfrac{27}{2\gamma^3}\left(\rho^{\frac{1}{n}}(\theta) - \tfrac{2}{27}\gamma^3\right)\right) - \tfrac{2\pi}{3}\right)\right),$$

$$R_3(\theta) = \tfrac{2}{3}\gamma\left(\cos\left(\tfrac{1}{3}\arccos\left(\tfrac{27}{2\gamma^3}\left(\rho^{\frac{1}{n}}(\theta) - \tfrac{2}{27}\gamma^3\right)\right) - \tfrac{4\pi}{3}\right)\right).$$

Going back through the changes of variables, we obtain :

$$r_1(\theta) = \sqrt{\tfrac{2}{3}\gamma\left(1 + \cos\left(\tfrac{1}{3}\arccos\left(\tfrac{27}{2\gamma^3}\left(\rho^{\frac{1}{n}}(\theta) - \tfrac{2}{27}\gamma^3\right)\right)\right)\right)}, \tag{13}$$

$$r_2(\theta) = \sqrt{\tfrac{2}{3}\gamma\left(1 + \cos\left(\tfrac{1}{3}\arccos\left(\tfrac{27}{2\gamma^3}\left(\rho^{\frac{1}{n}}(\theta) - \tfrac{2}{27}\gamma^3\right)\right) - \tfrac{2\pi}{3}\right)\right)},$$

$$r_3(\theta) = \sqrt{\tfrac{2}{3}\gamma\left(1 + \cos\left(\tfrac{1}{3}\arccos\left(\tfrac{27}{2\gamma^3}\left(\rho^{\frac{1}{n}}(\theta) - \frac{2}{27}\gamma^3\right)\right) - \tfrac{4\pi}{3}\right)\right)}$$

where

$$\rho(\theta) = e^{\alpha n\theta}\left(\frac{e^{2\alpha n\pi} f(2\pi)}{1 - e^{2\alpha n\pi}} + \int_0^\theta \frac{e^{-\alpha ns}}{(a + b\cos 2s)^n} ds\right).$$

From (7), there are three different values with the property $r(2\pi, r_0) = r_0 > 0$,

$$r_{*1} = \sqrt{\tfrac{2}{3}\gamma\left(1 + \cos\left(\tfrac{1}{3}\arccos\left(\tfrac{27}{2\gamma^3}\left(\rho_0^{\frac{1}{n}} - \tfrac{2}{27}\gamma^3\right)\right)\right)\right)},$$

$$r_{*2} = \sqrt{\tfrac{2}{3}\gamma\left(1 + \cos\left(\frac{1}{3}\arccos\left(\frac{27}{2\gamma^3}\left(\rho_0^{\frac{1}{n}} - \tfrac{2}{27}\gamma^3\right)\right) - \tfrac{2\pi}{3}\right)\right)},$$

$$r_{*3} = \sqrt{\tfrac{2}{3}\gamma\left(1 + \cos\left(\frac{1}{3}\arccos\left(\tfrac{27}{2\gamma^3}\left(\rho_0^{\frac{1}{n}} - \tfrac{2}{27}\gamma^3\right)\right) - \tfrac{4\pi}{3}\right)\right)},$$

which are solution of $r_0^2\left(r_0^2 - \gamma\right)^2 = \rho_0^{\frac{1}{n}}$ where $\rho_0 = \frac{e^{2\alpha n\pi} f(2\pi)}{1 - e^{2\alpha n\pi}}$.

The implicit form of the solution of (5) can be written as

$$F(r, \theta) = r^2(\theta)\left(r^2(\theta) - \gamma\right)^2 - \rho^{\frac{1}{n}}(\theta) = 0 \tag{14}$$

where

$$\rho(\theta) = e^{\alpha n\theta}\left(\tfrac{e^{2\alpha n\pi} f(2\pi)}{1 - e^{2\alpha n\pi}} + \int_0^\theta \tfrac{e^{-\alpha ns}}{(a + b\cos 2s)^n} ds\right).$$

To show that $r_i(\theta)$, $i = 1, 2, 3$ are periodic solutions, we have to show that

a) there does not exist any singular point of (14),

b) the functions $\theta \mapsto r_i(\theta)$, $i = 1, 2, 3$ are 2π-periodic,

c) $r_i(\theta) > 0$, $i = 1, 2, 3$ for all $\theta \in [0, 2\pi[$.

