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L^1-Convergence Of The Sine Series Whose Coefficients Belong To Some Generalized Classes Of Sequences*

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Abstract

In this paper, we have introduced three generalized classes of sequences. In addition, we have studied the L^1-convergence of sine series whose coefficients belong to them. Finally, we show that our results covers some results proved previously by others.

1 Introduction

We consider trigonometric sine series

$$g(x) = \sum_{k=1}^{\infty} a_k \sin kx,$$

with its partial sums

$$S_n^s(x) = \sum_{k=1}^{n} a_k \sin kx,$$

and

$$\lim_{n\to\infty} S_n^s(x) = g(x).$$

It is a well-known fact that if a trigonometric series converges in L^1-norm, then it is a Fourier series. In general, the converse of this statement is not always true. So, the question to be considered is how to make possible that the converse statement to be true? For this purpose a lot of researchers have introduced the so-called modified trigonometric cosine sums or modified trigonometric sine sums or both as well as some classes of sequences to which the coefficients of a trigonometric series belong to.

The most famous modified trigonometric sums appearing in literature are

$$f_n(x) = \frac{1}{2}\sum_{k=0}^{n} \Delta a_k + \sum_{k=1}^{n}\sum_{j=k}^{n} \Delta a_j \cos kx,$$

introduced in [4], and then the modified cosine and sine sums

$$g_n^c(x) = \frac{a_0}{2} + \sum_{k=1}^{n}\sum_{j=k}^{n} \Delta\left(\frac{a_j}{j}\right) k\cos kx$$

and

$$g_n^s(x) = \sum_{k=1}^{n}\sum_{j=k}^{n} \Delta\left(\frac{a_j}{j}\right) k\sin kx,$$

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introduced in [11], where $\Delta a_i := a_i - a_{i+1}$. We do not recall here all modified trigonometric sums introduced by others, however we suggest the interested reader to consult the references [5]-[9] of the present paper and references therein to find other modified trigonometric sums.

Seemingly, it was S. A. Telyakovskii [13] who introduced the class $\widetilde{S}$ of sequences and F. Móricz [10] who introduced the classes $\widetilde{BV}$ and $\widetilde{C}$ of sequences.

Very recently, the authors of [2] introduced the following modified sine sums

$$z_n^s(x) = \sum_{k=1}^{n} \left[\frac{a_{k+1}}{k+1} + \sum_{j=k}^{n} \Delta^2 \left(\frac{a_j}{j} \right) \right] k \sin kx,$$

where $\Delta^2 a_i := \Delta(\Delta a_i) = a_i - 2a_{i+1} + a_{i+2}$, and the following classes of sequences:

Definition 1 *A sequence (a_k) tending to zero belongs to the class $\widetilde{C}_r$, $r = 0, 1, 2, \dots$, if for every $\varepsilon > 0$, there exists $\delta > 0$ independent on n and such that for all n,*

$$\int_0^{\delta} \left| \sum_{k=n}^{\infty} \Delta b_k D_k^{(r+1)}(x) \right| dx \leq \varepsilon,$$

where $b_k = \frac{a_k}{k}$ and $D_k^{(r+1)}(x)$ denotes the $(r+1)$-th derivative of the Dirichlet's kernel

$$D_k(x) = \frac{1}{2} + \sum_{j=1}^{k} \cos jx = \frac{\sin\left(k + \frac{1}{2}\right) x}{2 \sin \frac{x}{2}}.$$

Definition 2 *A sequence (a_k) tending to zero belongs to the class $\widetilde{S}_r$, $r = 0, 1, 2, \dots$, if there exists a non-increasing sequence (B_k) of numbers so that, $|\Delta b_k| \leq B_k$, $\forall k \in \{1, 2, \dots\}$, and $\sum_{k=1}^{\infty} k^{r+1} B_k < \infty$, where $b_k = \frac{a_k}{k}$.*

Definition 3 *A sequence (a_k) tending to zero belongs to the class $\widetilde{BV}_r$, $r = 0, 1, 2, \dots$, if*

$$\sum_{k=1}^{\infty} k^{r+1} |\Delta b_k| < \infty,$$

where $b_k = \frac{a_k}{k}$.

In the same paper there has been proved the following results.

Theorem 1 ([2]) *$\widetilde{S}_r \subset \widetilde{C}_r \cap \widetilde{BV}_r$ for each $r \in \{0, 1, 2, \dots\}$.*

Theorem 2 ([2]) *Let $(a_n) \in \widetilde{C} \cap \widetilde{BV}$ and $\lim_{n\to\infty} a_n \log n = 0$. Then*

$$\lim_{n\to\infty} \|z_n^s - g\| = 0.$$

Theorem 3 ([2]) *Let $(a_n) \in \widetilde{C} \cap \widetilde{BV}$ and $\lim_{n\to\infty} a_n \log n = 0$. Then*

$$\lim_{n\to\infty} \|S_n^s - g\| = 0.$$

Theorem 4 ([2]) *Let $(a_n) \in \widetilde{C}_r \cap \widetilde{BV}_r$ and $\lim_{n\to\infty} n^r a_n \log n = 0$, $r \in \{0, 1, 2, \dots\}$. Then*

$$\lim_{n\to\infty} \|(z_n^s)^{(r)} - g^{(r)}\| = 0.$$

Theorem 5 **([2])** *Let* $(a_n) \in \widetilde{C}_r \cap \widetilde{BV}_r$ *and* $\lim_{n\to\infty} n^r a_n \log n = 0$, $r \in \{0, 1, 2, \dots\}$. *Then*

$$\lim_{n\to\infty} \|(S_n^s)^{(r)} - g^{(r)}\| = 0.$$

Corollary 1 **([2])** *Let* $(a_n) \in \widetilde{S}_r$ *and* $\lim_{n\to\infty} n^r a_n \log n = 0$, $r \in \{0, 1, 2, \dots\}$. *Then*

(i) $\lim_{n\to\infty} \|(z_n^s)^{(r)} - g^{(r)}\| = 0$.

