Applied Mathematics E-Notes, 20(2020), 133-141 © ISSN 1607-2510
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Kernel of A Diffusion Equation With Non-Constant Space-Time Diffusion Coefficient*

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Received 14 March 2019

Abstract

We compute a new solution of a diffusion equation with space and time dependent variable diffusion coefficients in terms of a special function known as a confluent hypergeometric (Kummer) function. This new solution generalizes the already existing well-known fundamental solution of the constant diffusion coefficient heat equation (the Gaussian heat kernel).

1 Introduction

The study of diffusion equation with diffusion coefficients is as old as Mathematics itself, see [2, 10, 16] and their references. Fa and Lenzi in [5, 6] and Fa [7] studied diffusion equation with space-time-dependent diffusion coefficients. Specifically, in [5], they considered a power law diffusion coefficient and anomalous diffusion equation in one-dimensional space of the type

$$\frac{\partial u(x,\bar{t})}{\partial \bar{t}} = D\frac{\partial}{\partial x}\left[|x|^{-\alpha}\frac{\partial u(x,\bar{t})}{\partial x}\right], \tag{1}$$

where $\bar{t}$ is a scaled time and a function of t (that is placed inside the derivative). The solution to the above equation (1) is given by

$$u(x,\bar{t}) = C\frac{e^{-\frac{|x|^{2+\alpha}}{D(2+\alpha)^2\bar{t}}}}{\bar{t}^{1/(2+\alpha)}},$$

with C a normalization constant.

This research was motivated by a paper [11] which studied a fractional order diffusion equation with a generalized diffusion constant with its numerical and modeling applications:

$$\frac{\partial^\alpha P(x,t)}{\partial t^\alpha} = D_{\alpha,\beta}\frac{\partial^\beta P(x,t)}{\partial |x|^\beta},$$

where $P(x,t)$ is a diffusion propagator, $\frac{\partial^\alpha}{\partial t^\alpha}$ represents the Caputo time-fractional derivative for $0 < \alpha \leq 1$, $\frac{\partial^\beta}{\partial |x|^\beta}$ represents the order of the Riesz space-fractional derivative for $1 < \beta \leq 2$ and $D_{\alpha,\beta}$ is the generalized diffusion constant (distant$^\beta$/time$^\alpha$).

In what follows, therefore, we consider a diffusion equation with both temporally and spatially-dependent (non-constant) diffusion coefficient

$$u_t(x,t) - \kappa(x,t)\Delta u(x,t) = 0,\ x\in\mathbb{R}^n,\ t>0, \tag{2}$$

where $\kappa(x,t)$ is a space-time diffusion coefficient. Diffusion equations with temporally (time-dependent) and spatially (space-dependent) variables diffusion coefficients have many modeling applications. For the

*Mathematics Subject Classifications: 35K05, 35K08, 33C15, 33C36.

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application of time-dependent diffusion coefficients see [3, 13, 15, 18, 19] and see [1, 17, 20] for the application of spatially variable diffusion coefficients. See also [8, 9, 14] for more applications. A special case of equation (2) is when $\kappa(x,t) = \kappa$ a constant coefficient, with the fundamental solution known as the usual heat kernel given by

$$u(x,t) = p(t,x) = \frac{1}{(4\kappa\pi t)^{n/2}} e^{-\frac{|x|^2}{4\kappa t}},\ x \in \mathbb{R}^n,\ t > 0, \tag{3}$$

where $|x|$ represents the Euclidean norm of x on $\mathbb{R}^n$.

There is no known kernel (analytic solution) in literature, to the best of our knowledge, for the diffusion coefficient of non-constant term. We therefore seek to give a fundamental solution for the equation for a given space-time dependent diffusion coefficient

$$\kappa(x,t) = |x|^a t^b,\ x \in \mathbb{R}^n,\, t > 0,\, a \neq 2,\, b \neq -1.$$

2 Remarks I

We assume that x should not be $\mathbf{0}$ when a is negative. We thus obtain the following result.