a) We first prove that there is no singular point of (14). In particular we prove that the curve (3) does not intersect the orbit (14). This curve in polar coordinates becomes $\left(3r^2-\gamma\right)\left(r^2-\gamma\right)^{2n-1}=0$. To show this, we have to show that the system

$$\begin{cases} r^2\left(\theta\right)\left(r^2\left(\theta\right)-\gamma\right)^2-\rho^{\frac{1}{n}}\left(\theta\right)=0, \\ \left(3r^2\left(\theta\right)-\gamma\right)\left(r^2\left(\theta\right)-\gamma\right)^{2n-1}=0 \end{cases} \tag{15}$$

has no solutions. Indeed the equation $\left(3r^2-\gamma\right)\left(r^2-\gamma\right)^{2n-1}=0$ implies that $r^2\left(\theta\right)=\frac{\gamma}{3}$ or $r^2\left(\theta\right)=\gamma$. If $r^2\left(\theta\right)=\gamma$ substituting $r^2\left(\theta\right)=\gamma$ into $r^2\left(\theta\right)\left(r^2\left(\theta\right)-\gamma\right)^2-\rho^{\frac{1}{n}}\left(\theta\right)=0$, we get $\rho^{\frac{1}{n}}\left(\theta\right)=0$ which is a contradiction. If $r^2\left(\theta\right)=\frac{\gamma}{3}$,then the system (15) can be written as $\frac{4}{27}\gamma^3-\rho^{\frac{1}{n}}\left(\theta\right)=0$, which is a contradiction because $\rho^{\frac{1}{n}}\left(\theta\right)<\frac{4}{27}\gamma^3$ for all $\theta\in\mathbb{R}$. Hence, (15) has no solution.

b) From (13), it follows that $r_i(\theta), i=1,2,3$ are 2π-periodic if and only if $\rho(\theta)$ is $2\pi-$periodic and we have

$$\rho\left(\theta+2\pi\right)=e^{2\alpha n\pi}e^{\alpha n\theta}\left(\tfrac{e^{2\alpha n\pi}f(2\pi)}{1-e^{2\alpha n\pi}}+f\left(\theta+2\pi\right)\right); \tag{16}$$

we have

$$f\left(\theta+2\pi\right)=\int_0^{\theta+2\pi}\tfrac{e^{-\alpha ns}}{(a+b\cos 2\theta)^n}ds=f\left(2\pi\right)+\int_{2\pi}^{\theta+2\pi}\tfrac{e^{-\alpha ns}}{(a+b\cos 2\theta)^n}ds;$$

making the change of variable $u=s-2\pi$ in the integral $\int_{2\pi}^{\theta+2\pi}\frac{e^{-\alpha ns}}{(a+b\cos 2\theta)^n}ds$, we get

$$f\left(\theta+2\pi\right)=f\left(2\pi\right)+\int_0^{\theta}\tfrac{e^{-\alpha n(u+2\pi)}}{(a+b\cos 2(u+2\pi))^n}du=f\left(2\pi\right)+e^{-2\alpha n\pi}f\left(\theta\right).$$

We replace $f\left(\theta+2\pi\right)$ by $f\left(2\pi\right)+e^{-2a\pi}f\left(\theta\right)$ in (16), and after some calculations we obtain $\rho(\theta+2\pi)=\rho(\theta)$. Since $\rho(\theta)$ is a 2π-periodic function, then $r_i(\theta)$, $i=1,2,3$ are also 2π-periodic functions.

c) Next we prove that $r_i(\theta)>0$, $i=1,2,3$ Indeed, since $0<\rho^{\frac{1}{n}}\left(\theta\right)<\frac{4\gamma^3}{27}$, it follows that $-1<\frac{27}{2\gamma^3}\left(\rho^{\frac{1}{n}}\left(\theta\right)-\frac{2}{27}\gamma^3\right)<1$ for all $\theta\in\mathbb{R}$. Since the function $x\mapsto\arccos x$ is strictly decreasing in the interval $]-1,1[$ then

$$0<\arccos\left(\tfrac{27}{2\gamma^3}\left(\rho^{\frac{1}{n}}\left(\theta\right)-\tfrac{2}{27}\gamma^3\right)\right)<\pi. \tag{17}$$