(ii) $\lim_{n\to\infty} \|(S_n^s)^{(r)} - g^{(r)}\| = 0$.

Now we introduce the following generalized modified sine sums

$$z_{n,m}^s(x) = \sum_{k=1}^{n} \left[\frac{a_{k+1}}{(k+1)^m} + \sum_{j=k}^{n} \Delta^2 \left(\frac{a_j}{j^m} \right) \right] k^m \sin kx, \ \ m \in \{1, 2, \dots\},$$

where again $\Delta^2 c_i := \Delta(\Delta c_i) := c_i - 2c_{i+1} + c_{i+2}$.

Remark 1 *Note that* $z_{n,1}^s(x) \equiv z_n^s(x)$ *which have been introduced for the fist time in [3].*

Further we generalize the classes $\widetilde{C}_r$, $\widetilde{S}_r$, and $\widetilde{BV}_r$, $(r \in \{0, 1, 2, \dots\})$ as follows:

Definition 4 *A sequence* (a_k) *tending to zero belongs to the class* $\widetilde{C}_{r,m}$, $(r = 0, 1, 2, \dots; \ m = 1, 2, \dots)$, *if for every* $\varepsilon > 0$, *there exists* $\delta > 0$ *independent on* n *and such that for all* n,

$$\int_0^\delta \left| \sum_{k=n}^{\infty} \Delta b_{k,m} D_k^{(r+m)}(x) \right| dx \le \varepsilon,$$

where $b_k = \frac{a_k}{k^m}$, $D_k^{(r+m)}(x)$ *denotes the* $(r+m)$*-th derivative of the Dirichlet kernel*

$$D_k(x) = \frac{1}{2} + \sum_{j=1}^{k} \cos jx = \frac{\sin\left(k + \frac{1}{2}\right)x}{2 \sin \frac{x}{2}}.$$

Remark 2 *It is clear that* $\widetilde{C}_{r+1,m} \subset \widetilde{C}_{r,m}$, $(r = 0, 1, 2, \dots; \ m = 1, 2, \dots)$, *however, the converse inclusion need not be true in general as shown in the next example.*

Example 1 *Define* $b_{n,m} = \sum_{k=n}^{\infty} \frac{1}{k^{r+m+2}}$, $(r = 0, 1, 2, \dots; \ m = 1, 2, \dots)$, *then* $\Delta b_{n,m} = \frac{1}{n^{r+m+2}}$ *and*

$$a_n = n b_{n,m} = n \sum_{k=n}^{\infty} \frac{1}{k^{r+m+2}} \le \sum_{k=n}^{\infty} \frac{k}{k^{r+m+2}} = \sum_{k=n}^{\infty} \frac{1}{k^{r+m+2}} \to 0, \quad \textit{when} \quad n \to \infty.$$

So, using Bernstein's inequality, the integral

$$\begin{aligned}
\int_0^\delta \left| \sum_{k=n}^{\infty} \Delta b_{k,m} D_k^{(r+1+m)}(x) \right| dx &\le \int_0^\pi \left| \sum_{k=n}^{\infty} \Delta b_{k,m} D_k^{(r+1+m)}(x) \right| dx \\
&\le \sum_{k=n}^{\infty} |\Delta b_{k,m}| \int_0^\pi \left| D_k^{(r+1+m)}(x) \right| dx \\
&\le \sum_{k=n}^{\infty} \frac{k^{r+1+m}}{k^{r+m+2}} \int_0^\pi |D_k(x)|\, dx = \mathcal{O}\left(\sum_{k=1}^{\infty} \frac{\log k}{k} \right),
\end{aligned}$$

is divergent, which means $(a_n) \notin \widetilde{C}_{r+1,m}$. On the other side, the integral

$$\begin{aligned}\int_0^{\delta}\left|\sum_{k=n}^{\infty}\Delta b_{k,m}D_k^{(r+m)}(x)\right|dx &\leq \int_0^{\pi}\left|\sum_{k=n}^{\infty}\Delta b_{k,m}D_k^{(r+m)}(x)\right|dx\\ &\leq \sum_{k=n}^{\infty}|\Delta b_{k,m}|\int_0^{\pi}\left|D_k^{(r+m)}(x)\right|dx\\ &\leq \sum_{k=n}^{\infty}\frac{k^{r+m}}{k^{r+m+2}}\int_0^{\pi}|D_k(x)|\,dx=\mathcal{O}\left(\sum_{k=1}^{\infty}\frac{\log k}{k^2}\right),\end{aligned}$$

is convergent, which means $(a_n) \in \widetilde{C}_{r,m}$.

