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Multiplicity Of Solutions For $\overrightarrow{p}(x)$-Laplacian Elliptic Kirchhoff Type Equations*

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Abstract

In this work, we deal with a class of anisotropic nonlocal type equations, by means of the variational approach, we prove the existence of infinitely many solutions under suitable assumptions.

1 Introduction

The investigation of the problems concerning anisotropic variable exponent has drawn the attention of many authors, since there are some physical phenomena which can be modelled by such kind of equations, the reader can find several models in mathematical physics where this class of problems appears, like those in electrorheological fluids [8], thermorheological fluids [2] and image restoration [5].

The purpose of the present paper is to study the nonlocal anisotropic $p(x)$-Laplacian Dirichlet problems of the form

$$\begin{gathered} -\sum_{i=1}^{N} M_i\left(\int_\Omega \frac{|\partial_{x_i} u|^{p_i(x)}}{p_i(x)}\,dx\right)\partial_{x_i}\left(|\partial_{x_i} u|^{p_i(x)-2}\partial_{x_i} u\right) = f(x,u), \quad \text{for } x\in\Omega \\ u=0, \quad \text{for } x\in\partial\Omega, \end{gathered} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is a bounded open set with smooth boundary, p_i, $i = 1, ...N$, are continuous functions on $\overline{\Omega}$, and for each $i = 1, ..., N$, $M_i : [0,\infty) \to [0,\infty)$, $f : \Omega\times\mathbb{R}\to\mathbb{R}$ are continuous functions with the potential

$$F(x,t) = \int_0^t f(x,s)\,ds.$$

This kind of equations with variable exponent growth conditions enable the study of equations with more complicated nonlinearities since the differential operator $\Delta_{\overrightarrow{p}(x)}(u) := \sum_{i=1}^N \partial_{x_i}\left(|\partial_{x_i}u|^{p_i(x)-2}\partial_{x_i}u\right)$ allows a distinct behavior for partial derivatives in various directions.

Some articles have been interested in the existence of solutions for this kind of problems in the case when the nonlinearity f verifies the Ambrosetti-Rabinowitz type conditions (AR), (see below), for instance in [3] and [7], the authors obtained the existence and multiplicity of solutions under the well-known Ambrosetti-Rabinowitz type condition:

$$(AR) \qquad \exists \nu > p_M^+, \quad K>0 \quad \text{such that} \quad x\in\Omega, |t|\geq K \Rightarrow 0\leq \nu F(x,t)\leq f(x,t)t.$$

The role of (AR) condition is to ensure the boundness of the Palais-Smale sequences of the Euler-Lagrange functional. This is very crucial in the applications of critical point theory, especially for the nonlocal equations. A distinguishing feature is that we use (C_c) Cermai condition which is weaker than the (P.S) condition.

Furthermore, problem (1) is related also to the stationary version of a model, the so-called Kirchhoff equation, which is introduced by Kirchhoff [9]. To be more precise, Kirchhoff established a model given by the equation

$$\rho\frac{\partial^2 u}{\partial t^2} - \Big(\frac{\rho_0}{h} + \frac{E}{2L}\int_0^L \big|\frac{\partial u}{\partial x}\big|^2\,dx\Big)\frac{\partial^2 u}{\partial x^2} = 0, \tag{2}$$

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which extends the classical D'Alembert's wave equation by considering the effects of the changes in the length of the strings during the vibrations. A distinct feature is that (2) contains a nonlocal coefficient $\frac{\rho_0}{h} + \frac{E}{2L}\int_0^L |\frac{\partial u}{\partial x}|^2\,dx$ which depends on the average $\frac{1}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx$, and hence the equation is no longer a pointwise equation.

This work is organized as follows: In section 2, we introduce some preliminary results on the anisotropic variable exponent Sobolev spaces. In section 3, we consider problem (1) and obtain some results on existence and multiplicity for (1) by using the variational method. Finally, we give the proof of the main results.

2 Preliminaries

In order to deal with the problem (1), we recall some auxiliary results. For convenience, we only recall some basic facts which will be used later, we refer to [4, 6, 10].

