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On A General Huygens-Wilker Inequality*

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Abstract

This note will present an extension of a general Wilker type inequality. The proofs rely basically on iteration of derivations for real functions.

1 Introduction

We set

$$f(x) := a\left(\frac{x}{\sin(x)}\right)^m + b\left(\frac{x}{\tan(x)}\right)^n$$

for any $x \in]0, \frac{\pi}{2}[$ where a and b are two positive real numbers, m and $n \neq 0$. The inequality $f(x) > a+b$ for $a = 2$, $b = 1$, $m = -1$ and $n = -1$ is known as Huygens inequality and for $a = b = 1$, $m = -2$, $n = -1$ we obtain Wilker's inequality ([2, 4, 5]). These and more related inequalities were extensively studied, reproved and generalized see [9, 3, 1, 8, 10, 6, 7].

Our main focus is on the general inequality $f(x) > a+b$ where it is proved that $f(x)$ is strictly increasing on $]0, \frac{\pi}{2}[$ under some conditions on the parameters a, b, m and n. Inverse inequality cases of $f(x) < a+b$ are also derived.

Lemma 1 *The derivative $f'(x)$ is equal to:*

$$P(x)\left[am(\sin(x) - x\cos(x)) - bn\left(\frac{x}{\sin(x)}\right)^{n-m}\cos(x)^{n-1}(x - \cos(x)\sin(x))\right]$$

where $P(x) = \dfrac{1}{\sin(x)x\cos(x)^2}\left(\dfrac{x}{\sin(x)}\right)^m$ *and* $f'(x) = 0$ *on* $]0, \frac{\pi}{2}[$ *if and only if:*

1. $$\frac{am}{bn} = \left(\frac{x}{\sin(x)}\right)^{n-m}\cos(x)^{n-1}\left(\frac{x - \cos(x)\sin(x)}{\sin(x) - x\cos(x)}\right) = L(x), \tag{1}$$

2. $$\frac{am}{bn} = \left(\frac{x}{\tan(x)}\right)^{n-1}\left(\frac{x}{\sin(x)}\right)^{1-m}\left(\frac{x - \cos(x)\sin(x)}{\sin(x) - x\cos(x)}\right) = L(x), \tag{2}$$

3. $$\frac{am}{bn} = \left(\frac{x}{\sin(x)}\right)^{n-m}\cos(x)^{n}\left(\frac{\frac{x}{\cos(x)} - \sin(x)}{\sin(x) - x\cos(x)}\right) = H(x), \tag{3}$$

4. $$\frac{am}{bn} = \left(\frac{x}{\tan(x)}\right)^{n}\left(\frac{\sin(x)}{x}\right)^{m}\left(\frac{\frac{x}{\cos(x)} - \sin(x)}{\sin(x) - x\cos(x)}\right) = H(x). \tag{4}$$

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Of course the four expressions are all equivalent but it is mandatory to separate them to conclude.

It is worth mentioning that when $0 < n < 1$, $f(x)$ isn't an increasing function on $]0, \frac{\pi}{2}[$ as it can be shown that $f(x)$ is at least decreasing on $]\xi, \frac{\pi}{2}[$ for some ξ (from Lemma 1). However with the boundary condition $a((\frac{\pi}{2})^m - 1) \geq b$ added to $am \geq 2bn$ the inequality $f(x) > a + b$ on $]0, \frac{\pi}{2}[$ seems to hold for any m and n of same sign. In fact with $am \geq 2bn > 0$ studying the case $am = 2bn$ is sufficient and for $a = b$ the inequality is already proven in [11]. Some special cases for particular values of a, b, m and n are proved among others in [12] and [13].

2 Main Results

Before stating the main theorem we have the following:

Lemma 2 *The function* $D(x) := \dfrac{x - \cos(x)\sin(x)}{\sin(x) - x\cos(x)}$ *is strictly decreasing on* $]0, \frac{\pi}{2}[$.