Definition 5 *A sequence (a_k) tending to zero belongs to the class $\widetilde{S}_{r,m}$, $(r = 0, 1, 2, \ldots; m = 1, 2, \ldots)$, if there exists a non-increasing sequence (B_k) so that, $|\Delta b_{k,m}| \leq B_k$, $\forall k \in \{1, 2, \ldots\}$, and $\sum_{k=1}^{\infty} k^{r+m}B_k < \infty$, where $b_k = \frac{a_k}{k^m}$.*

Remark 3 *It is clear that $\widetilde{S}_{r+1,m} \subset \widetilde{S}_{r,m}$, $(r = 0, 1, 2, \ldots; m = 1, 2, \ldots)$, however, the converse inclusion need not to be true as shown in the next example.*

Example 2 *Define $b_{n,m} = \sum_{k=n}^{\infty} \frac{1}{k^{r+m+2}}$, $(r = 0, 1, 2, \ldots; m = 1, 2, \ldots)$, then $\Delta b_{n,m} = \frac{1}{n^{r+m+2}}$ and*

$$a_n = nb_{n,m} = n\sum_{k=n}^{\infty}\frac{1}{k^{r+m+2}} \leq \sum_{k=n}^{\infty}\frac{k}{k^{r+m+2}} = \sum_{k=n}^{\infty}\frac{1}{k^{r+m+1}} \to 0, \quad \textit{when} \quad n \to \infty.$$

Choosing $B_n = \frac{1}{n^{r+m+2}}$, $(r = 0, 1, 2, \ldots; m = 1, 2, \ldots)$, then $B_n \downarrow 0$ and $|\Delta b_{n,m}| \leq B_n$. Now, the series

$$\sum_{k=1}^{\infty}k^{r+m}B_k = \sum_{k=1}^{\infty}k^{r+m}\frac{1}{k^{r+m+2}} = \sum_{k=1}^{\infty}\frac{1}{k^2} < \infty$$

is convergent, which means $(a_n) \in \widetilde{S}_{r,m}$. However, the series

$$\sum_{k=1}^{\infty}k^{r+1+m}B_k = \sum_{k=1}^{\infty}k^{r+1+m}\frac{1}{k^{r+m+2}} = \sum_{k=1}^{\infty}\frac{1}{k}$$

is divergent, which means $(a_n) \notin \widetilde{S}_{r+1,m}$.

Definition 6 *A zero sequence (a_k) belongs to the class $\widetilde{BV}_{r,m}$, $(r = 0, 1, 2, \ldots; m = 1, 2, \ldots)$, if*

$$\sum_{k=1}^{\infty}k^{r+m}|\Delta b_{k,m}| < \infty,$$

where $b_{k,m} = \frac{a_k}{k^m}$.

Remark 4 *It is clear that $\widetilde{BV}_{r+1,m} \subset \widetilde{BV}_{r,m}$, $(r = 0, 1, 2, \ldots; m = 1, 2, \ldots)$, however, the converse inclusion may not be true.*

Remark 5 *We note that $\widetilde{C}_{r,m} \equiv \widetilde{C}_r$, $\widetilde{S}_{r,m} \equiv \widetilde{S}_r$, and $\widetilde{BV}_{r,m} \equiv \widetilde{BV}_r$ for $m = 1$, and $\widetilde{C}_{r,m} \equiv \widetilde{C}$, $\widetilde{S}_{r,m} \equiv \widetilde{S}$, and $\widetilde{BV}_{r,m} \equiv \widetilde{BV}$ for $m = 1$ and $r = 0$.*

The objective of this paper is to prove some theorems more general than Theorems 1–5 and Corollary 1 involving new classes $\widetilde{C}_{r,m}$, $\widetilde{S}_{r,m}$, and $\widetilde{BV}_{r,m}$. To achieve this objective we need to recall some lemmas which have already proved elsewhere. Throughout this paper, for two positive quantities u and v, we write $u = \mathcal{O}(v)$, if there exists a positive constant C so that $u \leq Cv$.

2 Helpful Lemmas

Lemma 1 ([15]) *Let m be a non-negative integer. Then for all $0 < |x| \le \pi$ and all $n \ge 1$ the estimate $|\widetilde{D}_n^{(m)}(x)| \le \frac{4n^r\pi}{|x|}$ holds true, where $\widetilde{D}_n^{(m)}(x)$ denotes m-th derivative of the conjugate Dirichlet kernel*

$$\widetilde{D}_k(x) = \sum_{j=1}^{k} \sin jx = \frac{\cos\frac{x}{2} - \cos\left(k+\frac{1}{2}\right)x}{2\sin\frac{x}{2}}.$$

Lemma 2 ([12]) *Let m be a non-negative integer. Then for all $0 < \varepsilon \le x \le \pi$ and all $n \ge 1$ the estimate $|D_n^{(m)}(x)| \le \frac{Cn^r}{x}$ holds true, where C denotes a positive absolute constant.*

Lemma 3 ([12]) $\|D_n^{(m)}(x)\|_{L^1} = \mathcal{O}\left(n^m \log n\right)$, $m \in \{0,1,2,\dots\}$, *holds true, where $D_n^{(m)}(x)$ denotes m-th derivative of the Dirichlet kernel.*

Lemma 4 ([7]) *If $D_n(x)$, $\widetilde{D}_n(x)$, and $F_n(x)$ are the Dirichlet, the conjugate Dirichlet and the Fejér kernel respectively, then $\widetilde{D}'_n(x) = (n+1)D_n(x) - (n+1)F_n(x)$.*

Lemma 5 ([14]) *Let the real numbers α_i, $i = 1,2,\dots,k$, satisfy conditions $|\alpha_i| \le 1$. Then the following estimations hold true*

$$\int_0^\pi \left| \sum_{i=0}^{k} \alpha_i \frac{\sin\left(i+\frac{1}{2}\right)x}{2\sin\frac{x}{2}} \right| dx \le C(k+1),$$

where C is a positive constant.