For $q \in C_+(\overline{\Omega})$, we introduce the Lebesgue space with variable exponent defined by

$$L^{q(\cdot)}(\Omega) = \left\{u : u \text{ is a measurable real-valued function, } \int_\Omega |u(x)|^{q(x)}\,dx < \infty\right\},$$

where $C_+(\overline{\Omega}) = \{q \in C(\overline{\Omega}, \mathbb{R}) : \inf_{x\in\Omega} q(x) > 1\}$. This space, endowed with the Luxemburg norm,

$$|u|_{q(.)} = \|u\|_{L^{q(\cdot)}(\Omega)} = \inf\left\{\mu > 0 : \int_\Omega \left|\frac{u(x)}{\mu}\right|^{q(x)} dx \le 1\right\},$$

is a separable and reflexive Banach space. We also have an embedding result.

Proposition 1 *Assume that Ω is bounded and $q_1,\ q_2 \in C_+(\overline{\Omega})$ such that $q_1 \le q_2$ in Ω. Then the embedding $L^{q_2(\cdot)}(\Omega) \hookrightarrow L^{q_1(\cdot)}(\Omega)$ is continuous.*

Furthermore, the Hölder-type inequality

$$\left|\int_\Omega u(x)v(x)\,dx\right| \le 2\|u\|_{L^{q(\cdot)}(\Omega)}\|v\|_{L^{q'(\cdot)}(\Omega)} \tag{3}$$

holds for all $u \in L^{q(\cdot)}(\Omega)$ and $v \in L^{q'(\cdot)}(\Omega)$, where $L^{q'(\cdot)}(\Omega)$ the conjugate space of $L^{q(\cdot)}(\Omega)$, with $1/q(x) + 1/q'(x) = 1$. Moreover, we denote

$$q^+ = \sup_{x\in\Omega} q(x), \quad q^- = \inf_{x\in\Omega} q(x)$$

and for $u \in L^{q(\cdot)}(\Omega)$, we have the following properties:

$$\|u\|_{L^{q(\cdot)}(\Omega)} < 1\ (=1;\ >1) \ \Leftrightarrow\ \int_\Omega |u(x)|^{q(x)}\,dx < 1\ (=1;\ >1);$$

$$\|u\|_{L^{q(\cdot)}(\Omega)} > 1 \ \Rightarrow\ \|u\|^{q^-}_{L^{q(\cdot)}(\Omega)} \le \int_\Omega |u(x)|^{q(x)}\,dx \le \|u\|^{q^+}_{L^{q(\cdot)}(\Omega)}; \tag{4}$$

$$\|u\|_{L^{q(\cdot)}(\Omega)} < 1 \ \Rightarrow\ \|u\|^{q^+}_{L^{q(\cdot)}(\Omega)} \le \int_\Omega |u(x)|^{q(x)}\,dx \le \|u\|^{q^-}_{L^{q(\cdot)}(\Omega)}; \tag{5}$$

$$\|u\|_{L^{q(\cdot)}(\Omega)} \to 0\ (\to \infty) \ \Leftrightarrow\ \int_\Omega |u(x)|^{q(x)}\,dx \to 0\ (\to \infty).$$

To recall the definition of the isotropic Sobolev space with variable exponent, $W^{1,q(\cdot)}(\Omega)$, we set

$$W^{1,q(\cdot)}(\Omega) = \{u \in L^{q(\cdot)}(\Omega) : \partial_{x_i} u \in L^{q(\cdot)}(\Omega) \text{ for all } i \in \{1,\dots,N\}\},$$

endowed with the norm

$$\|u\|_{W^{1,q(\cdot)}(\Omega)} = \|u\|_{L^{q(\cdot)}(\Omega)} + \sum_{i=1}^{N} \|\partial_{x_i} u\|_{L^{q(\cdot)}(\Omega)}.$$

The space $\left(W^{1,r(\cdot)}(\Omega),\ \|\cdot\|_{W^{1,r(\cdot)}(\Omega)}\right)$ is a separable and reflexive Banach space.

Now, we consider $\vec{p}: \overline{\Omega} \to \mathbb{R}^N$ to be the vectorial function

$$\vec{p}(x) = (p_1(x), \ldots, p_N(x))$$

with $p_i \in C_+(\overline{\Omega})$ for all $i \in \{1, \ldots, N\}$ and we put

$$p_M(x) = \max\{p_1(x), \ldots, p_N(x)\}, \quad p_m(x) = \min\{p_1(x), \ldots, p_N(x)\}.$$