Theorem 1 *For $\kappa(x,t) = |x|^a t^b$, $x \in \mathbb{R}^n, t > 0, a \neq 2, b \neq -1$, then the general solution to (2) is a confluent hypergeometric (Kummer) function given by*

$$\begin{aligned} u(x,t) &= \frac{C_1}{t^{\frac{n(1+b)}{2-a}}} {}_1F_1\left[\frac{n}{2-a}, \frac{n-a}{2-a}, -\frac{1+b}{(2-a)^2}\frac{|x|^{2-a}}{t^{1+b}}\right] \\ &+ C_2\left(\frac{a}{2-a}\right)^{\left(\frac{n-2}{a-2}\right)} a^{\left(\frac{n-2}{2-a}\right)} \left(\frac{2-a}{1+b}\right)^{\left(\frac{n-2}{2-a}\right)} \frac{t^{\frac{(1+b)(n-2)}{2-a}}}{|x|^{n-2}} \\ &\times\ {}_1F_1\left[\frac{2}{2-a}, 1 - \frac{n-2}{2-a}, -\frac{1+b}{(2-a)^2}\frac{|x|^{2-a}}{t^{1+b}}\right], \end{aligned} \tag{4}$$

where C_1 and C_2 are some normalization constants (which can be determined uniquely by assuming some initial conditions on $u(x,t)$).

3 Remarks II

1. The above solution (4) is not always positive or non-negative for all the parameters a, b, n and therefore for now represents a mathematical result (since the negative solutions do not represent any known applicable diffusion processes like density, temperature, probability density functions, etc). Thus the solution has known physical applications for the positive values of $u(x,t)$ and possible future applications when $u(x,t)$ assumes negative values.

2. Our solution (4) agrees with solution (3) when $a = b = 0$ by the property of the confluent hypergeometric function in (5), that is,

$$\frac{C_1}{t^{\frac{n}{2}}} {}_1F_1\left[\frac{n}{2}, \frac{n}{2}, -\frac{1}{4}\frac{|x|^2}{t}\right] = p(t,x)$$

   is the Gaussian kernel with C_1 a normalization constant and thus equation (4) generalizes solution (3).

We briefly consider the confluent hypergeometric function and their properties.

4 Preliminary

For second-order differential equations of the standard form

$$\frac{d^2w}{dz^2}+p(z)\frac{dw}{dz}+q(z)w(z)=0$$

with $p(z)$ and $q(z)$ some given complex-valued functions, they have some special functions as its solutions. In particular is the confluent hypergeometric equation

$$z\frac{d^2w}{dz^2}+(b-z)\frac{dw}{dz}-aw(z)=0$$

whose solution is given by

$${}_1F_1(a;b;z):=\sum_{n=0}^{\infty}\frac{(a)_n}{n!(b)_n}z^n,\ \ (b\neq 0,-1,-2,...)$$

known as the confluent hypergeometric function (Kummer function) with the following integral representations

$${}_1F_1(a;b;z)=\frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)}\int_0^1 e^{zu}u^{a-1}(1-u)^{b-a-1}du,$$

where $\Re(b)>\Re(a)>0,\ \arg u=\arg(1-u)=0$; and

$${}_1F_1(a;b;z)=\frac{1}{2\pi i}\frac{\Gamma(b)}{\Gamma(a)}\int_{-i\infty}^{i\infty}\frac{\Gamma(a+s)\Gamma(-s)}{\Gamma(b+s)}(-z)^s ds,$$

where $a\neq 0,-1,-2,...,\ |\arg(-z)|<\frac{\pi}{2}$. The confluent hypergeometric function has the following properties relating to some special and elementary functions, thus,

$${}_1F_1(a;a;z)=e^z, \tag{5}$$

$${}_1F_1\left(\nu+\frac{1}{2};2\nu+1;2z\right)=\left(\frac{z}{2}\right)^{-\nu}e^zI_\nu(z),$$