3 Main Results

At first, pertaining to the $\widetilde{BV}_{r,m}$ class, $r \in \{0,1,2,\dots\}$ and $m \in \{1,2,\dots\}$, we can raise the following natural question: What about inclusion of classes $\widetilde{BV}_{r,m}$ with respect to m? The answer is given in next simple proposition.

Proposition 1 *If*

$$\sum_{k=1}^{\infty} (k+1)^r |\Delta a_k| < \infty,$$

then

$$\widetilde{BV}_{r,m} \subset \widetilde{BV}_{r,m+1},$$

for all $r \in \{0,1,2,\dots\}$ and $m \in \{1,2,\dots\}$.

Proof. We have

$$\begin{aligned}
\sum_{k=1}^{\infty} k^{r+m+1}|\Delta b_{k,m+1}| &\leq \sum_{k=1}^{\infty} k^{r+m+1}\left|\frac{a_k}{k^{m+1}} - \frac{a_{k+1}}{k(k+1)^m}\right| \\
&\quad + \sum_{k=1}^{\infty} k^{r+m+1}\left|\frac{a_{k+1}}{k(k+1)^m} - \frac{a_{k+1}}{(k+1)^{m+1}}\right| \\
&\leq \sum_{k=1}^{\infty} k^{r+m}\left|\frac{a_k}{k^m} - \frac{a_{k+1}}{(k+1)^m}\right| + \sum_{k=1}^{\infty} k^{r-1}\left|a_{k+1}\right| \\
&\leq \sum_{k=1}^{\infty} k^{r+m}\left|\Delta b_{k,m}\right| + \sum_{k=1}^{\infty} k^{r-1}\sum_{j=k+1}^{\infty}\left|\Delta a_j\right| \\
&= \sum_{k=1}^{\infty} k^{r+m}\left|\Delta b_{k,m}\right| + \sum_{j=1}^{\infty}\left|\Delta a_j\right|\sum_{k=1}^{j+1} k^{r-1} \\
&\leq \sum_{k=1}^{\infty} k^{r+m}\left|\Delta b_{k,m}\right| + \sum_{j=1}^{\infty}(j+1)^r\left|\Delta a_j\right|,
\end{aligned}$$

which implies that $\widetilde{BV}_{r,m} \subset \widetilde{BV}_{r,m+1}$, for all $r \in \{0,1,2,\dots\}$ and $m \in \{1,2,\dots\}$. The proof is completed. ∎

Theorem 6 *The following relation holds true* $\widetilde{S}_{r,m} \subset \widetilde{C}_{r,m} \cap \widetilde{BV}_{r,m}$ *for each* $r \in \{0,1,2,\dots\}$ *and* $m \in \{1,2,\dots\}$.

Proof. Let $(a_k) \in \widetilde{S}_{r,m}$, $(r = 0,1,2,\dots;\ m = 1,2,\dots)$. Then there exists a non-increasing sequence (B_k) of numbers so that, $|\Delta b_{k,m}| \leq B_k$, $\forall k \in \{1,2,\dots\}$, and $\sum_{k=1}^{\infty} k^{r+m}B_k < \infty$. Whence, we clearly have

$$\sum_{k=1}^{\infty} k^{r+m}|\Delta b_{k,m}| \leq \sum_{k=1}^{\infty} k^{r+m}B_k < \infty, \tag{1}$$

which means that $\widetilde{S}_{r,m} \subset \widetilde{BV}_{r,m}$ for each $r \in \{0,1,2,\dots\}$ and $m \in \{1,2,\dots\}$. So, it remains to prove the inclusion $\widetilde{S}_{r,m} \subset \widetilde{C}_{r,m}$ for each $r \in \{0,1,2,\dots\}$ and $m \in \{1,2,\dots\}$. Let $(a_k) \in \widetilde{S}_{r,m}$. Then applying Abel's transformation we get

$$\begin{aligned}
\int_0^{\pi}\left|\sum_{k=n}^{\infty}\Delta b_{k,m}D_k^{(r+m)}(x)\right|dx &\leq \lim_{s\to\infty}\left[\sum_{k=n}^{s-1}\Delta B_k\int_0^{\pi}\left|\sum_{j=0}^{k}\frac{\Delta b_{j,m}}{B_j}D_j^{(r+m)}(x)\right|dx\right. \\
&\quad + B_s\int_0^{\pi}\left|\sum_{j=0}^{s}\frac{\Delta b_{j,m}}{B_j}D_j^{(r+m)}(x)\right|dx \\
&\quad \left. + B_n\int_0^{\pi}\left|\sum_{j=0}^{n-1}\frac{\Delta b_{j,m}}{B_j}D_j^{(r+m)}(x)\right|dx\right].
\end{aligned}$$

Applying, in last inequality, the well-known Bernstein's inequality and Lemma 5, we obtain

$$\begin{aligned}\int_0^\pi\left|\sum_{k=n}^{\infty}\Delta b_{k,m}D_k^{(r+m)}(x)\right|dx &\le \lim_{s\to\infty}\left[\sum_{k=n}^{s-1}k^{r+m}\Delta B_k\int_0^\pi\left|\sum_{j=0}^{k}\frac{\Delta b_{j,m}}{B_j}D_j(x)\right|dx\right.\\ &\quad +s^{r+m}B_s\int_0^\pi\left|\sum_{j=0}^{s}\frac{\Delta b_{j,m}}{B_j}D_j(x)\right|dx\\ &\quad \left.+(n-1)^{r+m}B_n\int_0^\pi\left|\sum_{j=0}^{n-1}\frac{\Delta b_{j,m}}{B_j}D_j(x)\right|dx\right]\\ &\le C\lim_{s\to\infty}\left[\sum_{k=n}^{s-1}(k+1)^{r+m+1}\Delta B_k\right.\\ &\quad \left.+s^{r+m+1}B_s+n^{r+m+1}B_n\right].\end{aligned}$$