The anisotropic space with variable exponent is

$$W^{1,\vec{p}(\cdot)}(\Omega) = \{u \in L^{p_M(\cdot)}(\Omega) : \partial_{x_i} u \in L^{p_i(\cdot)}(\Omega) \text{ for all } i \in \{1, \ldots, N\}\}$$

and it is endowed with the norm

$$\|u\|_{W^{1,\vec{p}(\cdot)}(\Omega)} = \|u\|_{L^{p_M(\cdot)}(\Omega)} + \sum_{i=1}^{N} \|\partial_{x_i} u\|_{L^{p_i(\cdot)}(\Omega)}.$$

The space $\left(W^{1,\vec{p}(\cdot)}(\Omega), \|\cdot\|_{W^{1,\vec{p}(\cdot)}(\Omega)}\right)$ is a reflexive Banach space. Furthermore, an embedding theorem takes place for all the exponents that are strictly less than a variable critical exponent, which is introduced with the help of the notations

$$\bar{p}(x) = \frac{N}{\sum_{i=1}^{N} 1/p_i(x)}, \quad q^{\star}(x) = \begin{cases} Nq(x)/[N - q(x)] & \text{if } q(x) < N, \\ \infty & \text{if } q(x) \geq N. \end{cases}$$

Proposition 2 *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set and $p_i \in C_+(\overline{\Omega})$ for all $i \in \{1, \ldots, N\}$. If $q \in C(\overline{\Omega}, \mathbb{R})$, $1 \leq q(x) < \max\{\overline{p}^*(x), p_M(x)\}$ for all $x \in \overline{\Omega}$, then we have the compact and continuous embedding $W^{1,\vec{p}(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$.*

We denote by $X = W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,\vec{p}(\cdot)}(\Omega)$. The space $\left(W_0^{1,\vec{p}(\cdot)}(\Omega), \|u\|_{W_0^{1,\vec{p}(\cdot)}(\Omega)}\right)$ is a reflexive Banach space, where

$$\|u\| = \|u\|_{W_0^{1,\vec{p}(\cdot)}(\Omega)} = \sum_{i=1}^{N} \|\partial_{x_i} u\|_{L^{p_i(\cdot)}(\Omega)}.$$

3 Main Results

Proposition 3 ([7]) *Putting*

$$I(u) = \sum_{i=1}^{N} I_i(u) = \sum_{i=1}^{N} \int_\Omega \frac{1}{p_i(x)} |\partial_{x_i} u)|^{p_i(x)}\, dx,$$

then $I \in C^1(X, \mathbb{R})$ and the derivative operator I' of I is

$$I'(u).v = \sum_{i=1}^{N} \int_\Omega |\partial_{x_i} u|^{p_i(x)-2} \partial_{x_i} u\ \partial_{x_i} v\, dx.$$

(i) The functional I' is of (S_+) type, where I' is the Gâteaux derivative of the functional I.

(ii) I' is a bounded homeomorphism and strictly monotone operator.

For the function M_i, $i = 1, ..., N$, we make the following assumptions.

(M_0) $\exists\, m_0 > 0$ such that
$$M_i(t) \geq m_0 \quad \text{for all } t \geq 0. \tag{6}$$

(M_1) $\exists\, 0 < \mu < 1$ such that
$$\widehat{M_i}(t) \geq (1-\mu) M_i(t) t \quad \text{for all } t \geq 0, \tag{7}$$
where $\widehat{M_i}(t) = \int_0^t M_i(s) ds$.

(M_2) M_i is a differentiable and decreasing function on $\mathbb{R}^+$.

For the function f we assume the following conditions are satisfied.

(f_0) $|f(x,t)| \leq C(1+|t|^{q(x)-1})$ for all $(x,t) \in \Omega \times \mathbb{R}$ with $C \geq 0$ and $1 < q(x) < p^*(x)$, where $p^*(x) = \frac{Np(x)}{N-p(x)}$ if $p(x) < N$, $p^*(x) = +\infty$ if $p(x) \geq N$.

(f_1) $\lim_{|t| \to \infty} \frac{F(x,t)}{|t|^{\frac{p_M^+}{1-\mu}}} = +\infty$, uniformly for a.e. $x \in \Omega$, where $F(x,t) = \int_0^t f(x,s) ds$.

(f_2) There exists $\theta \geq 1$ such that $\theta G(x,t) \geq G(x,st)$ for $(x,t) \in \Omega \times \mathbb{R}$ and $s \in [0,1]$, where $G(x,t) = f(x,t)t - \frac{p_M^+}{1-\mu} F(x,t)$.

