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Sampling Theorem Associated with Dirac System of the Hahn Difference Operator*

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Abstract

The sampling theory and its applications and extensions have been the subject of intensive research, both by mathematicians and by communication engineers. In this paper, we give an extension of this sampling theory to the Hahn difference operator. In other words, we state and prove the sampling theorem associated with q,ω-Dirac system. As a special case, we give two examples.

1 Introduction

The sampling theory showed up in communication systems and signal processing. Shannon [22] was the first who introduced the sampling theorem to information theory, but the basis for the theorem was laid by several others. Whittaker [27] studied the problem of finding an analytic expression of a function, whose values are known. In the Russian literature an equivalent statement was given by Kotelnikov [20]. Therefore the sampling theorem is sometimes referred as the WKS sampling theorem. This WKS sampling theorem states that if $f \in PW_{\pi}(\mathbb{R})$, which is known as the Paley-Wiener space, then f can be determined from its values at the integers via the sampling series

$$f(t) = \sum_{n=-\infty}^{\infty} f(n) \frac{\sin \pi (t-n)}{\pi (t-n)},\ t \in \mathbb{C}.$$

This series is absolutely and uniformly convergent on compact subsets of $\mathbb{C}$, uniformly convergent on $\mathbb{R}$, (see [10, 11, 18, 28]).

This theorem has been extended in many different ways. Sampling theory associated with boundary value problems was first observed by Weiss in [26] and outlined in general terms by Kramer in [21]. But a more systematic approach to this subject was first introduced by Zayed et al. in [29]. Since then, many authors have studied on sampling theory associated with discontinuous Sturm-Liouville problems [5, 16, 17, 23], and Dirac systems [1, 24, 25]. In recent years, these results have been extended to several types of difference operators, like the q-difference operators [2, 3, 4, 9, 14, 19], and the q,ω-Hahn difference operators [8]. In this paper, we present another example of extending the q,ω-Hahn difference operators. That is, we derive sampling theory associated with q,ω-Dirac system defined below. Finally, as a special case, we give two examples showing sampling formula. Consider the q,ω-Dirac system

$$\begin{cases} -\dfrac{1}{q} D_{\frac{1}{q},\frac{-\omega}{q}} y_2 + p(t)\, y_1 = \lambda y_1, \\ D_{q,\omega} y_1 + r(t)\, y_2 = \lambda y_2, \end{cases} \tag{1}$$

with the boundary conditions

$$B_1(y) := k_{11} y_1(\omega_0) + k_{12} y_2(\omega_0) = 0, \tag{2}$$

$$B_2(y) := k_{21} y_1(a) + k_{22} y_2\left(h^{-1}(a)\right) = 0, \tag{3}$$

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where $\omega_0 \leq t \leq a < \infty$, k_{ij} $(i,j=1,2)$ are real numbers, $\lambda \in \mathbb{C}$, $p(.)$ and $r(.)$ are real-valued continuous functions and $h(t)$ is a function defined below. Here $D_{q,\omega}$ is the Hahn difference operator which is defined by

$$D_{q,\omega} f(t) := \begin{cases} \dfrac{f(qt+\omega)-f(t)}{(qt+\omega)-t}, & t \neq \omega_0, \\ f'(\omega_0), & t = \omega_0, \end{cases}$$

where $q \in (0,1)$, $\omega > 0$ are fixed and $\omega_0 := \omega/(1-q)$, (see [12, 13]). This is valid if f is differentiable at ω_0. The function f needs to be defined on I; an interval of $\mathbb{R}$ containing ω_0. The q,ω-Dirac system (1)-(3) was introduced in [15]. The author investigated the existence and uniqueness of solutions and also obtained some spectral properties such as the eigenvalues $\{\lambda_n\}_{n=-\infty}^{\infty}$ are real and simple, the eigenfunctions that belong to different eigenvalues are orthogonal. The function h is defined by

$$h(t) := qt+\omega,\ t \in I.$$

The kth order iteration of $h(t)$ is given by

$$h^k(t) = q^k t + \omega [k]_q,\ t \in I,$$

where the sequence $h^k(t)$ is uniformly convergent to ω_0 on I and

$$[k]_q = \frac{1-q^k}{1-q},$$

are the q-numbers.