Since (B_k) is a non-increasing sequence and $\sum_{k=1}^{\infty}k^{r+m}B_k<\infty$, we see that $k^{r+m+1}B_k\to 0$ as $k\to\infty$, and thus

$$\begin{aligned}\int_0^\pi\left|\sum_{k=n}^{\infty}\Delta b_{k,m}D_k^{(r+m)}(x)\right|dx &\le C\left[\sum_{k=n}^{\infty}(k+1)^{r+m+1}\Delta B_k+n^{r+m+1}B_n\right]\\ &\le C\left\{\sum_{k=n}^{\infty}B_k\left[(k+1)^{r+m+1}-k^{r+m+1}\right]+n^{r+m+1}B_n\right\}\\ &\le C(r,m)\left\{\sum_{k=n}^{\infty}k^{r+m}B_k+n^{r+m+1}B_n\right\}\le\frac{\varepsilon}{2},\end{aligned}$$

for n large enough, say $s\ge n>n_0$.

Finally, using the fact that

$$\left|D_k^{(r+m)}(x)\right|=\left|\sum_{j=1}^{k}j^{(r+m)}\sin\left(jx+\frac{(r+m)\pi}{2}\right)\right|\le k^{r+m+1},$$

for any $1\le n\le s$, we can write as follows

$$\begin{aligned}\int_0^\delta\left|\sum_{k=n}^{s}\Delta b_{k,m}D_k^{(r+m)}(x)\right|dx &\le \int_0^\delta\left|\sum_{k=n}^{n_0}\Delta b_{k,m}D_k^{(r+m)}(x)\right|dx\\ &\quad +\int_0^\pi\left|\sum_{k=n_0+1}^{s}\Delta b_{k,m}D_k^{(r+m)}(x)\right|dx\\ &\le \delta\sum_{k=n}^{n_0}k^{r+m+1}\left|\Delta b_{k,m}\right|\\ &\quad +\int_0^\pi\left|\sum_{k=n_0+1}^{\infty}\Delta b_{k,m}D_k^{(r+m)}(x)\right|dx\\ &\le \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon,\end{aligned}$$

for δ small enough. This means that $\widetilde{S}_{r,m}\subset\widetilde{C}_{r,m}$ for each $r\in\{0,1,2,\dots\}$ and $m\in\{1,2,\dots\}$. The proof is completed. ■

Remark 6 *For $m=1$ Theorem 6 reduces to Theorem 1.*

Theorem 7 *Let $(a_n) \in \widetilde{C}_m \cap \widetilde{BV}_m$ for each $m \in \{1, 2, \dots\}$, and $\lim_{n\to\infty} a_n \log n = 0$. Then*

$$\lim_{n\to\infty} \|z^s_{n,m} - g\| = 0.$$

Proof. We have

$$\begin{aligned} z^s_{n,m}(x) &= \sum_{k=1}^{n}\left[\frac{a_{k+1}}{(k+1)^m} + \sum_{j=k}^{n}\Delta^2\left(\frac{a_j}{j^m}\right)\right]k^m \sin kx \\ &= \sum_{k=1}^{n} a_k \sin kx + \left[\frac{a_{n+2}}{(n+2)^m} - \frac{a_{n+1}}{(n+1)^m}\right]\sum_{k=1}^{n} k^m \sin kx \\ &= S^s_n(x) - \Delta\left(b_{n+1,m}\right)\sum_{k=1}^{n} k^m \sin kx \end{aligned} \tag{2}$$

After some transformation we have found that

$$S^s_n(x) = \begin{cases} -\sum_{k=1}^{n} b_{k,m}(\cos kx)^{(m)}, & \text{if } m = 4p-3; \\ -\sum_{k=1}^{n} b_{k,m}(\sin kx)^{(m)}, & \text{if } m = 4p-2; \\ +\sum_{k=1}^{n} b_{k,m}(\cos kx)^{(m)}, & \text{if } m = 4p-1; \\ +\sum_{k=1}^{n} b_{k,m}(\sin kx)^{(m)}, & \text{if } m = 4p, \end{cases} \tag{3}$$

and

$$\sum_{k=1}^{n} k^m \sin kx = \begin{cases} -D^{(m)}_n(x), & \text{if } m = 4p-3; \\ -\widetilde{D}^{(m)}_n(x), & \text{if } m = 4p-2; \\ +D^{(m)}_n(x), & \text{if } m = 4p-1; \\ +\widetilde{D}^{(m)}_n(x), & \text{if } m = 4p, \end{cases} \tag{4}$$

where in all cases $p \in \mathbb{N}$. Combining (2) along with (3) and (4) we obtain

$$z^s_{n,m}(x) = \begin{cases} -\sum_{k=1}^{n} b_{k,m}(\cos kx)^{(m)} + \Delta\left(b_{n+1,m}\right) D^{(m)}_n(x), & \text{if } m = 4p-3; \\ -\sum_{k=1}^{n} b_{k,m}(\sin kx)^{(m)} + \Delta\left(b_{n+1,m}\right) \widetilde{D}^{(m)}_n(x), & \text{if } m = 4p-2; \\ +\sum_{k=1}^{n} b_{k,m}(\cos kx)^{(m)} - \Delta\left(b_{n+1,m}\right) D^{(m)}_n(x), & \text{if } m = 4p-1; \\ +\sum_{k=1}^{n} b_{k,m}(\sin kx)^{(m)} - \Delta\left(b_{n+1,m}\right) \widetilde{D}^{(m)}_n(x), & \text{if } m = 4p, \end{cases} \tag{5}$$