Since $\gamma>0$ and the function $x\mapsto\cos x$ is strictly decreasing in the interval $\left]0,\frac{\pi}{3}\right[$, then

$$1>\cos\left(\frac{1}{3}\arccos\left(\frac{27}{2\gamma^3}\left(\rho^{\frac{1}{n}}\left(\theta\right)-\tfrac{2}{27}\gamma^3\right)\right)\right)>\frac{1}{2},$$

and $r_1(\theta)>0$ for all $\theta\in\mathbb{R}$. Taking into account (17) we have

$$-\tfrac{2\pi}{3}<\frac{1}{3}\arccos\left(\tfrac{27}{2\gamma^3}\left(\rho^{\frac{1}{n}}\left(\theta\right)-\tfrac{2}{27}\gamma^3\right)\right)-\tfrac{2\pi}{3}<-\tfrac{\pi}{3},$$

since $\gamma>0$ and the function $x\mapsto\cos x$ is strictly increasing in the interval $\left]-\frac{2\pi}{3},-\frac{\pi}{3}\right[$ then

$$-\tfrac{1}{2}<\cos\left(\tfrac{1}{3}\arccos\left(\tfrac{27}{2\gamma^3}\left(\rho^{\frac{1}{n}}\left(\theta\right)-\tfrac{2}{27}\gamma^3\right)\right)-\tfrac{2\pi}{3}\right)<\tfrac{1}{2},$$

and $r_2(\theta)>0$ for all $\theta\in\mathbb{R}$. Taking into account (17) we have

$$-\tfrac{4\pi}{3}<\tfrac{1}{3}\arccos\left(\tfrac{27}{2\gamma^3}\left(\rho^{\frac{1}{n}}\left(\theta\right)-\tfrac{2}{27}\gamma^3\right)\right)-\tfrac{4\pi}{3}<-\pi,$$

since $\gamma>0$ and the function $x\mapsto\cos x$ is strictly decreasing in the interval $\left]-\frac{4\pi}{3},-\pi\right[$ then

$$-\tfrac{1}{2}>\cos\left(\frac{1}{3}\arccos\left(\tfrac{27}{2\gamma^3}\left(\rho^{\frac{1}{n}}\left(\theta\right)-\frac{2}{27}\gamma^3\right)\right)-\tfrac{2\pi}{3}\right)>-1,$$

and $r_3(\theta) > 0$ for all $\theta \in \mathbb{R}$.

In order to prove that the periodic orbits are hyperbolic limit cycles, we consider

$$\begin{aligned} r_1(\theta,\lambda) &= \sqrt{\tfrac{2\gamma}{3}\left(1+\cos\left(\tfrac{1}{3}\arccos\left(\tfrac{27}{2\gamma^3}\left(\rho^{\frac{1}{n}}(\theta,\lambda)-\tfrac{2\gamma^3}{27}\right)\right)\right)\right)}, \\ r_2(\theta,\lambda) &= \sqrt{\tfrac{2\gamma}{3}\left(1+\cos\left(\tfrac{1}{3}\arccos\left(\tfrac{27}{2\gamma^3}\left(\rho^{\frac{1}{n}}(\theta,\lambda)-\tfrac{2\gamma^3}{27}\right)\right)-\tfrac{2\pi}{3}\right)\right)}, \\ r_3(\theta,\lambda) &= \sqrt{\tfrac{2\gamma}{3}\left(1+\cos\left(\tfrac{1}{3}\arccos\left(\tfrac{27}{2\gamma^3}\left(\rho^{\frac{1}{n}}(\theta,\lambda)-\tfrac{2\gamma^3}{27}\right)\right)-\tfrac{4\pi}{3}\right)\right)} \end{aligned}$$

where $\rho(\theta,\lambda) = e^{\alpha n\theta}\left(\lambda^{2n}\left(\lambda^2-\gamma\right)^{2n}+f(\theta)\right)$, and introduce the Poincaré return map $\lambda \mapsto \Pi(2\pi,\lambda) = r(2\pi,\lambda)$. Therefore, periodic orbits of system (2) are hyperbolic limit cycles if and only if

$$\left.\frac{dr_i(2\pi,\lambda)}{d\lambda}\right|_{\lambda=r_{*i}} \neq 1, \ i=1,2,3$$