$${}_1F_1(1;n+1;z)=nz^{-n}e^z\gamma(n,z),$$

$${}_1F_1\left(\frac{1}{2};\frac{3}{2};-z^2\right)=\frac{\sqrt{\pi}}{2z}erf(z)$$

with $I_\nu(z)$ the modified Bessel function, $\gamma(n,z)$ the incomplete gamma function and $erf(z)$ the error function, respectively defined by

$$I_\nu(z)=i^{-\nu}J_\nu(iz)=i^{-\nu}\sum_{k=0}^{\infty}\frac{1}{k!\Gamma(\nu+k+1)}\left(\frac{iz}{2}\right)^{2k+\nu},$$

$$\gamma(n,z)=\int_0^z t^{n-1}e^{-t}dt,$$

$$erf(z)=\frac{2}{\sqrt{2}}\int_0^z e^{-t^2}dt.$$

See [12] and its respective references for more on confluent hypergeometric function.

5 Proof of Theorem 1

We assume a solution of the structure type (see [4])

$$u(x,t) = \lambda^{\alpha} u(\lambda^{\beta} x, \lambda t) = \frac{1}{t^{\alpha}} u\left(\frac{x}{t^{\beta}}, 1\right) = \frac{1}{t^{\alpha}} v\left(\frac{x}{t^{\beta}}\right), \ x \in \mathbb{R}^n, \ t > 0,$$

with $\lambda = \frac{1}{t}$ and $v(y) = u(y,1)$, where the constants α, β and the function $v : \mathbb{R}^n \to \mathbb{R}$ are to be found. Then from (2), we have

$$u_t(x,t) - |x|^a t^b \Delta u(x,t) = u_t(x,t) - \left(\frac{|x|}{t^{\beta}}\right)^a t^{b+\beta a} \Delta u(x,t) = 0$$

and thus for $y = \frac{x}{t^{\beta}}$ (and $|y| = \frac{|x|}{t^{\beta}}$),

$$-\alpha t^{-(\alpha+1)} v(y) - \beta t^{-(\alpha+1)} y.Dv(y) - t^{-(\alpha+2\beta)} t^{b+\beta a} |y|^a \Delta v(y) = 0,$$

which implies that

$$\alpha t^{-(\alpha+1)} v(y) + \beta t^{-(\alpha+1)} y.Dv(y) + t^{-[(\alpha-b)+\beta(2-a)]} |y|^a \Delta v(y) = 0.$$

To simplify the equation, we let $(\alpha - b) + \beta(2-a) = \alpha + 1 \Rightarrow \beta = \frac{1+b}{2-a}$. Thus with $\beta = \frac{1+b}{2-a}$ we have

$$\alpha v(y) + \left(\frac{1+b}{2-a}\right) y.Dv(y) + |y|^a \Delta v(y) = 0.$$

We further let v be a radially symmetrical solution, that is, $v(y) = w(|y|)$ for some $w : \mathbb{R} \to \mathbb{R}$. Therefore for $r = |y|$,

$$\alpha w + \left(\frac{1+b}{2-a}\right) r w' + r^a w'' + \frac{n-1}{r^{1-a}} w' = \alpha w + \left[\frac{n-1}{r^{1-a}} + \left(\frac{1+b}{2-a}\right) r\right] w' + r^a w'' = 0.$$

Multiply through by r^{n-1} and let $\alpha = n\left(\frac{1+b}{2-a}\right)$ to obtain

$$\left(\frac{1+b}{2-a}\right)(r^n w)' + (r^{a+n-1} w')' - a r^{a+n-2} w' = 0.$$

We first consider the case of time-dependent diffusion coefficient, $a = 0$ to obtain

$$\left(\frac{1+b}{2}\right)(r^n w)' + (r^{n-1} w')' = 0$$

and

$$w(r) = C\, e^{-\left(\frac{1+b}{4}\right) r^2}$$

and the fundamental solution given by

$$u(x,t) = \frac{1}{t^{\alpha}} v\left(\frac{x}{t^{\beta}}\right) = \frac{C}{t^{\alpha}} e^{-\left(\frac{1+b}{4}\right)\frac{|x|^2}{t^{2\beta}}} = \frac{C}{t^{n(1+b)/2}} e^{-\frac{1+b}{4t^{1+b}}|x|^2} = \frac{1}{(4\pi t)^{n(1+b)/2}} e^{-\frac{1+b}{4t^{1+b}}|x|^2},$$