(f_3) $\lim_{t \to 0} \frac{F(x,t)}{|t|^{p_M^+}} = 0$, uniformly for a.e. $x \in \Omega$.

Under (AR), any Palais Smale (PS) sequence of the corresponding energy functional is bounded, which plays an important role in the application of variational methods. Indeed, there are many superlinear functions which do not satisfy (AR) condition. For instance the function below does not satisfy (AR)
$$f(x,t) = \frac{p_M^+}{1-\mu} |t|^{\frac{p_M^+}{1-\mu}-2} t ln(|t|), \tag{8}$$
where $\mu \in (0,1)$. But it is easy to see the above function (8) satisfies (f_0)–(f_3).

Now we are ready to state our result.

Theorem 4 *Assume that (M_0)–(M_2) and (f_0)–(f_3) are satisfied. Moreover, we assume that*

(f_4) $f(x,-t) = -f(x,t)$ for all $x \in \Omega$ and $t \in \mathbb{R}$.

If $q^- > p_M^+$, then problem (1) has a sequence of weak solutions $\{\pm u_k\}_{k=1}^{\infty}$ such that $J(\pm u_k) \to +\infty$ as $k \to +\infty$.

Example 1 *For $i = 1, ..., N$, $M_i(t) = 2 + \frac{1}{1+t^2}$, $i = 1, ..., N$. It is easy to see that M_i satisfies the conditions (M_0)–(M_2). It is clear that the above function (8) satisfies (f_0)–(f_3).*

Define
$$J(u) = \sum_{i=1}^{N} \widehat{M_i}(I_i(u)) - \int_\Omega F(x,u)\, dx, \quad u \in X,$$
where $\widehat{M_i}(t) = \int_0^t M_i(s)\, ds$.

Definition 1 *We call the weak solution for problem (1) a function $u \in X$ satisfying:*
$$\sum_{i=1}^{N} M_i \left(\int_\Omega \frac{1}{p_i(x)} |\partial_{x_i} u|^{p_i(x)}\, dx \right) \int_\Omega |\partial_{x_i} u|^{p_i(x)-2} \partial_{x_i} u\, \partial_{x_i} v\, dx - \int_\Omega f(x,u) v\, dx = 0$$
for all $v \in X$.

4 Proofs

First of all, we start with the following compactness result which plays a crucial role.

Lemma 5 *Under assumptions* (M_0)–(M_2) *and* (f_0)–(f_2), J *verifies the Cerami condition.*

Proof. For all $c \in R$, let's prove that J satisfies the first assertion (i) of Cerami condition. Let $\{u_n\} \subset X$ be bounded, $J(u_n) \to c$ and $J'(u_n) \to 0$. Without loss of generality, we assume that $u_n \rightharpoonup u$, then $J'(u_n)(u_n - u) \to 0$. Thus we have

$$\begin{aligned} J'(u_n)(u_n - u) &= \sum_{i=1}^{N} M_i \left(\int_\Omega \frac{|\partial_{x_i} u_n|^{p_i(x)}}{p_i(x)} dx \right) \int_\Omega |\partial_{x_i} u_n|^{p_i(x)-2} \partial_{x_i} u_n (\partial_{x_i} u_n - \partial_{x_i} u) dx \\ &\quad - \int_\Omega f(x, u_n)(u_n - u) dx \to 0. \end{aligned}$$

In view of (f_0) and Propositions 2, it follows that $\int_\Omega f(x, u_n)(u_n - u)dx \to 0$. Therefore, we have

$$\sum_{i=1}^{N} M_i \left(\int_\Omega \frac{|\partial_{x_i} u_n|^{p_i(x)}}{p_i(x)} dx \right) \int_\Omega |\partial_{x_i} u_n|^{p_i(x)-2} \partial_{x_i} u_n (\partial_{x_i} u_n - \partial_{x_i} u) dx \to 0.$$

From (M_0), we infer that

$$\int_\Omega |\partial_{x_i} u_n|^{p_i(x)-2} \partial_{x_i} u_n (\partial_{x_i} u_n - \partial_{x_i} u) dx \to 0.$$

This shows that $u_n \to u$ in X. Afterwards, we claim that J satisfies the assertion (ii) of Cerami condition. Otherwise, there exist $c \in R$ and $\{u_n\} \subset X$ such that:

$$J(u_n) \to c, \qquad \|u_n\| \to \infty, \qquad \|J'(u_n)\| \|u_n\| \to 0. \tag{9}$$