where

$$\begin{aligned} r_{*1} &= \sqrt{\tfrac{2}{3}\gamma\left(1+\cos\left(\tfrac{1}{3}\arccos\left(\tfrac{27}{2\gamma^3}\left(\rho_0^{\frac{1}{n}}-\tfrac{2}{27}\gamma^3\right)\right)\right)\right)}, \\ r_{*2} &= \sqrt{\tfrac{2}{3}\gamma\left(1+\cos\left(\tfrac{1}{3}\arccos\left(\tfrac{27}{2\gamma^3}\left(\rho_0^{\frac{1}{n}}-\tfrac{2}{27}\gamma^3\right)\right)-\tfrac{2\pi}{3}\right)\right)}, \\ r_{*3} &= \sqrt{\tfrac{2}{3}\gamma\left(1+\cos\left(\tfrac{1}{3}\arccos\left(\tfrac{27}{2\gamma^3}\left(\rho_0^{\frac{1}{n}}-\frac{2}{27}\gamma^3\right)\right)-\tfrac{4\pi}{3}\right)\right)} \end{aligned}$$

where $\rho_0 = \frac{e^{2\alpha n\pi}f(2\pi)}{1-e^{2\alpha n\pi}}$. After some calculations, we obtain that

$$\left.\frac{dr_1(2\pi,\lambda)}{d\lambda}\right|_{\lambda=r_{*1}} = \left.\frac{dr_2(2\pi,\lambda)}{d\lambda}\right|_{\lambda=r_{*2}} = \left.\frac{dr_3(2\pi,\lambda)}{d\lambda}\right|_{\lambda=r_{*3}} = e^{2\pi n\alpha} < 1.$$

Consequently the limit cycles of the differential equation (5) are hyperbolic and stable, for more details see [8].

If $b \neq 0$, clearly the curve $(r(\theta)\cos\theta, r(\theta)\sin\theta)$ in the (x,y) plane with

$$F(r,\theta) = r^{2n}(\theta)\left(r^2(\theta)-\gamma\right)^{2n} - e^{\alpha n\theta}\left(\tfrac{e^{2\alpha n\pi}f(2\pi)}{1-e^{2\alpha n\pi}}+f(\theta)\right) = 0 \tag{18}$$

is not algebraic due to the expression $\rho(\theta) = e^{\alpha n\theta}\left(\frac{e^{2a\pi}f(2\pi)}{1-e^{2a\pi}}+f(\theta)\right)$. More precisely, in Cartesian coordinates the curve defined by this limit cycle is

$$\left(x^2+y^2\right)^n\left(x^2+y^2-\gamma\right)^{2n} - e^{\alpha n\arctan\frac{y}{x}}\left(\frac{e^{2\alpha n\pi}f(2\pi)}{1-e^{2\alpha n\pi}} + \int_0^{\arctan\frac{y}{x}} \frac{e^{-\alpha ns}}{(a+b\cos 2s)^n}ds\right) = 0.$$

If the limit cycle is algebraic this curve must be given by a polynomial, but a polynomial $F(x,y)$ in the variables x and y satisfies that there is a positive integer n such that $\dfrac{\partial^n F}{\partial x^n} = 0$ and this is not the case because in the derivative $\dfrac{\partial F}{\partial x}$ appears again the expression

$$e^{\alpha n\arctan\frac{y}{x}}\left(\frac{e^{2\alpha n\pi}f(2\pi)}{1-e^{2\alpha n\pi}} + \int_0^{\arctan\frac{y}{x}} \frac{e^{-\alpha ns}}{(a+b\cos 2s)^n}ds\right),$$

which already appears in $F(x,y)$, and this expression will appear in the partial derivative at any order.

Proof of statement (2) of Theorem 1. If we take $b=0$ in (8), we obtain $\rho(\theta)=-\frac{1}{a^n n\alpha}$. Going back through the changes of variables we obtain $r^{2n}\left(r^2-\gamma\right)^{2n}=-\frac{1}{a^n n\alpha}$. By passing to Cartesian coordinates (x,y), we deduce the

$$\left(x^2+y^2\right)^n\left(x^2+y^2-\gamma\right)^{2n}+\frac{1}{a^n n\alpha}=0.$$

This completes the proof of statement (2). ■

3 Examples

The following examples illustrate our result.

Example 1 *When $\gamma=2, a=2, b=1, \alpha=-1$, system (2) reads*

$$\begin{array}{l} x'=x+n\left(-x\left(x^2+y^2-2\right)-2y\left(3x^2+3y^2-2\right)\right)\left(x^2+y^2-2\right)^{2n-1}\left(3x^2+y^2\right)^n, \\ y'=y+n\left(-y\left(x^2+y^2-2\right)+2x\left(3x^2+3y^2-2\right)\right)\left(x^2+y^2-2\right)^{2n-1}\left(3x^2+y^2\right)^n \end{array} \tag{19}$$

It is easy to verify that all conditions of statement (1) of Theorem 1 are satisfied. We conclude that system (19) has three non-algebraic limit cycles as shown on the Poincaré disc in Figure 1.