There exists the following relation between $D_{q,\omega}$ and its adjoint operator

$$(D_{q,\omega} f)\left(h^{-1}(t)\right) = D_{\frac{1}{q},\frac{-\omega}{q}} f(t).$$

The q,ω-type product formula is given by

$$D_{q,\omega}(fg)(t) = D_{q,\omega}(f(t))\, g(t) + f(qt+\omega)\, D_{q,\omega} g(t).$$

The q,ω-integral is introduced in [6] to be the Jackson-Nörlund sum

$$\int_a^b f(t)\, d_{q,\omega} t = \int_{\omega_0}^b f(t)\, d_{q,\omega} t - \int_{\omega_0}^a f(t)\, d_{q,\omega} t,$$

where $\omega_0 < a < b,\ a,b \in I$, and

$$\int_{\omega_0}^x f(t)\, d_{q,\omega} t = (x(1-q)-\omega) \sum_{k=0}^{\infty} q^k f\left(xq^k + \omega [k]_q\right),\ x \in I,$$

provided that the series converges. The fundamental theorem of q,ω- calculus given in [6] states that if $f : I \to \mathbb{R}$ is continuous at ω_0, and

$$F(t) := \int_{\omega_0}^t f(x)\, d_{q,\omega} x,\ x \in I,$$

then F is continuous at ω_0. Furthermore, $D_{q,\omega} F(t)$ exists for every $t \in I$ and

$$D_{q,\omega} F(t) = f(t).$$

Conversely,

$$\int_a^b D_{q,\omega} f(t)\, d_{q,\omega} t = f(b) - f(a), \text{ for all } a, b \in I.$$

The q,ω-integration by parts for continuous functions $f,\ g$ is given by

$$\int_a^b f(t)\, D_{q,\omega} g(t)\, d_{q,\omega} t = f(t)\, g(t)\big|_a^b - \int_a^b D_{q,\omega}(f(t))\, g(qt+\omega)\, d_{q,\omega} t,\ \ a, b \in I.$$

Trigonometric functions of q,ω-cosine and sine are defined by

$$C_{q,\omega}(t,\mu) := \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2} \left(\mu\left(t(1-q)-\omega\right)\right)^{2n}}{(q;q)_{2n}},\ \ t \in \mathbb{C},$$

$$S_{q,\omega}(t,\mu) := \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)} \left(\mu\left(t(1-q)-\omega\right)\right)^{2n+1}}{(q;q)_{2n+1}},\ \ t \in \mathbb{C}.$$

Here $(q;q)_k$ is the q-shifted factorial

$$(q;q)_k := \begin{cases} 1, & k = 0, \\ \prod\limits_{j=1}^{k} \left(1-q^j\right), & k = 1, 2, \ldots. \end{cases}$$

Furthermore, these functions have real and simple zeros $\{\pm x_n\}_{n=1}^{\infty}$ and $\{\pm y_n\}_{n=1}^{\infty}$, respectively,

$$x_n = \omega_0 + q^{-n+1/2}(1-q)^{-1}\left(1 + O(q^n)\right),$$

$$y_n = \omega_0 + q^{-n}(1-q)^{-1}\left(1 + O(q^n)\right),$$

where $n \geq 1$, see [7].

Let $C^2_{q,\omega}(\omega_0, a)$ be the subspace of $L^2_{q,\omega}(\omega_0, a)$, which consists of all functions $y(.)$ for which $y(.)$ and $D_{q,\omega} y(.)$ are continuous at ω_0. Let $H_{q,\omega}$ be the Hilbert space

$$H_{q,\omega} := \left\{ y(.) = \begin{pmatrix} y_1(.) \\ y_2(.) \end{pmatrix},\ y_1, y_2 \in C^2_{q,\omega}(\omega_0, a) \right\}.$$

The inner product of $H_{q,\omega}$ is defined by

$$\langle y(.),\ z(.) \rangle := \int_{\omega_0}^{a} y^{\top}(t)\, \overline{z(t)} d_{q,\omega} t,$$

where $\top$ denotes the matrix transpose. Let

$$y(.) = \begin{pmatrix} y_1(.) \\ y_2(.) \end{pmatrix},\ \ z(.) = \begin{pmatrix} z_1(.) \\ z_2(.) \end{pmatrix} \in H_{q,\omega}.$$

Then the q,ω-Wronskian of these functions is defined by

$$W(y,z)(t) := y_1(t)\, z_2\left(h^{-1}(t)\right) - z_1(t)\, y_2\left(h^{-1}(t)\right). \tag{4}$$