By (9), we have that $J(u_n) - \frac{1-\mu}{p_M^+} J'(u_n)(u_n) \to c$ as $n \to +\infty$. Denote $w_n = \frac{u_n}{\|u_n\|}$, then $\{w_n\}$ is bounded. Up to a subsequence, for some $w \in X$, we obtain

$$w_n \rightharpoonup w \quad \text{in } X, w_n \to w \quad \text{in } L^{q(x)}(\Omega), w_n(x) \to w(x) \quad \text{a.e. in } \Omega.$$

If $w \equiv 0$, we can define a sequence $\{t_n\} \subset R$ such that

$$J(t_n u_n) = \max_{t \in [0,1]} J(t u_n).$$

For any $B > 0$, putting $b_n = \left(2BN^{p_m^- - 1} p_M^+\right)^{\frac{1}{p_m^-}} w_n = K\omega_n$, since $b_n \to 0$ in $L^{q(x)}(\Omega)$ and $|F(x,t)| \leq C(1 + |t|^{q(x)})$, by the continuity of the Nemytskii operator, we see that $F(., b_n) \to 0$ in $L^1(\Omega)$ as $n \to +\infty$, therefore

$$\lim_{n \to \infty} \int_\Omega F(x, b_n) dx = 0. \tag{10}$$

Putting

$$\sigma_{i,n} = \begin{cases} p_M^+ & \text{if } \|\partial_{x_i} u_n\|_{L^{p_i(x)}} < 1, \\ p_m^- & \text{if } \|\partial_{x_i} u_n\|_{L^{p_i(x)}} > 1. \end{cases}$$

Similarly, from Remark 2.1 in [11], by using (4) and (5) we have for any n,

$$\begin{aligned}
\sum_{i=1}^{N}\int_{\Omega}|\partial_{x_i}u_n|^{p_i(x)}\,dx &\geq \sum_{i=1}^{N}\|\partial_{x_i}u_n\|_{L^{p_i(x)}(\Omega)}^{\sigma_{i,n}} \\
&\geq \sum_{i=1}^{N}\|\partial_{x_i}u_n\|_{L^{p_i(x)}(\Omega)}^{p_m^-} - \sum_{\{i:\ \sigma_{i,n}=p_M^+\}}\Big(\|\partial_{x_i}u_n\|_{L^{p_i(x)}(\Omega)}^{p_m^-} - \|\partial_{x_i}u_n\|_{L^{p_i(x)}(\Omega)}^{p_M^+}\Big) \\
&\geq N\left(\frac{\sum_{i=1}^{N}\|\partial_{x_i}u_n\|_{L^{p_i(x)}}}{N}\right)^{p_m^-} - N \\
&= \frac{\|u_n\|^{p_m^-}}{N^{p_m^- -1}} - N.
\end{aligned}$$

For n large enough, $\left(2BN^{p_m^- -1}p_M^+\right)^{\frac{1}{p_m^-}}/\|u_n\| \in (0,1)$. From (M_0) and by virtue of the last inequalities we obtain

$$\begin{aligned}
J(t_nu_n) &\geq J(b_n) \\
&\geq \sum_{i=1}^{N}\int_0^{I_i(b_n)} m_0\,ds - \int_{\Omega}F(x,b_n)dx \\
&\geq \frac{m_0}{p_M^+}\frac{\|b_n\|^{p_m^-}}{N^{p_m^- -1}} - \frac{N}{p_M^+} - \int_{\Omega}F(x,b_n)dx \\
&\geq m_0\left(\frac{\|K\omega_n\|^{p_m^-}}{p_M^+N^{p_m^- -1}} - \frac{N}{p_M^+}\right) - \int_{\Omega}F(x,b_n)dx \\
&\geq m_0\left(\frac{K^{p_m^-}}{2p_M^+N^{p_m^- -1}} - \frac{N}{p_M^+}\right).
\end{aligned}$$