with $\alpha = \frac{n(1+b)}{2}$, $\beta = \frac{1+b}{2}$ and C a normalization constant. This solution corresponds to (3) with $\kappa = 1$ when $b = 0$. Next for the general case a, $b \neq 0$, we solve for w in the second order differential equation

$$r^a w'' + \left[\frac{n-1}{r^{1-a}} + \left(\frac{1+b}{2-a}\right) r\right] w' + n\left(\frac{1+b}{2-a}\right) w = 0$$

to have

$$\begin{aligned} w(r) &= C_{1\,1}F_1\left[\frac{n}{2-a},\frac{n-a}{2-a},-\frac{1+b}{(2-a)^2}r^{2-a}\right] \\ &+C_2\left(\frac{a}{2-a}\right)^{\left(\frac{n-2}{a-2}\right)}a^{\left(\frac{n-2}{2-a}\right)}\left(\frac{2-a}{1+b}\right)^{\left(\frac{n-2}{2-a}\right)}r^{-(n-2)} \\ &\times{}_1F_1\left[\frac{2}{2-a},1-\frac{n-2}{2-a},-\frac{1+b}{(2-a)^2}r^{2-a}\right] \end{aligned}$$

and with $u(x,t)=\frac{1}{t^\alpha}w(\frac{|x|}{t^\beta}),\ \alpha=\frac{n(1+b)}{2-a},\ \beta=\frac{1+b}{2-a}$ to get

$$\begin{aligned} u(x,t) &= \frac{C_1}{t^{\frac{n(1+b)}{2-a}}}{}_1F_1\left[\frac{n}{2-a},\frac{n-a}{2-a},-\frac{1+b}{(2-a)^2}\frac{|x|^{2-a}}{t^{1+b}}\right] \\ &+C_2\left(\frac{a}{2-a}\right)^{\left(\frac{n-2}{a-2}\right)}a^{\left(\frac{n-2}{2-a}\right)}\left(\frac{2-a}{1+b}\right)^{\left(\frac{n-2}{2-a}\right)}\frac{t^{\frac{(1+b)(n-2)}{2-a}}}{|x|^{n-2}} \\ &\times{}_1F_1\left[\frac{2}{2-a},1-\frac{n-2}{2-a},-\frac{1+b}{(2-a)^2}\frac{|x|^{2-a}}{t^{1+b}}\right], \end{aligned}$$

where C_1 and C_2 are normalization constants.

6 Example

When $a=b=1$ and $n=2$, we have the solution given by

$$u(x,t)=\frac{C_1}{t^4}{}_1F_1\left[2,1,-2\frac{|x|}{t^2}\right]+C_2\ {}_1F_1\left[2,1,-2\frac{|x|}{t^2}\right]=\left[\frac{C_1}{t^4}+C_2\right]{}_1F_1\left[2,1,-2\frac{|x|}{t^2}\right].$$

Next we plot graphs (see Pages 6 & 7) of the solution for $C_1=C_2=1,2,3,4,5,6$ when $t=1$ and $t=2$ for different x values in the following interval $[-1,1]$, $[-2,2]$, $[-3,3]$, $[-4,4]$, $[-5,5]$, $[-10,10]$ and $[-100,100]$.

7 Remarks III

The above example is non-negative for all $|x|\le\frac{t^2}{2}$, that is;

$$u(x,t)=\left[\frac{C_1}{t^4}+C_2\right]{}_1F_1\left[2,1,-2\frac{|x|}{t^2}\right]\ge 0,\ \forall\ 2\frac{|x|}{t^2}\le 1\ \Leftrightarrow |x|\le\frac{t^2}{2}.$$

This implies that for a fixed $x\in\mathbb{R}^n$ and large t (that is, t growing very large, tending to infinity), then the solution becomes non-negative.