Let

$$y_1(t,\lambda_1) = \begin{pmatrix} y_{11}(t,\lambda_1) \\ y_{12}(t,\lambda_1) \end{pmatrix} \ \text{ and } \ y_2(t,\lambda_2) = \begin{pmatrix} y_{21}(t,\lambda_2) \\ y_{22}(t,\lambda_2) \end{pmatrix}$$

be two solutions of (1): hence

$$\begin{cases} -\dfrac{1}{q} D_{\frac{1}{q},\frac{-\omega}{q}} y_{12} + \{p(t) - \lambda_1\} y_{11} = 0, \\ D_{q,\omega} y_{11} + \{r(t) - \lambda_1\} y_{12} = 0, \end{cases} \tag{5}$$

and

$$\begin{cases} -\dfrac{1}{q} D_{\frac{1}{q},\frac{-\omega}{q}} y_{22} + \{p(t) - \lambda_2\} y_{21} = 0, \\ D_{q,\omega} y_{21} + \{r(t) - \lambda_2\} y_{22} = 0. \end{cases} \tag{6}$$

Multiplying (5) by y_{21} and y_{22} and (6) by $-y_{11}$ and $-y_{22}$, respectively, and adding together, we have

$$\begin{aligned} & D_{q,\omega} \left\{ y_{11}(t,\lambda_1) y_{22}\left(h^{-1}(t), \lambda_2\right) - y_{12}\left(h^{-1}(t), \lambda_1\right) y_{21}(t,\lambda_2) \right\} \\ = \; & (\lambda_1 - \lambda_2) \left\{ y_{11}(t,\lambda_1) y_{21}(t,\lambda_2) + y_{12}(t,\lambda_1) y_{22}(t,\lambda_2) \right\}. \end{aligned} \tag{7}$$

Let us consider the q,ω-Dirac equation (1) together with the following initial conditions

$$y_1(\omega_0,\lambda) = k_{12}, \quad y_2(\omega_0,\lambda) = -k_{11}. \tag{8}$$

By virtue of Theorem 2.3 in [15], this problem has a unique solution $\phi(t,\lambda) = \begin{pmatrix} \phi_1(t,\lambda) \\ \phi_2(t,\lambda) \end{pmatrix}$. It is obvious that $\phi(t,\lambda)$ satisfies the boundary condition (2) and this function is uniformly bounded on the subsets of the form $[\omega_0, a] \times \Omega$ where $\Omega \subset \mathbb{C}$ is compact set. To find the eigenvalues of the q,ω-Dirac system (1)–(3) we have to insert this function into the boundary condition (3) and find the roots of the obtained equation. So, putting the function $\phi(t,\lambda)$ into the boundary condition (3) we get the following equation whose zeros are the eigenvalues which are all real and simple (see [15])

$$\Delta(\lambda) = -k_{21}\phi_1(a,\lambda) - k_{22}\phi_2\left(h^{-1}(a), \lambda\right). \tag{9}$$

It is also known that if $\{\phi_n(.)\}_{n=-\infty}^{\infty}$ denotes a set of vector-valued eigenfunctions corresponding $\{\lambda_n\}_{n=-\infty}^{\infty}$, then $\{\phi_n(.)\}_{n=-\infty}^{\infty}$ is a complete orthogonal set of $H_{q,\omega}$. Since the eigenvalues are real, we can take the eigenfunctions to be real vector-valued.

2 The Sampling Theory

In this section, we state and prove q,ω-analogue of sampling theorem associated with q,ω-Dirac system (1)–(3).

Theorem 1 *Let $g(t) = \begin{pmatrix} g_1(t) \\ g_2(t) \end{pmatrix} \in H_{q,\omega}$ and $F(\lambda)$ be the q,ω-type transform*

$$F(\lambda) = \int_{\omega_0}^{a} g^{\top}(t)\, \phi(t,\lambda)\, d_{q,\omega}t, \quad \lambda \in \mathbb{C}, \tag{10}$$

where $\phi(t,\lambda)$ is the solution defined above. Then $F(\lambda)$ can be reconstructed using its values at the points $\{\lambda_n\}_{n=-\infty}^{\infty}$ by means of the sampling form

$$F(\lambda) = \sum_{n=-\infty}^{\infty} F(\lambda_n) \frac{\Delta(\lambda)}{(\lambda - \lambda_n)\Delta'(\lambda_n)}, \tag{11}$$

where $\Delta(\lambda)$ is defined in (9). The sampling series (11) converges absolutely on $\mathbb{C}$ and uniformly on compact subsets of $\mathbb{C}$.