for all $p \in \mathbb{N}$. The use of Abel's transformation in (5), implies

$$z^s_{n,m}(x) = \begin{cases} -\sum_{k=1}^{n} \Delta b_{k,m} D^{(m)}_k(x) & \\ \qquad -b_{n,m} D^{(m)}_n(x) + \Delta\left(b_{n+1,m}\right) D^{(m)}_n(x), & \text{if } m = 4p-3; \\ -\sum_{k=1}^{n} \Delta b_{k,m} \widetilde{D}^{(m)}_k(x) & \\ \qquad -b_{n,m} \widetilde{D}^{(m)}_n(x) + \Delta\left(b_{n+1,m}\right) \widetilde{D}^{(m)}_n(x), & \text{if } m = 4p-2; \\ \sum_{k=1}^{n} \Delta b_{k,m} D^{(m)}_k(x) & \\ \qquad +b_{n,m} D^{(m)}_n(x) - \Delta\left(b_{n+1,m}\right) D^{(m)}_n(x), & \text{if } m = 4p-1; \\ \sum_{k=1}^{n} \Delta b_{k,m} \widetilde{D}^{(m)}_k(x) & \\ \qquad +b_{n,m} \widetilde{D}^{(m)}_n(x) - \Delta\left(b_{n+1,m}\right) \widetilde{D}^{(m)}_n(x), & \text{if } m = 4p, \end{cases} \tag{6}$$

for all $p \in \mathbb{N}$. Applying Abel's transformation in (3), we also get

$$S_n^s(x) = \begin{cases} -\sum_{k=1}^n \Delta b_{k,m} D_k^{(m)}(x) - b_{n,m} D_n^{(m)}(x), & \text{if } m = 4p-3; \\ -\sum_{k=1}^n \Delta b_{k,m} \widetilde{D}_k^{(m)}(x) - b_{n,m} \widetilde{D}_n^{(m)}(x), & \text{if } m = 4p-2; \\ +\sum_{k=1}^n \Delta b_{k,m} D_k^{(m)}(x) + b_{n,m} D_n^{(m)}(x), & \text{if } m = 4p-1; \\ +\sum_{k=1}^n \Delta b_{k,m} \widetilde{D}_k^{(m)}(x) + b_{n,m} \widetilde{D}_n^{(m)}(x), & \text{if } m = 4p, \end{cases} \tag{7}$$

for all $p \in \mathbb{N}$. Using Lemmas 1 and 2, in (6) and (7), we have that

$$|z_{n,m}^s(x)| \le \mathcal{O}\left(x^{-1}\right) \left(\sum_{k=1}^n k^m |\Delta b_{k,m}| + |a_n| + |a_{n+1}| + |a_{n+2}| \right), \tag{8}$$

and

$$|S_n^s(x)| \le \mathcal{O}\left(x^{-1}\right) \left(\sum_{k=1}^n k^m |\Delta b_{k,m}| + |a_n| \right), \tag{9}$$

for all $m \in \mathbb{N}$.

Whence, letting $n \to \infty$ in (8) and (9), and taking into account that $(a_k) \in \widetilde{BV}_{r,m}$, $m = 1, 2, \dots$), we conclude that series $\sum_{k=1}^\infty \Delta b_{k,m} D_k^{(m)}(x)$ and $\sum_{k=1}^\infty \Delta b_{k,m} \widetilde{D}_k^{(m)}(x)$ converge absolutely, and

$$\lim_{n\to\infty} z_{n,m}^s(x) = \lim_{n\to\infty} S_n^s(x) = g(x)$$

exists for all $x \in [\varepsilon, \pi]$, where $\varepsilon > 0$ as small as. Now, we have

$$g(x) - z_{n,m}^s(x) = \begin{cases} -\sum_{k=n+1}^\infty \Delta b_{k,m} D_k^{(m)}(x) & \\ \qquad +b_{n,m} D_n^{(m)}(x) - \Delta\left(b_{n+1,m}\right) D_n^{(m)}(x), & \text{if } m = 4p-3; \\ -\sum_{k=n+1}^\infty \Delta b_{k,m} \widetilde{D}_k^{(m)}(x) & \\ \qquad +b_{n,m} \widetilde{D}_n^{(m)}(x) - \Delta\left(b_{n+1,m}\right) \widetilde{D}_n^{(m)}(x), & \text{if } m = 4p-2; \\ +\sum_{k=n+1}^\infty \Delta b_{k,m} D_k^{(m)}(x) & \\ \qquad -b_{n,m} D_n^{(m)}(x) + \Delta\left(b_{n+1,m}\right) D_n^{(m)}(x), & \text{if } m = 4p-1; \\ +\sum_{k=n+1}^\infty \Delta b_{k,m} \widetilde{D}_k^{(m)}(x) & \\ \qquad -b_{n,m} \widetilde{D}_n^{(m)}(x) + \Delta\left(b_{n+1,m}\right) \widetilde{D}_n^{(m)}(x), & \text{if } m = 4p, \end{cases} \tag{10}$$