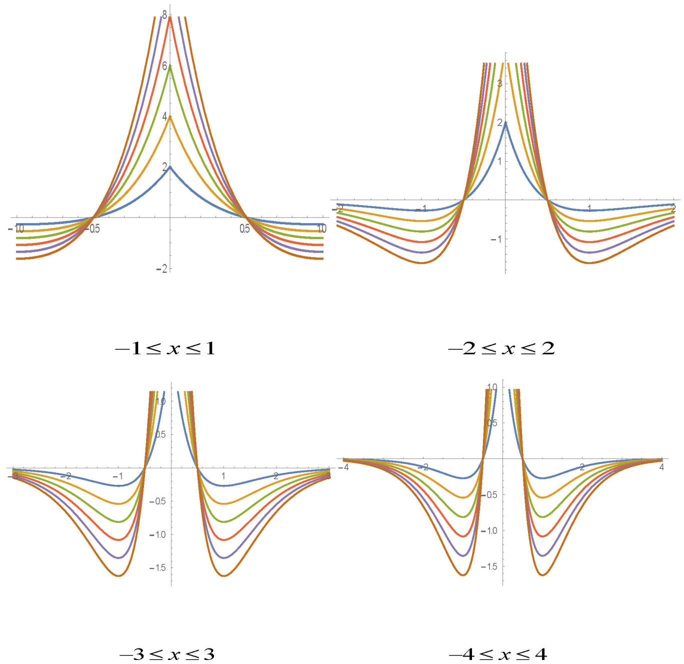


Figure 1: Graphs of the solution $u(x,t) = \left[\frac{C_1}{t^4} + C_2\right] {}_1F_1\left[2, 1, -2\frac{|x|}{t^2}\right]$ where $C_1 = C_2 = 1, 2, ..., 6$ and $t = 1$.

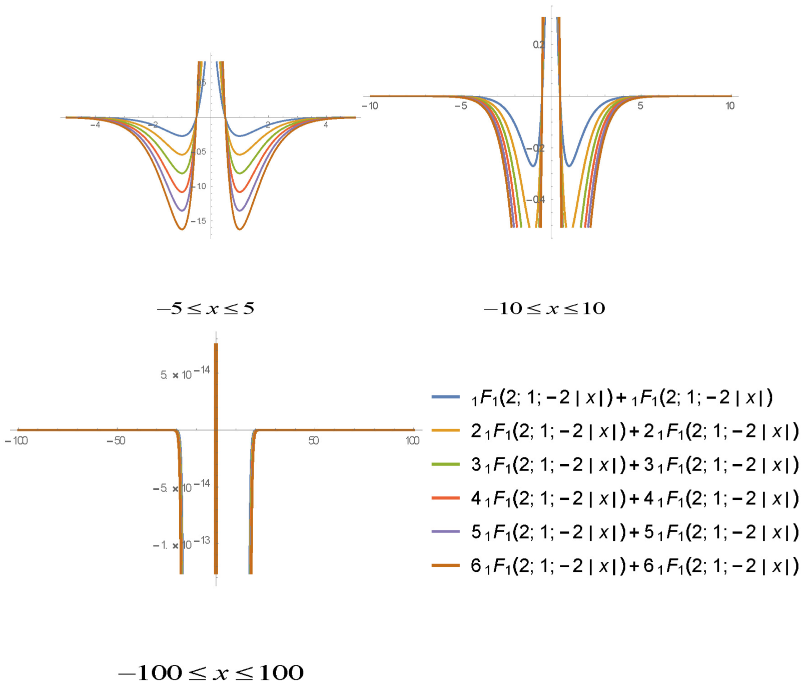
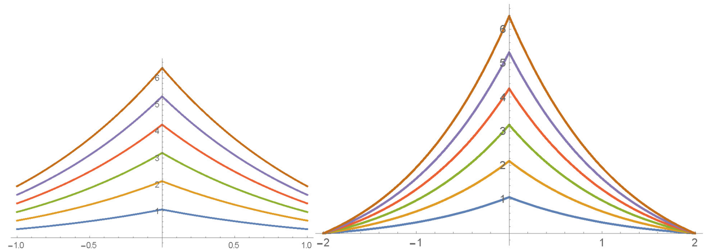
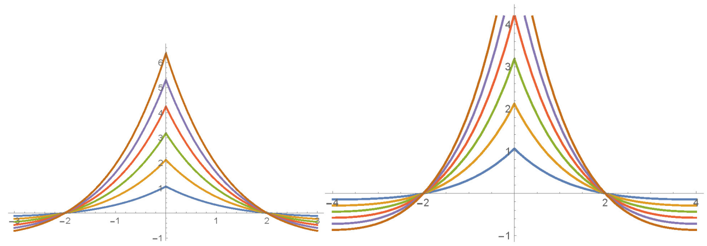