Proof. Since $\phi(t,\lambda)$ is in $H_{q,\omega}$ for any λ, we have

$$\phi(t,\lambda)=\sum_{n=-\infty}^{\infty}\widehat{\phi}_n\frac{\phi_n(t)}{\|\phi_n\|^2},$$

where

$$\begin{aligned}\widehat{\phi}_n &= \int_{\omega_0}^{a}\phi^{\top}(t,\lambda)\,\phi_n(t)\,d_{q,\omega}t\\ &= \int_{\omega_0}^{a}\left\{\phi_1(t,\lambda)\,\phi_{n,1}(t)+\phi_2(t,\lambda)\,\phi_{n,2}(t)\right\}d_{q,\omega}t,\end{aligned} \tag{12}$$

$\phi_n^{\top}(t)=\left(\phi_{n,1}(t),\ \phi_{n,2}(t)\right)$ is the vector-valued eigenfunction corresponding to the eigenvalue λ_n and

$$\|\phi_n\|^2=\int_{\omega_0}^{a}\phi_n^{\top}(t)\,\phi_n(t)\,d_{q,\omega}t=\int_{\omega_0}^{a}\left(\phi_{n,1}^2(t)+\phi_{n,2}^2(t)\right)d_{q,\omega}t.$$

Since g is in $H_{q,\omega}$, it has the Fourier expansion

$$g(t)=\sum_{n=-\infty}^{\infty}\widehat{g}_n\frac{\phi_n(t)}{\|\phi_n\|^2},$$

where

$$\widehat{g}_n=\int_{\omega_0}^{a}g^{\top}(t)\,\phi_n(t)\,d_{q,\omega}t=\int_{\omega_0}^{a}\left\{g_1(t)\,\phi_{n,1}(t)+g_2(t)\,\phi_{n,2}(t)\right\}d_{q,\omega}t.$$

By Parseval's equality, (10) turns out to be

$$F(\lambda)=\sum_{n=-\infty}^{\infty}F(\lambda_n)\frac{\widehat{\phi}_n}{\|\phi_n\|^2}. \tag{13}$$

Let $\lambda\in\mathbb{C}$, $\lambda\neq\lambda_n$ and $n\in\mathbb{N}$ be fixed. From relation (7), with $y_{11}(t,\lambda_1)=\phi_1(t,\lambda)$, $y_{12}(t,\lambda_1)=\phi_2(t,\lambda)$ and $y_{21}(t,\lambda_2)=\phi_{n,1}(t)$, $y_{22}(t,\lambda_2)=\phi_{n,2}(x)$, we obtain

$$\begin{aligned}&(\lambda-\lambda_n)\int_{\omega_0}^{a}\left\{\phi_1(t,\lambda)\,\phi_{n,1}(t)+\phi_2(t,\lambda)\,\phi_{n,2}(t)\right\}d_{q,\omega}t\\ =\ & W(\phi,\phi_n)(t)|_{t=a}-W(\phi,\phi_n)(t)|_{x=\omega_0}.\end{aligned}$$

Since $\phi(.,\lambda)$ and $\phi_n(.)$ satisfy the same initial conditions (8) and from (4), we have

$$\begin{aligned}&(\lambda-\lambda_n)\int_{\omega_0}^{a}\left\{\phi_1(t,\lambda)\,\phi_{n,1}(t)+\phi_2(t,\lambda)\,\phi_{n,2}(t)\right\}d_{q,\omega}t\\ =\ & \phi_1(a,\lambda)\,\phi_{n,2}\left(h^{-1}(a)\right)-\phi_{n,1}(a)\,\phi_2\left(h^{-1}(a),\lambda\right).\end{aligned} \tag{14}$$

Assume that $k_{22}\neq 0$. Since $\phi_n(.)$ is an eigenfunction, then it satisfies (3). Hence

$$\phi_{n,2}\left(h^{-1}(a)\right)=-\frac{k_{21}}{k_{22}}\phi_{n,1}(a). \tag{15}$$