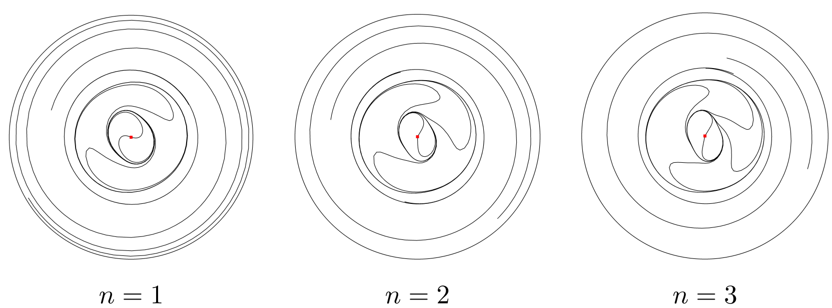

Figure 1: The phase portrait in the Poincaré disc of the polynomial system (19)

Example 2 *When $\gamma=2,\ a=2,\ b=0,\ \alpha=-1$ system (2) reads*

$$\begin{array}{l} x'=x+n\left(-x\left(x^2+y^2-2\right)-2y\left(3x^2+3y^2-2\right)\right)\left(x^2+y^2-2\right)^{2n-1}\left(2x^2+2y^2\right)^n, \\ y'=y+n\left(-y\left(x^2+y^2-2\right)+2x\left(3x^2+3y^2-2\right)\right)\left(x^2+y^2-2\right)^{2n-1}\left(2x^2+2y^2\right)^n \end{array} \tag{20}$$

It is easy to verify that all conditions of statement (2) of Theorem 1 are satisfied. We conclude that system (20) has three algebraic limit cycles as shown on the Poincaré disc in Figure 2.

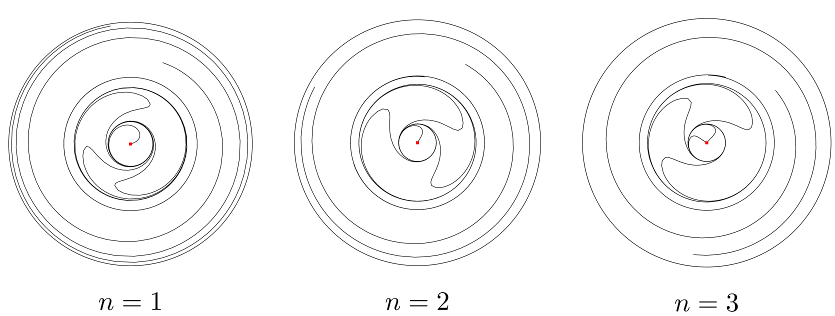

Figure 2: The phase portrait in the Poincaré disc of the polynomial system (20)

References

[1] K. I. T. Al-Dosary, Non-algebraic limit cycles for parametrized planar polynomial systems, Int. J. Math., 18(2007), 179–189.

[2] A. Bendjeddou, A. Berbache and R. Cheurfa, Exact algebraic and non-algebraic limit cycles for a class of integrable quintic and planar polynomial differential systems, An. Stiint. Univ. Al. I. Cuza Ia si. Mat. (N.S.) Tomul LXIV, 2018, f. 2.

[3] R. Benterki and J. Llibre, Polynomial differential systems with explicit non-algebraic limit cycles, Elect. J. of Diff. Equ., 78(2012), 1–6.

[4] A. Gasull, H. Giacomini and J. Torregrosa, Explicit non-algebraic limit cycles for polynomial systems, J. Comput. Appl. Math., 200(2007), 448–457.

[5] J. Giné and M. Grau, A note on: relaxation oscillator with exact limit cycles, J.Math. Anal. and Appl., 324(2006), 739–745.

[6] J. Giné and M. Grau, Coexistence of algebraic and non-algebraic limit cycles, explicitly given, using Riccati equations, Nonlinearity., 19(2006), 1939–1950.

[7] K. Odani, The limit cycle of the van der Pol equation is not algebraic, J. Differential Equations., 115(1995), 146–152.

[8] L. Perko, Differential Equations and Dynamical Systems, Third Edition. Texts in Applied Mathematics, 7. Springer Verlag, New York, 2001.

[9] S. Shelbey, CRC Standard Mathematical Tables, CRC Press 1975.