for all $p \in \mathbb{N}$. Thus, based on (10), we have

$$\|g - z_{n,m}^s\| \le \begin{cases} \int_0^\pi \left| \sum_{k=n+1}^\infty \Delta b_{k,m} D_k^{(m)}(x) \right| dx & \\ +|b_{n,m}| \int_0^\pi \left| D_n^{(m)}(x) \right| dx & \\ +|\Delta\left(b_{n+1,m}\right)| \int_0^\pi \left| D_n^{(m)}(x) \right| dx, & \text{if } m = 4p-3 \vee m = 4p-1; \\ \int_0^\pi \left| \sum_{k=n+1}^\infty \Delta b_{k,m} \widetilde{D}_k^{(m)}(x) \right| dx & \\ +|b_{n,m}| \int_0^\pi \left| \widetilde{D}_n^{(m)}(x) \right| dx & \\ +|\Delta\left(b_{n+1,m}\right)| \int_0^\pi \left| \widetilde{D}_n^{(m)}(x) \right| dx, & \text{if } m = 4p-2 \vee m = 4p, \end{cases} \tag{11}$$

for all $p \in \mathbb{N}$. Let us estimate the terms in right hand side of (11). Namely, since $(a_n) \in \widetilde{C}_m \cap \widetilde{BV}_m$ for each $m \in \{1, 2, \dots\}$, then for $\varepsilon > 0$ there exists $\delta > 0$, such that

$$\int_0^\delta \left| \sum_{k=n+1}^\infty \Delta b_{k,m} D_k^{(m)}(x) \right| dx \le \frac{\varepsilon}{2},$$

for all $n \geq 0$. Consequently, for $m = 4p-3 \vee m = 4p-1$ and Bernstein's inequality we get

$$\begin{aligned}
\int_0^\pi \left| \sum_{k=n+1}^{\infty} \Delta b_{k,m} D_k^{(m)}(x) \right| dx &= \int_0^\delta \left| \sum_{k=n+1}^{\infty} \Delta b_{k,m} D_k^{(m)}(x) \right| dx \qquad (12)\\
&\quad + \int_\delta^\pi \left| \sum_{k=n+1}^{\infty} \Delta b_{k,m} D_k^{(m)}(x) \right| dx \\
&\leq \frac{\varepsilon}{2} + \sum_{k=n+1}^{\infty} |\Delta b_{k,m}| \int_\delta^\pi \left| D_k^{(m)}(x) \right| dx \\
&\leq \frac{\varepsilon}{2} + \sum_{k=n+1}^{\infty} k^{m-1} |\Delta b_{k,m}| \int_\delta^\pi |D_k'(x)|\, dx \\
&\leq \frac{\varepsilon}{2} + C \sum_{k=n+1}^{\infty} k^{m} |\Delta b_{k,m}| \int_\delta^\pi \frac{dx}{x^2} \\
&\leq \frac{\varepsilon}{2} + \frac{C}{\delta} \sum_{k=n+1}^{\infty} k^{m} |\Delta b_{k,m}| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \qquad (13)
\end{aligned}$$

and in a similar way, for $m = 4p-2 \vee m = 4p$,

$$\int_0^\pi \left| \sum_{k=n+1}^{\infty} \Delta b_{k,m} \widetilde{D}_k^{(m)}(x) \right| dx < \varepsilon. \qquad (14)$$

The other terms also tend to zero, since they can be estimated as $a_n \log n$ (using Lemma 3). Using these facts, (11), (12) and (14), we have proved that

$$\lim_{n\to\infty} \|z_{n,m}^s - g\| = 0.$$

The proof is completed. ∎

Remark 7 *For $m = 1$ Theorem 7 reduces to Theorem 2.*

Theorem 8 *Let $(a_n) \in \widetilde{C}_m \cap \widetilde{BV}_m$ for each $m \in \{1, 2, \dots\}$, and $\lim_{n\to\infty} a_n \log n = 0$. Then*

$$\lim_{n\to\infty} \|S_n^s - g\| = 0.$$

Proof. Using Theorem 7, equalities (6), and equalities (7), we get

$$\begin{aligned}
\|S_n^s - g\| &= \|S_n^s - z_{n,m}^s + z_{n,m}^s - g\| \\
&\leq \|z_{n,m}^s - S_n^s\| + \|z_{n,m}^s - g\| \\
&\leq \begin{cases} |\Delta\left(b_{n+1,m}\right)| \int_0^\pi \left|D_n^{(m)}(x)\right| dx + o(1), & \text{if } m = 4p-3 \wedge m = 4p-1; \\ |\Delta\left(b_{n+1,m}\right)| \int_0^\pi \left|\widetilde{D}_n^{(m)}(x)\right| dx + o(1), & \text{if } m = 4p-2 \wedge m = 4p; \end{cases}
\end{aligned}$$

for all $p \in \mathbb{N}$. Now applying Lemma 3, Bernstein's inequality, Lemma 4, and conditions of our theorem, we obtain

$$\begin{aligned}
&\|S_n^s - g\| \\
\leq\; & \begin{cases} C(n+1)^m |\Delta\left(b_{n+1,m}\right)| \log(n+1) + o(1), & \text{if } m = 4p-3 \wedge m = 4p-1; \\ (n+1)|\Delta\left(b_{n+1,m}\right)| & \\ \times \left(\int_0^\pi \left|D_n^{(m-1)}(x)\right|dx + \int_0^\pi \left|F_n^{(m-1)}(x)\right|dx\right) + o(1), & \text{if } m = 4p-2 \wedge m = 4p; \end{cases} \\
\leq\; & \begin{cases} C(n+1)^m |\Delta\left(b_{n+1,m}\right)| \int_0^\pi \left|D_n(x)\right|dx + o(1), & \text{if } m = 4p-3 \wedge m = 4p-1; \\ (n+1)^m |\Delta\left(b_{n+1,m}\right) & \\ \times| \left(\int_0^\pi \left|D_n(x)\right|dx + \int_0^\pi \left|F_n(x)\right|dx\right) + o(1), & \text{if } m = 4p-2 \wedge m = 4p; \end{cases} \\
=\; & \mathcal{O}(a_{n+1}\log(n+1) + a_{n+2}\log(n+2) + o(1)) = o(1) \quad \text{as} \quad n \to \infty.
\end{aligned}$$

The proof is completed. ■

Remark 8 *For $m = 1$ Theorem 8 reduces to Theorem 3.*

The following statements also hold true.