Figure 2: Graphs of the solution $u(x,t) = \left[\frac{C_1}{t^4} + C_2\right] {}_1F_1\left[2, 1, -2\frac{|x|}{t^2}\right]$ where $C_1 = C_2 = 1, 2, ..., 6$ and $t = 1$.

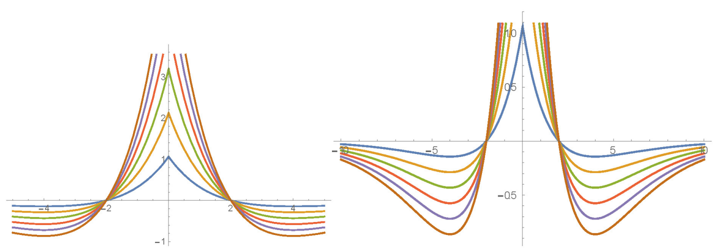


$-1 \le x \le 1$ $\quad$ $-2 \le x \le 2$

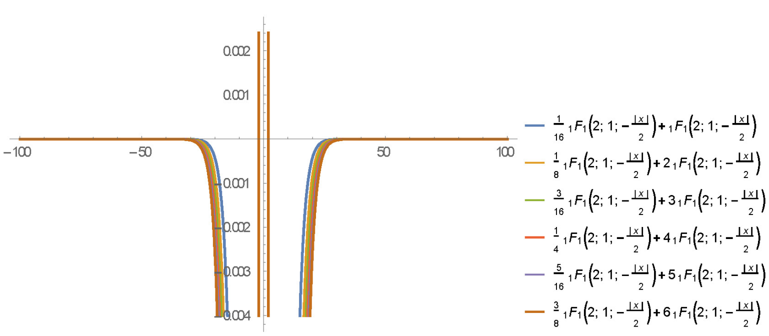


$-3 \le x \le 3$ $\quad$ $-4 \le x \le 4$

Figure 3: Graphs of the solution $u(x,t) = \left[\frac{C_1}{t^4} + C_2\right] {}_1F_1\left[2, 1, -2\frac{|x|}{t^2}\right]$ where $C_1 = C_2 = 1, 2, ..., 6$ and $t = 2$.



$-5 \le x \le 5$ $\quad$ $-10 \le x \le 10$



$-100 \le x \le 100$

Figure 4: Graphs of the solution $u(x,t) = \left[\frac{C_1}{t^4} + C_2\right] {}_1F_1\left[2, 1, -2\frac{|x|}{t^2}\right]$ where $C_1 = C_2 = 1, 2, ..., 6$ and $t = 2$.

8 Conclusion

This solution arose from the curiosity of what happens to the heat equation when the diffusion coefficient becomes both time and space dependent. The result gave a confluent hypergeometric function which application of its property agrees with established result and thus the classical heat kernel in (3) is a particular case of this new kernel.

Acknowledgment. We want to appreciate the anonymous referee whose invaluable contribution and insight improved the content and context of the paper. The first author would want to acknowledge the support of the University of Hafr Al Batin, KSA, and also for providing a research friendly environment.

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