Substituting from (15) in (14), we obtain

$$\begin{aligned}
&(\lambda-\lambda_n)\int_{\omega_0}^{a}\left\{\phi_1(t,\lambda)\phi_{n,1}(t)+\phi_2(t,\lambda)\phi_{n,2}(t)\right\}d_{q,\omega}t\\
=\quad & -\phi_{n,1}(a)\left\{\frac{k_{21}}{k_{22}}\phi_1(a,\lambda)+\phi_2\left(h^{-1}(a),\lambda\right)\right\}=\frac{\Delta(\lambda)\phi_{n,1}(a)}{k_{22}}
\end{aligned}\tag{16}$$

provided that $k_{22}\neq 0$. Similarly, we can show that

$$(\lambda-\lambda_n)\int_{\omega_0}^{a}\left\{\phi_1(t,\lambda)\phi_{n,1}(t)+\phi_2(t,\lambda)\phi_{n,2}(t)\right\}d_{q,\omega}t=\frac{\Delta(\lambda)\phi_{n,2}\left(h^{-1}(a)\right)}{k_{21}}\tag{17}$$

provided that $k_{21}\neq 0$. Differentiating with respect to λ and taking the limit as $\lambda\to\lambda_n$, we obtain

$$\|\phi_n\|^2=\int_{\omega_0}^{a}\phi_n^{\top}(t)\phi_n(t)\,d_{q,\omega}t=\frac{\Delta'(\lambda_n)\phi_{n,1}(a)}{k_{22}},\tag{18}$$

and

$$\|\phi_n\|^2=\int_{\omega_0}^{a}\phi_n^{\top}(t)\phi_n(t)\,d_{q,\omega}t=\frac{\Delta'(\lambda_n)\phi_{n,2}\left(h^{-1}(a)\right)}{k_{21}}.\tag{19}$$

From (12), (16) and (18), we have for $k_{22}\neq 0$,

$$\frac{\widehat{\phi}_n}{\|\phi_n\|^2}=\frac{\Delta(\lambda)}{(\lambda-\lambda_n)\Delta'(\lambda_n)},\tag{20}$$

and if $k_{21}\neq 0$, we use (12), (17) and (19) to obtain the same result. Therefore from (13) and (20) we get (11) when λ is not an eigenvalue. Now we investigate the convergence of (11). Using Cauchy-Schwarz inequality for $\lambda\in\mathbb{C}$.

$$\begin{aligned}
&\sum_{n=-\infty}^{\infty}\left|F(\lambda_n)\frac{\Delta(\lambda)}{(\lambda-\lambda_n)\Delta'(\lambda_n)}\right|\\
=\quad&\sum_{n=-\infty}^{\infty}\left|\widehat{g}_n\frac{\widehat{\phi}_n}{\|\phi_n\|^2}\right|\\
\leq\quad&\left(\sum_{n=-\infty}^{\infty}\left|\frac{\widehat{g}_n}{\|\phi_n\|}\right|^2\right)^{1\backslash 2}\left(\sum_{n=-\infty}^{\infty}\left|\frac{\widehat{\phi}_n}{\|\phi_n\|}\right|^2\right)^{1\backslash 2}<\infty,
\end{aligned}$$

since $g(.),\ \phi(.,\lambda)\in H_{q,\omega}$. Thus (11) converge absolutely on $\mathbb{C}$. As for uniform convergence on Ω where Ω is a compact subsets of $\mathbb{C}$. Using Cauchy-Schwarz' inequality, we obtain that for $\lambda\in\Omega$,

$$\begin{aligned}
\Gamma_N(\lambda)\quad&=\quad\left|F(\lambda)-\sum_{n=-N}^{N}F(\lambda_n)\frac{\Delta(\lambda)}{(\lambda-\lambda_n)\Delta'(\lambda_n)}\right|\\
&\leq\quad\left(\sum_{n=-N}^{N}\frac{|\widehat{g}_n|^2}{\|\phi_n\|^2}\right)^{1\backslash 2}\left(\sum_{n=-N}^{N}\frac{\left|\widehat{\phi}_n\right|^2}{\|\phi_n\|^2}\right)^{1\backslash 2}.
\end{aligned}$$

In view of Bessel's inequality

$$\sum_{n=-N}^{N} \frac{\left|\widehat{\phi}_n\right|^2}{\|\phi_n\|^2} \leq \|\phi(.,\lambda)\|^2 .$$

Since the function $\phi(.,\lambda)$ is uniformly bounded on the subsets of $\mathbb{C}$, we can find a positive constant C_Ω which is independent of λ such that $\|\phi(.,\lambda)\| \leq C_\Omega$, $\lambda \in \Omega$. Thus

$$\Gamma_N(\lambda) \leq C_\Omega \left(\sum_{n=-N}^{N} \frac{|\widehat{g}_n|^2}{\|\phi_n\|^2} \right)^{1\backslash 2} \to 0 \text{ as } N \to \infty.$$