Theorem 9 *Let $(a_n) \in \widetilde{C}_{r,m} \cap \widetilde{BV}_{r,m}$, $r \in \{0, 1, \dots\}$, $m \in \{1, 2, \dots\}$, and $\lim_{\to\infty} n^r a_n \log n = 0$. Then*

$$\lim_{n\to\infty} \|[z_{n,m}^s]^{(r)} - g^{(r)}\| = 0.$$

Proof. The proof is similar to the proof of Theorem 7. Therefore, we omit it. ■

Remark 9 *For $r = 0$, Theroem 8 reduces to Theorem 4*

Theorem 10 *Let $(a_n) \in \widetilde{C}_{r,m} \cap \widetilde{BV}_{r,m}$, $r \in \{0, 1, \dots\}$, $m \in \{1, 2, \dots\}$, and $\lim_{n\to\infty} n^r a_n \log n = 0$. Then*

$$\lim_{n\to\infty} \|[S_n^s]^{(r)} - g^{(r)}\| = 0.$$

Proof. The proof is similar to the proof of Theorem 8. Therefore, we omit it. ■

Remark 10 *For $r = 0$, Theorem 10 reduces to Theorem 5.*

Using Theorems 9 and 10, we obtain next consequence.

Corollary 2 *Let $(a_n) \in \widetilde{S}_{r,m}$, $r \in \{0, 1, \dots\}$, $m \in \{1, 2, \dots\}$, and $\lim_{n\to\infty} n^r a_n \log n = 0$. Then*

(i) $\lim_{n\to\infty} \|[z_{n,m}^s]^{(r)} - g^{(r)}\| = 0.$

(ii) $\lim_{n\to\infty} \|[S_n^s]^{(r)} - g^{(r)}\| = 0.$

For $m = 1$ we have:

Corollary 3 *([2]) Let $(a_n) \in \widetilde{S}_r$, $r \in \{0, 1, \dots\}$, and $\lim_{n\to\infty} n^r a_n \log n = 0$. Then*

(i) $\lim_{n\to\infty} \|[z_n^s]^{(r)} - g^{(r)}\| = 0.$

(ii) $\lim_{n\to\infty} \|[S_n^s]^{(r)} - g^{(r)}\| = 0.$

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References

[1] N. L. Braha and Xh. Z. Krasniqi, On L_1-convergence of certain cosine sums, Bull. Math. Anal. Appl., 1(2009), 55–61.

[2] S. K. Chouhan, J. Kaur and S. S. Bhatia, Extensions of Móricz classes and convergence of trigonometric sine series in L^1-norm, Mathematics, 6(2018), 292.

[3] S. K. Chouhan, J. Kaur and S. S. Bhatia, Convergence and Summability of Fourier sine and cosine series with its applications, Proc. Nat. Acad. Sci. India Sect. A, 89(2019), 141–148.

[4] J. W. Garrett and C. V. Stanojevic, On L^1 convergence of certain cosine sums, Proc. Amer. Math. Soc., 54(1976), 101–105.

[5] N. Hooda, B. Ram and S. S. Bhatia, On L^1-convergence of a modified cosine sum. Soochow J. Math., 28(2002), 305–310.

[6] K. Kaur, S. S. Bhatia and B. Ram, Integrability and L^1-convergence of modified sine sums, Georgian Math. J., 11(2004), 99–104.

[7] J. Kaur and S. S. Bhatia, Convergence of new modified trigonometric sums in the metric space L, J. Nonlinear Sci. Appl., 1(2008), 179–188.

[8] Xh. Z. Krasniqi, A note on L^1-convergence of the sine and cosine trigonometric series with semi-convex coefficients, Int. J. Open Probl. Comput. Sci. Math., 2(2009), 231–239.

[9] Xh. Z. Krasniqi, Some new modified cosine sums and L^1-convergence of cosine trigonometric series, Arch. Math., 49(2013), 43–50.

[10] F. Móricz, On the integrability and L^1-convergence of sine series, Studia Math., 92(1989), 187–200.

[11] B. Ram and S. Kumari, On L^1-convergence of certain trigonometric sums, Indian J. Pure Appl. Math., 20(1989), 908–914.

[12] S. Y.. Sheng, The extension of the theorems of C. V. Stanojevic and V. B. Stanojevic, Proc. Amer. Math. Soc., 110(1990), 895–904.

[13] S. A. Telyakovskii, On the integrability of sine series, Trudy Mat. Inst. Steklov. 163(1984), 229–233.

[14] S. A. Telyakovskii, A certain sufficient condition of Sidon for the integrability of trigonometric series, (Russian) Mat. Zametki, 14(1973), 317–328.

[15] Z. Tomovski, Some results on L^1-approximation of the r-th derivate of Fourier series, JIPAM. J. Inequal. Pure Appl. Math., 3(2002), 11 pp.