That is, $J(t_nu_n)\to+\infty$. From $J(0)=0$ and $J(u_n)\to c$, we know that $t_n\in(0,1)$ and

$$\langle J'(t_nu_n),t_nu_n\rangle = t_n\frac{d}{dt}\Big|_{t=t_n}J(tu_n)=0.$$

According to (M_2) and (f_2), we get

$$\begin{aligned}
c &= \lim_{n\to+\infty}\left[J(u_n) - \frac{1-\mu}{p_M^+}\langle J'(u_n),u_n\rangle\right] \\
&= \lim_{n\to+\infty}\left[\sum_{i=1}^{N}\widehat{M}_i\Big(I_i(u_n)\Big) - \frac{1-\mu}{p_M^+}\sum_{i=1}^{N}M_i\Big(I_i(u_n)\Big)\int_{\Omega}|\partial_{x_i}u_n|^{p_i(x)}dx + \frac{1-\mu}{p_M^+}\int_{\Omega}G(x,u_n)dx\right] \\
&\geq \lim_{n\to+\infty}\left[\sum_{i=1}^{N}\widehat{M}_i\Big(I_i(t_nu_n)\Big) - \frac{1-\mu}{p_M^+}\sum_{i=1}^{N}M_i\Big(I_i(t_nu_n)\Big)\int_{\Omega}t_n^{p_i(x)}|\partial_{x_i}u_n|^{p_i(x)}dx + \frac{1-\mu}{p_M^+}\int_{\Omega}\frac{G(x,t_nu_n)}{\theta}dx\right] \\
&= \lim_{n\to+\infty}\frac{1}{\theta}\left[J(t_nu_n) - \frac{1-\mu}{p_M^+}\langle J'(t_nu_n),t_nu_n\rangle\right] = +\infty,
\end{aligned}$$

which is impossible. If $w\neq 0$, then the set $\Lambda=\{x\in\Omega : w(x)\neq 0\}$ has positive Lebesgue measure, for $x\in\Lambda$ we have $|u_n(x)|\to+\infty$ as $n\to+\infty$. Hence by (f_1) we deduce

$$\frac{F(x,u_n)}{|u_n(x)|^{\frac{p_M^+}{1-\mu}}}|w_n(x)|^{\frac{p_M^+}{1-\mu}} \to +\infty \qquad \text{as} \quad n\to+\infty. \tag{11}$$

Noting that $1 < \frac{p_M^+}{1-\mu}$. Then

$$\frac{\|u_n\|}{\|u_n\|^{\frac{p_M^+}{1-\mu}}} \to 0.$$

Now, because $J(u_n) \to c$, by (11), we deduce via the Fatou Lemma that

$$\begin{aligned}
\frac{C_1}{(p_m^-)^{\frac{1}{1-\mu}}} - \frac{c+o(1)}{\|u_n\|^{\frac{p_M^+}{1-\mu}}} &\geq \int_\Omega \frac{F(x,u_n)}{\|u_n\|^{\frac{p_M^+}{1-\mu}}} dx \\
&= \int_{w_n \neq 0} \frac{F(x,u_n)}{|u_n(x)|^{\frac{p_M^+}{1-\mu}}} |w_n(x)|^{\frac{p_M^+}{1-\mu}} dx + \int_{w_n = 0} \frac{F(x,u_n)}{|u_n(x)|^{\frac{p_M^+}{1-\mu}}} |w_n(x)|^{\frac{p_M^+}{1-\mu}} dx \\
&= \int_{w_n \neq 0} \frac{F(x,u_n)}{|u_n(x)|^{\frac{p_M^+}{1-\mu}}} |w_n(x)|^{\frac{p_M^+}{1-\mu}} dx \to +\infty,
\end{aligned}$$

which is absurd. ■

Because X is a separable and reflexive Banach space [7], there exist $\{e_j\}_{j=1}^\infty \subset X$ and $\{f_j\}_{j=1}^\infty \subset X^*$ such that

$$f_i(e_j) = \delta_{i,j} = \begin{cases} 1, & \text{if } i = j; \\ 0, & \text{if } i \neq j. \end{cases}$$

$$X = \overline{span}\{e_j : j = 1, 2, \dots\}, \quad X^* = \overline{span}^{w^*}\{f_j : j = 1, 2, \dots\}.$$

For $k = 1, 2, \dots$ denote by

$$X_j = span\{e_j\}, \quad Y_k = \oplus_{j=1}^k X_j, \quad Z_k = \overline{\oplus_{j=k}^\infty X_j}.$$

Lemma 6 ([1]) *For $q \in C_+(\Omega)$, $q(x) < p^*(x)$ for any $x \in \Omega$, define $\beta_k = \sup\{|u|_{q(x)} : \|u\| = 1,\ u \in Z_k\}$. Then $\lim_{k\to\infty} \beta_k = 0$.*

Proof of Theorem 4. We use the Fountain Theorem. According to Lemma 5 and (f_4), J is an even functional and satisfies condition (C). We show that for k large enough, there exist $\rho_k > r_k > 0$ such that:

(A1) $b_k := \inf\{J(u) :\ u \in Z_k, \|u\| = r_k\} \to +\infty$ as $k \to +\infty$;

(A2) $a_k := \max\{J(u) :\ u \in Y_k, \|u\| = \rho_k\} \leq 0$ as $k \to +\infty$.