Hence (11) converges uniformly on compact subsets of $\mathbb{C}$, thus the proof is complete. ■

3 Examples

Example 1 *Consider q,ω-Dirac system (1)–(3) in which $p(t) = 0 = r(t)$:*

$$\begin{cases} -\dfrac{1}{q} D_{\frac{1}{q},\frac{-\omega}{q}} y_2 = \lambda y_1, \\ D_{q,\omega} y_1 = \lambda y_2, \end{cases} \tag{21}$$

$$y_1(\omega_0) = 0, \tag{22}$$

$$y_2\left(h^{-1}(\pi)\right) = 0. \tag{23}$$

It is easy to see that a solution of (21) and (22) is given by

$$\phi^{\intercal}(t,\lambda) = \left(-S_{q,\omega}(t,\lambda),\ -C_{q,\omega}(t,\sqrt{q}\lambda)\right).$$

By substituting this solution into (23), we obtain $\Delta(\lambda) = C_{q,\omega}\left(h^{-1}(\pi), \sqrt{q}\lambda\right)$, hence, the eigenvalues are $\lambda_n = \dfrac{q^{-n+1}}{(1-q)(\pi-\omega_0)}(1 + O(q^n))$. Applying Theorem 1, the q,ω-transforms

$$F(\lambda) = \int_{\omega_0}^{\pi} \{g_1(t) S_{q,\omega}(t,\lambda) + g_2(t) C_{q,\omega}(t,\sqrt{q}\lambda)\} d_{q,\omega}t,$$

for some $g_1, g_2 \in C_q^2(\omega_0,\pi)$, then it has the sampling formula

$$F(\lambda) = \sum_{n=-\infty}^{\infty} F(\lambda_n) \frac{C_{q,\omega}\left(h^{-1}(\pi), \sqrt{q}\lambda\right)}{\sqrt{q}(\lambda-\lambda_n) C'_{q,\omega}\left(h^{-1}(\pi), \sqrt{q}\lambda_n\right)},$$

where $C'_{q,\omega}\left(h^{-1}(\pi), \sqrt{q}\lambda_n\right) = \left(\frac{d}{d\lambda} C_{q,\omega}\left(h^{-1}(\pi), \sqrt{q}\lambda\right)\right)\Big|_{\lambda=\lambda_n}$.

Example 2 *Consider q,ω-Dirac equation (21) together with the following boundary conditions*

$$y_2(\omega_0) = 0,$$

$$y_2\left(h^{-1}(\pi)\right) = 0.$$

In this case $\phi^{\intercal}(t,\lambda) = \left(C_{q,\omega}(t,\lambda),\ -\sqrt{q}S_{q,\omega}\left(t,\sqrt{q}\lambda\right)\right)$. *Since* $\Delta(\lambda) = \sqrt{q}S_{q,\omega}\left(h^{-1}(\pi),\sqrt{q}\lambda\right)$, *then the eigenvalues are given by* $\lambda_n = \dfrac{q^{-n+1/2}}{(1-q)(\pi-\omega_0)}\left(1+O\left(q^n\right)\right)$. *Applying Theorem 1 above to the* q,ω*-transform*

$$F(\lambda) = \int_{\omega_0}^{\pi} \left\{g_1(t)\,C_{q,\omega}(t,\lambda) - g_2(t)\sqrt{q}S_{q,\omega}\left(t,\sqrt{q}\lambda\right)\right\} d_{q,\omega}t, \tag{55}$$

for some $g_1, g_2 \in C_q^2(\omega_0,\pi)$, *then we obtain*

$$F(\lambda) = \sum_{n=-\infty}^{\infty} F(\lambda_n)\,\frac{S_{q,\omega}\left(h^{-1}(\pi),\sqrt{q}\lambda\right)}{\sqrt{q}\left(\lambda-\lambda_n\right)S'_{q,\omega}\left(h^{-1}(\pi),\sqrt{q}\lambda_n\right)}, \tag{56}$$

where

$$S'_{q,\omega}\left(h^{-1}(\pi),\sqrt{q}\lambda_n\right) = \left(\frac{d}{d\lambda}S_{q,\omega}\left(h^{-1}(\pi),\sqrt{q}\lambda\right)\right)\Bigg|_{\lambda=\lambda_n}.$$

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