Proof. First by applying a succession of Hospital's rule one can show that

$$\lim_{x \to 0} D(x) = 2 \;\text{ and }\; D'(x) = \frac{-\sin(x)(-2 + x^2 + 2\cos(x)^2 + \sin(x)x\cos(x))}{(\sin(x) - x\cos(x))^2}.$$

Then

$$S(x) := -2 + x^2 + 2\cos(x)^2 + \sin(x)x\cos(x) > 0 \;\text{ for } x \in]0, \frac{\pi}{2}[$$

since $S(0) = S'(0) = S''(0) = S'''(0) = 0$ and $S'''(x) = 2(\sin(2x) - 2x\cos(2x)) > 0$ on $]0, \frac{\pi}{2}[$. ■

Lemma 3 *The function* $I(x) := \dfrac{\frac{x}{\cos(x)} - \sin(x)}{\sin(x) - x\cos(x)}$ *is strictly increasing on* $]0, \frac{\pi}{2}[$.

Proof. Similarly to the precedent proof we have $\lim_{x\to 0} I(x) = 2$ and

$$I'(x) = \frac{-\sin(x)(\cos(x)^3 - \cos(x) + 2x^2\cos(x) - x\sin(x))}{\cos(x)^2(\sin(x) - x\cos(x))^2}.$$

If $C(x) := \cos(x)^3 - \cos(x) + 2x^2\cos(x) - x\sin(x)$, then we need to show that $x\tan(x) + 1 - \cos(x)^2 - 2x^2 \geq 0$ for all $x \in]0, \frac{\pi}{2}[$. Set $R(x) := x\tan(x) + 1 - \cos(x)^2 - 2x^2$, $R(0) = R'(0) = R''(0) = 0$, upon computing $R^{(3)}(x)$ we get $R^{(3)}(x) > 0$ on $]0, \frac{\pi}{2}[$ since

$$3\tan(x) + 3x\tan(x)^2 + x > 3\cos(x)^3\sin(x) + \cos(x)^3\sin(x)$$

and the result follows. ■

Theorem 1 *Let $a \geq 0$ and $b \geq 0$. If $am \geq 2bn$, m and n are of same sign not equal to zero and $0 > \min(m, n)$ or $\min(m, n) \geq 1$, then $f(x)$ is strictly increasing on $]0, \frac{\pi}{2}[$ consequently:*

$$f(x) := a\left(\frac{x}{\sin(x)}\right)^m + b\left(\frac{x}{\tan(x)}\right)^n > a + b \;\;\textit{for all } x \in]0, \frac{\pi}{2}[.$$

Proof. The inequality when $0 > \min(m, n)$, $m < 0$ and $n < 0$ was already proved in [1], $\dfrac{am}{bn} \leq 2$ and $H(x)$ as in (3) or (4) is strictly increasing on $]0, \frac{\pi}{2}[$ with $\lim_{x\to 0} H(x) = 2$, to see this from Lemma 3 consider (3) when $n \geq m$ and (4) for any $m < 0$, $n < 0$. If $\min(m, n) \geq 1$, $\dfrac{am}{bn} \geq 2$ but $L(x) < 2$ on $]0, \frac{\pi}{2}[$ as in (1) when $m \geq n \geq 1$. Also $L(x) < 2$ on $]0, \frac{\pi}{2}[$ as in (2) when $n \geq m \geq 1$ (by Lemma 2). ■

Corollary 1 *For the function $f(x)$ given, if $0 < n < 1$, $m \geq 1$, $am \geq 2b$, then $f(x) > a + b$ on $]0, \frac{\pi}{2}[$.*

Corollary 2 *Let $a \geq 0$ and $b \geq 0$. If $am \leq 2bn$, $a((\frac{\pi}{2})^m - 1) \leq b$, m and n are of same sign not equal to zero and $0 > \min(m, n)$ or $\min(m, n) \geq 1$, then*

$$f(x) := a\left(\frac{x}{\sin(x)}\right)^m + b\left(\frac{x}{\tan(x)}\right)^n < a + b \quad \textit{for all } x \in]0, \frac{\pi}{2}[.$$

Proof. From Lemma 1 and Theorem 1, it is easy to see that: under stated conditions f has at most one single critical point (minimum) on $]0, \frac{\pi}{2}[$; by the regularity of f and its boundary limit values $f(x) < a + b$ for all $x \in]0, \frac{\pi}{2}[$. ■

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