(A1): Let $u \in Z_k$ with $\|u\| = r_k > 1$ (r_k will be specified below), by using condition (f_0), we have

$$\begin{aligned}
J(u) &= \sum_{i=1}^N \widehat{M}\Big(I_i(u)\Big) - \int_\Omega F(x,u)dx \\
&\geq \frac{m_0}{p_M^+}\|u\|^{p_m^-} - \int_\Omega C(|u| + |u|^{q(x)})dx \\
&\geq \frac{m_0}{p_M^+}\|u\|^{p_m^-} - C\|u\|^{q(\xi)}_{L^{q(x)}(\Omega)} - C\|u\|, \quad \text{where } \ \xi \in \Omega, \\
&\geq \begin{cases} \frac{m_0}{p_M^+}\|u\|^{p_m^-} - C - C\|u\| & \text{if } \|u\|_{L^{q(x)}(\Omega)} \leq 1 \\ \frac{m_0}{p_M^+}\|u\|^{p_m^-} - C(\beta_k\|u\|)^{q^+} & \text{if } \|u\|_{L^{q(x)}(\Omega)} > 1 \end{cases} \\
&\geq \frac{m_0}{p_M^+}\|u\|^{p_m^-} - C(\beta_k\|u\|)^{q^+} - C\|u\| \\
&= r_k^{p_m^-}\left(\frac{m_0}{p_M^+} - C\beta_k^{q^+} r_k^{q^+ - p_m^-}\right) - Cr_k.
\end{aligned}$$

Define $r_k = \left(\frac{Cq^+\beta_k^{q^+}}{m_0}\right)^{\frac{1}{p_m^- - q^+}}$, therefore

$$J(u) \geq m_0 r_k^{p_m^-}\left(\frac{1}{p_M^+} - \frac{1}{q+}\right) - Cr_k - C.$$

The fact $1 < p_m^- \leq p_M^+ < q^+$ implies that $r_k \to +\infty$, when $k \to +\infty$. Thus, $J(u) \to +\infty$ as $\|u\| \to +\infty$ with $u \in Z_k$.

(A2): Since $dim Y_k < \infty$ and all norms are equivalent in the finite-dimensional space, there exists $d_k > 0$, for all $u \in Y_k$ with $\|u\| \geq 1$, we have

$$\Phi(u) = \sum_{i=1}^{N} \widehat{M_i}\Big(I_i(u)\Big) \leq C_1 \frac{\|u\|^{\frac{p_M^+}{1-\mu}}}{(p_m^-)^{\frac{1}{1-\mu}}} \leq d_k \|u\|^{\frac{p_M^+}{1-\mu}}_{L^{\frac{p_M^+}{1-\mu}}(\Omega)}. \tag{12}$$

By conditions (f_0), (f_1) and (f_3) we have

$$F(x,u) \geq 2d_k|u|^{\frac{p_M^+}{1-\mu}} - \epsilon|u|^{p_M^+}, \quad \forall (x,u) \in \Omega \times \mathbb{R}. \tag{13}$$

Combining (12) and (13), for $u \in Y_k$ such that $\|u\| = \rho_k > r_k$, we infer that

$$\begin{aligned} J(u) &\leq -d_k\|u\|^{\frac{p_M^+}{1-\mu}}_{L^{\frac{p_M^+}{1-\mu}}(\Omega)} + \epsilon\|u\|^{p_M^+}_{L^{p_M^+}(\Omega)} \\ &\leq -\frac{C_2}{(p_m^-)^{\frac{1}{1-\mu}}}\|u\|^{\frac{p_M^+}{1-\mu}} + \epsilon C_3\|u\|^{p_M^+}. \end{aligned}$$

Therefore, for ρ_k large enough ($\rho_k > r_k$), we get from the above that

$$a_k := \max\{J(u) : \ u \in Y_k, \|u\| = \rho_k\} \leq 0.$$

The assertion (A2) holds, and this completes the proof of Theorem 4. ■

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